Thin layer approximation for a coupled bulk-surface PDE

BIOMATH 2024

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Motivated by recent findings on Rho GTPases's self-organisation and pattern formation for the cell function in cell biology, we study a system of coupled bulk-surface partial differential equations, describing changes in concentration of these proteins. When activated, they are bound to the plasma membrane where they diffuse and react with the inactive species; inactivated species diffuse inside the cell cortex and these react with the activated species when they are close to the cell membrane. The cortex is modeled as an annulus-shaped domain with a relatively small thickness, while the plasma membrane is its outer circular boundary.

In the first part of the talk we show that the master equation for the changes in concentration of Rho GTPases is the Kolmogorow forward equation for an underlying stochastic process, and, in particular, the related Cauchy problem is well-posed in an L^1-type space and is governed by a Markov semigroup generator. Moreover, this process possesses additional regularity properties, as the corresponding backward equation describes dynamics of a fellerian nature—there exists the related semigroup in a space of continuous functions.

In the second part, since the cell cortex is typically rather thin, we investigate the limit as the thickness converges to 0. To this end, we observe that letting the thickness of the annulus to zero is equivalent to keeping it constant but increasing the rate of radial diffusion. Consequently, an appropriate rescaling of coefficients and finding a common reference space, allows deriving the form of the limit equation. The bulk solutions gradually lose dependence on the radial variable and in the limit they may be regarded as functions on the circle. Thus the limit equations can be seen as describing surface diffusion on two copies of the circle with jumps from one copy to the other: transmission conditions featuring in the approximating equations become integral part of the limit master equation. It is worth mentioning that these results are obtained

formally as convergence theorems in two different settings: for both Markov and Feller semigroups.

References

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