Shift operators on product of Banach spaces

Let $X$ and $Y$ be Banach spaces over the field $\mathbb{F}$, where $\mathbb{F}$ is the real or complex numbers. A linear operator from $X$ into $Y$ is a mapping $T : X \to Y$ such that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all $x, y \in X$ and for all $\alpha, \beta \in \mathbb{F}$.

If $T$ is one-to-one, then we say that $X$ and $Y$ are linearly isomorphic (or isomorphic) and we call $T$ a linear isomorphic (or isomorphism). Moreover, if $\|Tx\| = \|x\|$ for all $x \in X$, then we say that $X$ and $Y$ are isometrically isomorphic and call $T$ an isometric isomorphism (isometry).

Definition ([2]) A shift operator (or shift) is a linear operator $T$ from $X$ into $Y$ with the following properties: (i) $T$ is an isometry, (ii) $T$ has codimension 1, and (iii) $\bigcap_{n=1}^{\infty} = \{0\}$.

It is well-known in the literature that the product of two Banach spaces does not necessarily admit a shift even if each factor has a shift. In [3], it is shown that if each Banach space $X$ and $Y$ have a shift and $X$ is isometric with $Y$, then $X \times Y$ has a shift. This result leading to the following natural question: What if $X$ does not is not isometric with $Y$? An example of two Banach spaces each having a shift but their product not having shift was given in [4]. The simple case in the complex numbers was considered in [2], where $X = c$, the space of all convergent sequences and $Y = c_0$, the space of all sequences converging to zero. It was shown that if we take the max norm on the product $c \times c_0$, of $c$ and $c_0$, this product admit a shift.

In the proposed study, we consider the shift on $c \times c_0$, where $c$ and $c_0$ are as defined above, with a different norm, for example the sum norm, that is

$$\|(x, y)\| = \|x\|_c + \|y\|_{c_0}.$$ 

It is well-known that the norms in infinite dimensional spaces are not equivalent. Therefore, it will be interesting to see if the result obtained in [2], is still true with
this norm? For further research, one can consider such study on Banach spaces over the real numbers, to try to extend this result in that case or find counter example.

References


