Quantitative uncertainty principles and the first non–zero eigenvalue on graphs

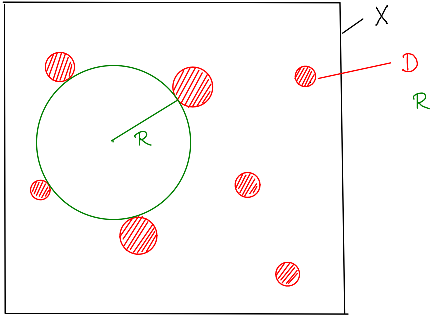
Peter Stollmann, joint work with Daniel Lenz, Marcel Schmidt and Gunter Stolz

DFG-AIMS Workshop “EVOLUTIONARY PROCESSES ON NETWORKS” AIMS Rwanda

March 2018

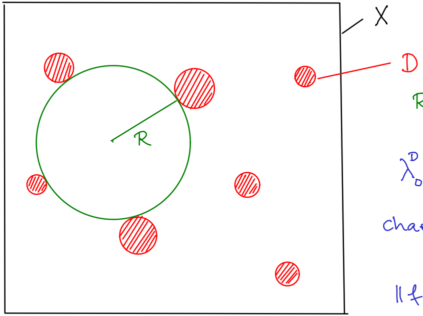
TECHNISCHE UNIVERSITÄT
CHEMNITZ

- Uncertainty principles – unique continuation: classical topic . . .
  functions with low energy are spread out in space
- Uncertainty principles: important for random Schrödinger operators . . .
- . . . many recent results: starting with Bourgain and Kenig, then Klein, Rojas-Molina, Veselić . . .
- . . . unique continuation **NOT TRUE** for graphs, but . . .
- a spectral uncertainty principle gives nice uniform results for a large class of graph laplacians.
- KEY: [Spectral] geometry for graphs – lower bounds for Dirichlet Laplacians on subsets



$\Omega = X \setminus D$

R = inradius of $\Omega < \infty$



$\Omega = X \setminus D$

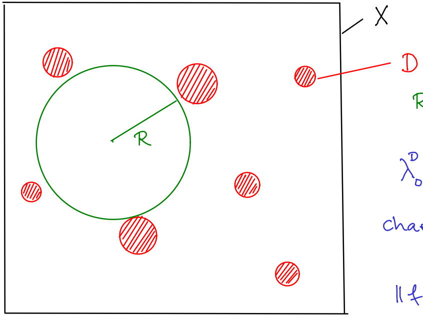
R = inradius of $\Omega < \infty$

$\lambda_0^D(\Omega)$ = 1st Dirichlet e.v.

characterized by

$\|f\|^2 \cdot \lambda_0^D(\Omega) \leq \|\nabla f\|^2$

for all f s.t. $f = 0$ on D



$\Omega = X \setminus D$

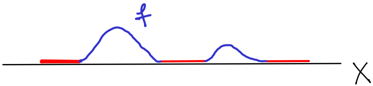
R = inradius of $\Omega < \infty$

$\lambda_0^D(\Omega)$ = 1st Dirichlet e.v.

characterized by

$\|f\|^2 \cdot \lambda_0^D(\Omega) \le \|\nabla f\|^2$

for all f s.t. $f = 0$ on D

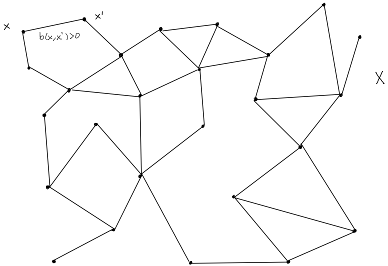


POINCARÉ INEQUALITY

Topology – Geometry:

A weighted graph (X, b) is given by

- a countable set X, finite or infinite;
- a symmetric $b : X \times X \to [0, \infty)$ called the *weight function* ($=$ **conductance**) satisfying $b(x, x) = 0$ for all $x \in X$.
- **edge** stands for positive weight

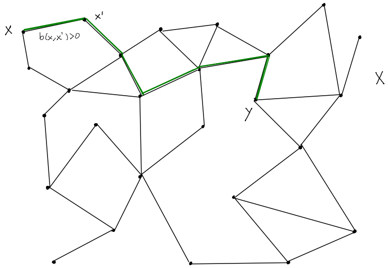


Measuring length:

- *path*: $\gamma = (x_0, x_1, ..., x_k)$ where $b(x_j, x_{j+1}) > 0$ for all $j = 0, ..., k-1$
- length of a path γ:

$$L(\gamma) := \sum_{j=0}^{k-1} \frac{1}{b(x_j, x_{j+1})}.$$

- *distance* between x and y:



$$d(x,y) := \inf\{L(\gamma) \mid \gamma \text{ a path from } x \text{ to } y\}; \qquad d(x,x) := 0.$$

Assume that (X, b) is connected$\Rightarrow d(x,y) < \infty$; d is pseudo−metric.

Example

Classical undirected graphs: X at most countable, $b : X \times X \to \{0, 1\}$ indicates whether there is an edge or not.
In this case: $d(x, y)$ is the **combinatorial distance**, counting the minimal number of edges for a path connecting x and y.

Our results are interesting and new for combinatorial graphs. One last quantity: The **energy** of $f \in \mathcal{F}(X) := \mathbb{R}^X$:

$$\mathcal{E}(f) := \frac{1}{2} \sum_{x,y \in X} b(x, y)|f(x) - f(y)|^2 \in [0, \infty].$$

Under appropriate conditions: $\mathcal{E}(f) < \infty$ for any

$$f \in \mathcal{F}_c = \{f \in \mathcal{F}(X) \mid \mathrm{supp}(f) \text{ is finite } \}.$$

- A topological Poincaré inequality
- Introducing a measure
- Lower bounds in the finite volume case
- Lower bounds in the infinite case
- The uncertainty principle
- Outlook

Proposition (d satisfies a topological Poincaré inequality)

Let $x, y \in X$ be arbitrary. Then for any path $\gamma = (x_0, \ldots, x_k)$ from x to y and $f \in \mathcal{D}$ the inequality

$$|f(x) - f(y)|^2 \leq L(\gamma) \sum_{j=0}^{k-1} b(x_j, x_{j+1})(f(x_j) - f(x_{j+1}))^2$$

holds. In particular

$$|f(x) - f(y)|^2 \leq d(x, y)\mathcal{E}(f)$$

is valid.

Proof:

It suffices to show the first inequality. Take a path $\gamma = (x_0, \ldots, x_k)$ from $x = x_0$ to $y = x_k$:

$$
\begin{aligned}
|f(x) - f(y)| &\leq \sum_{j=0}^{k-1} |f(x_j) - f(x_{j+1})| \\
&= \sum_{j=0}^{k-1} \frac{1}{b(x_j, x_{j+1})^{\frac{1}{2}}} \cdot b(x_j, x_{j+1})^{\frac{1}{2}} |f(x_j) - f(x_{j+1})| \\
&\leq L(\gamma)^{1/2} \left(\sum_{j=0}^{k-1} b(x_j, x_{j+1})(f(x_j) - f(x_{j+1}))^2 \right)^{\frac{1}{2}},
\end{aligned}
$$

using the triangle inequality and the Cauchy-Schwarz inequality. □

Let $m : X \to (0, \infty)$ and denote the corresponding measure by m as well.
This gives a Hilbert space $\ell^2(X, m)$ and we can view $\mathcal{E}$ as a symmetric (bilinear) form on $\ell^2(X, m)$, giving an associated operator H ...
Now we can introduce:

$$\lambda_1^N = \lambda_1^N(X, b, m) := \inf\{\mathcal{E}(f) \mid \|f\|_2 = 1, f \perp 1\}$$

“first [non-zero] Neumann eigenvalue”

and for $\Omega \subsetneq X$:

$$\lambda_0^D = \lambda_0^D(\Omega; X, b, m) := \inf\{\mathcal{E}(f) \mid \|f\|_2 = 1, f = 0 \text{ on } D := X \setminus \Omega\}.$$

“first [non-zero] Dirichlet eigenvalue” on Ω.

Both correspond to the graph Laplacian induced by $\mathcal{E}$.

Now consider the case that X (Ω, respectively) has finite volume $m(X) < \infty$ and finite diameter $\mathrm{diam}_d(X) := \sup\{d(x,y) \mid x, y \in X\} < \infty$.

Theorem

Assume $\mathrm{diam}_d(X) < \infty$. Then, for any finite measure m on X of full support,

$$\lambda_1^N \geq \frac{4}{\mathrm{diam}_d(X) \cdot m(X)}.$$

... follows by a clever decomposition of $f \perp 1$

$$\|f\|_2^2 \leq \frac{1}{4} sup_{x,y\in X} \left(f(x) - f(y)\right)^2 m(X) \leq \frac{1}{4} \mathrm{diam}_d(X) m(X) \mathcal{E}(f)$$

and the pointwise bound from the Proposition.

Theorem

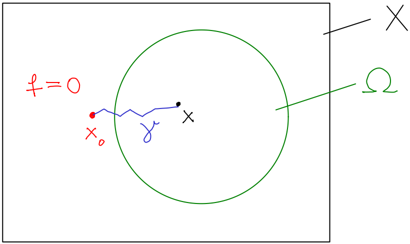
Assume $\Omega \subsetneq X$, $\text{Inr}(\Omega) < \infty$ *and* $m(\Omega) < \infty$. *Then*

$$\lambda_0^D \geq \frac{1}{\text{Inr}(\Omega) m(\Omega)}.$$

- Let $R > \text{Inr}(\Omega)$, $x \in \Omega$.
- $\exists x_0 \in X \setminus \Omega$, path γ from x to x_0 s.t. $L(\gamma) < R$.
- Proposition gives

$$|f(x)|^2 \leq L(\gamma)\mathcal{E}(f).$$

- Summing over Ω gives the estimate.



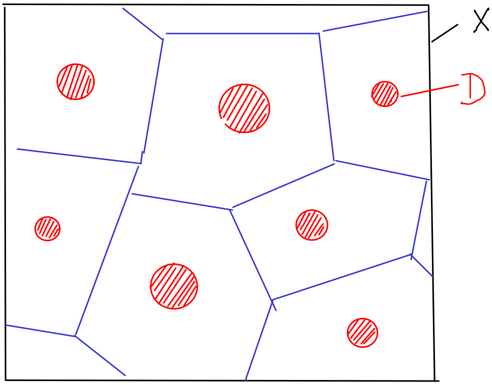
The estimate

$$\lambda_0^D \geq \frac{1}{\mathrm{Inr}(\Omega) m(\Omega)}$$

does not help, if $m(\Omega) = \infty$! Of course, we assume $\mathrm{Inr}(\Omega) < \infty$. Way out: Voronoi decomposition.

- Let the geometry be nice and $R > \mathrm{Inr}(\Omega)$:
- $\exists$ decomposition $V_\alpha, \alpha \in A$ of X.
- such that $V_\alpha \cap D \neq \emptyset$
- and $\mathrm{diam}(V_\alpha) < R$
- apply finite estimate on each the cells,

$$\|f\, 1_{V_\alpha}\|_2^2 \leq R m(V_\alpha) \mathcal{E}(f\, 1_{V_\alpha}).$$



Theorem

Assume $\Omega \subsetneq X$, $\mathrm{Inr}(\Omega) < \infty$ *and*

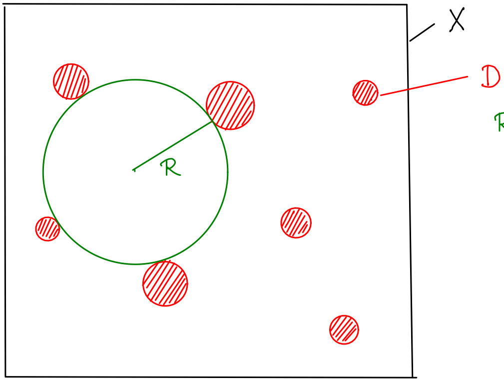
$$\mathrm{vol}^\sharp[\mathrm{Inr}(\Omega)] := \inf_{s > \mathrm{Inr}(\Omega)} \sup\{m(U_s(x)) \mid x \in X\} < \infty$$

Then

$$\lambda_0^D \geq \frac{1}{\mathrm{Inr}(\Omega)\mathrm{vol}^\sharp[\mathrm{Inr}(\Omega)]}.$$

We need that Ω is relatively dense, $\mathrm{Inr}(\Omega) < \infty$ and that the corresponding relative volume $\mathrm{vol}^\sharp[\mathrm{Inr}(\Omega)]$ is bounded. New result, even for very special cases like the combinatorial lattice $\mathbb{Z}^d$, for which there had been earlier estimates by Rojas–Molina and Elgart/Klein.

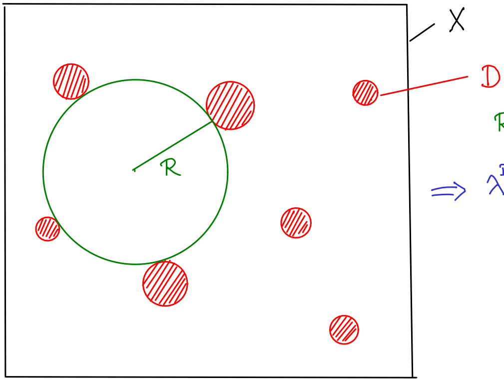
The uncertainty principle



$\Omega = X \setminus D$

R = inradius of $\Omega < \infty$

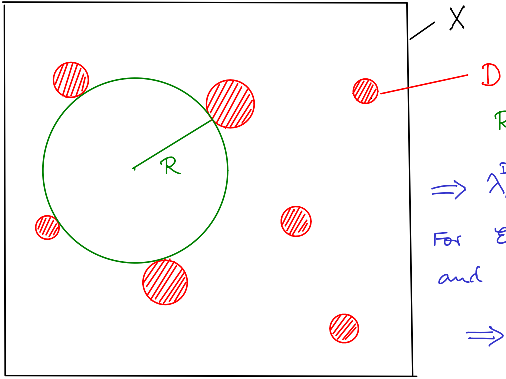
The uncertainty principle



$\Omega = X \setminus D$

R = inradius of $\Omega < \infty$

$\Longrightarrow \lambda_0^D(\Omega) \geq \frac{1}{R \cdot \mathrm{vol}^{\#}(R)}$



$\Omega = X \setminus D$

R = inradius of $\Omega < \infty$

$\Rightarrow \lambda_0^D(\Omega) \geq \frac{1}{R \cdot \mathrm{vol}^\#(R)}$

For $\mathcal{E}(f) <$ — " —

and $f|_D = 0$

$\Rightarrow f = 0$

For what follows we **assume** that (X, b, m) is such that the corresponding energy form $\mathcal{E}$ and hence the associated graph Laplacian H is bounded, and that the geometry is nice enough, namely

(B) Assume $\sup_{x\in X} \frac{1}{m(x)} \sum_{y\in X} b(x,y) =: \delta < \infty$,

so that $(Hf)(x) = \frac{1}{m(x)} \sum_{y\in X} b(x,y)(f(x) - f(y))$ is bounded by 2δ.

(M) Moreover, assume $m_{max} := \sup_{x\in X} m(x) < \infty$

Corollary (Qualitative version)

Let (X, b, m) be as above, $D \subset X$ relatively dense, i.e. $\Omega := X \setminus D$ has finite inradius. Let f have "small energy" in the sense that $f \in Range(P_I(H))$, where $\max I < \frac{1}{\mathrm{Inr}(\Omega)\cdot \mathrm{vol}^\sharp[\mathrm{Inr}(\Omega)]}$. *Then*

$$\|f 1_D\|^2 \geq \kappa \|f\|^2.$$

Corollary (Quantitative version)

Let (X, b, m) *be as above,* $D \subset X$ *relatively dense, i.e.* $\Omega := X \setminus D$ *has finite inradius. Let* $I \subset \mathbb{R}$ *such that* $\max I < \frac{1}{\mathrm{Inr}(\Omega)\cdot \mathrm{vol}^\sharp[\mathrm{Inr}(\Omega)]}$ *and* $f \in Range(P_I(H))$*. Then*

$$\|f 1_D\|^2 \geq \frac{\left(\frac{1}{\mathrm{Inr}(\Omega)\cdot \mathrm{vol}^\sharp[\mathrm{Inr}(\Omega)]} - \max I\right)^2}{16\|H+1\|^4}\|f\|^2.$$

- A **suitable distance** $d(x,y)$ provides insight into **spectral properties of graphs**.
- This leads to lower bounds on eigenfunctions **of low energy**.
- Possible extensions to spectral geometry on **quantum/metric graphs**.
- Applications of lower bounds for functions of low energy to more complicated networks.

- M. Barlow, T. Coulhon and A. Grigor'yan, *Manifolds and graphs with slow heat kernel decay*, Invent. Math. **144** (2001), 609–649
- A. Boutet de Monvel, D. Lenz and P. Stollmann, *An uncertainty principle, Wegner estimates and localization near fluctuation boundaries*, Math. Z. **269** (2011), 663–670.
- A. Elgart and A. Klein, *Ground state energy of trimmed discrete Schrödinger operators and localization for trimmed Anderson models*, J. Spectr. Theory **4** (2014), 391–413.
- D. Lenz, M. Schmidt, and P. Stollmann, *Topological Poincaré type inequalities and lower bounds on the infimum of the spectrum for graphs*, arXiv preprint, arXiv: 1801.09279 (2018)
- D. Lenz, P. Stollmann and Gunter Stolz: An uncertainty principle and lower bounds for the Dirichlet Laplacian on graphs. arXiv: 1606.07476, to appear in J. Spectr. Theory
- C. Rojas-Molina, *The Anderson model with missing sites*, Oper. Matrices **8** (2014), 287–299.