Singular diffusion on graphs with “sticky” vertices

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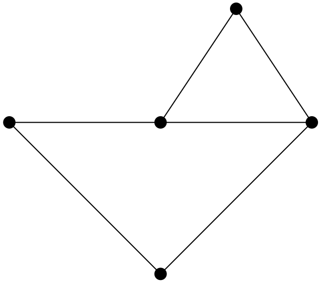
joint with J. Voigt (Dresden, Germany)

Motivation: What is singular (or gap) diffusion?



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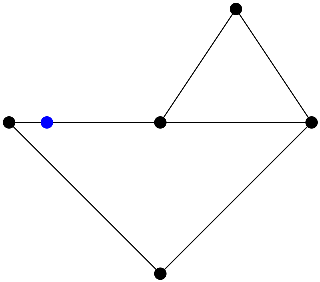

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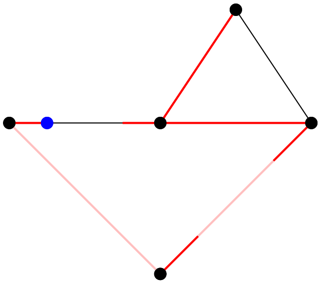

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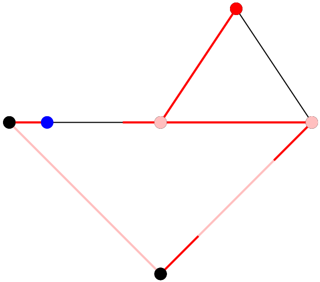

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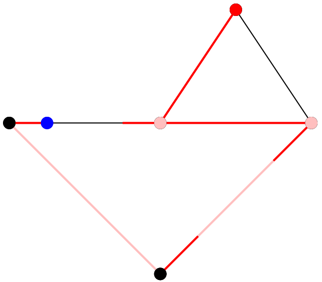

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Task: Describe singular diffusion within a functional analytic framework!

Second motivation

For graphs: two types of models

- discrete graphs: functions on the vertices
- metric graphs: functions on the edges

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For graphs: two types of models

- discrete graphs: functions on the vertices
- metric graphs: functions on the edges

Question: Combine both?!

History on the topic

- Feller 1954: generalized diffusion processes
- Lumer 1980: diffusion on graphs
- Langer, Schenk 1990: gap diffusion
- Weber 2000: gap diffusion and birth-and-death-processes on graphs
- Kant, Klauß, Voigt, Weber 2009: Dirichlet forms for singular diffusion
- S', Voigt 2011: Dirichlet forms for singular diffusion

Only one edge: Singular diffusion on an interval

Let $a, b \in \mathbb{R}$, $a < b$, μ a finite Borel measure on $[a, b]$, called speed measure, such that $a, b \in \operatorname{spt} \mu$ and $\mu(\{a, b\}) = 0$.

Define

$$C_\mu[a, b] := \{f \in C[a, b];\ f \text{ affine linear on the components of } [a, b] \setminus \operatorname{spt} \mu\},$$

$$W^1_{2,\mu}(a, b) := W^1_2(a, b) \cap C_\mu[a, b].$$

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$$\begin{aligned} C_\mu[a, b] &:= \{f \in C[a, b];\ f \text{ affine linear on the components of } [a, b] \setminus \operatorname{spt}\mu\}, \\ &= \{f \in C[a, b];\ f \text{ harmonic on } [a, b] \setminus \operatorname{spt}\mu\}, \\ W^1_{2,\mu}(a, b) &:= W^1_2(a, b) \cap C_\mu[a, b]. \end{aligned}$$

The imbedding

Let $\kappa\colon W_2^1(a,b)\cap C[a,b]\to L_2([a,b],\mu)$, $\kappa f := f$.

Lemma

$\operatorname{ran}(\kappa) = \operatorname{ran}(\kappa|_{W^1_{2,\mu}(a,b)})$ *and* $\kappa|_{W^1_{2,\mu}(a,b)}$ *is injective.*

Definition

Define $\iota := \left(\kappa|_{W^1_{2,\mu}(a,b)}\right)^{-1}$, i.e.

$$D(\iota) = \Big\{f \in L_2([a,b],\mu);\ \text{there exists } g \in W_2^1\ (a,b)\cap C[a,b] \\ \text{such that } g = f\ \mu\text{-a.e.}\Big\},$$

$$\iota f = g \in W^1_{2,\mu}(a,b).$$

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$$\iota f = g \in W^1_{2,\mu}(a,b).$$

The form

Let $X \subseteq \mathbb{K}^2$ be a subspace, L self-adjoint in X. Define a form τ in $L_2([a,b],\mu)$ by

$$D(\tau) := \left\{f \in D(\iota);\ \mathrm{tr}(\iota f) := \big(\iota f(a), \iota f(b)\big) \in X\right\},$$

$$\tau(f,g) := \int_a^b (\iota f)'(x)\overline{(\iota g)'(x)}\, d\mu(x) + (L\,\mathrm{tr}(\iota f) \,|\, \mathrm{tr}(\iota g)) .$$

Example

- $X = \{0\}$ yields Dirichlet boundary conditions.
- $X = \mathbb{K}^2$ and $L = 0$ yields Neumann boundary conditions.

Proposition

τ is densely defined, symmetric, bounded from below and closed.

The associated operator

Theorem (first representation theorem)

Let τ be densely defined, symmetric, bounded from below and closed. Then there exists a unique self-adjoint lower semibounded operator H such that

$$\tau(f, g) = (Hf \mid g) \quad (f \in D(H), g \in D(\tau)).$$

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Definition

Let $g \in L_{1,\mathrm{loc}}(a,b)$. If distributional derivative g' has density w.r.t. μ, then $\partial_\mu g := \frac{dg'}{d\mu}$.

Let Q be the orthogonal projection onto X.

Theorem

$$\begin{aligned} D(H) = \big\{ f \in L_2((a,b),\mu);\ & f \in D(\iota),\, \mathrm{tr}(\iota f) \in X, \\ & \partial_\mu(\iota f)' \text{ exists}, \\ & Q\big(-(\iota f)'(b-), (\iota f)'(a+)\big) = L\,\mathrm{tr}(\iota f)\big\}, \end{aligned}$$

$$Hf = -\partial_\mu(\iota f)'.$$

Remarks

- H is associated to classical Dirichlet form with imbedding $\kappa|_{W^1_{2,\mu}(a,b)}$.
- Depending on properties of X and L the form τ is a Dirichlet form.
- If τ is a Dirichlet form, then it is the trace of the classical Dirichlet form w.r.t. $\kappa|_{W^1_{2,\mu}(a,b)}$.
- Analogues in higher dimensions are available.

Examples

- Let $a = 0$, $b = 3$, μ supported on $[0,1] \cup [2,3]$, $\mu(\{1,2\}) = 0$. For $f \in D(H)$: $(\iota f)'$ continuous at 1 and at 2 and

$$(\iota f)'(1) = (\iota f)'(2) = \frac{(\iota f)(2) - (\iota f)(1)}{2 - 1}.$$

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$$(\iota f)'(1) = (\iota f)'(2) = \frac{(\iota f)(2) - (\iota f)(1)}{2 - 1}.$$
- Let $a = 0$, $b = 2$, $\mu = \lambda + c\delta_1$ for some $c > 0$. For $f \in D(H) \subseteq C^1[0,2]$:
$$Hf = -f'' \quad \text{a.e.},$$
$$Hf(1) = -\frac{f'(1+) - f'(1-)}{c}.$$

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$$Hf = -f'' \quad \text{a.e.},$$
$$Hf(1) = -\frac{f'(1+) - f'(1-)}{c}.$$
- Let $a = 0$, $b = 4$, μ supported on $[0,1] \cup \{2\} \cup [3,4]$, $\mu(\{2\}) = c > 0$. For $f \in D(H)$:
$$\begin{aligned} Hf(2) &= \frac{(\iota f)'(2+) - (\iota f)'(2-)}{c} \\ &= -\frac{1}{c}\left(\frac{(\iota f)(3) - (\iota f)(2)}{3-2} - \frac{(\iota f)(2) - (\iota f)(1)}{2-1}\right). \end{aligned}$$

Notation for (finite) graphs

Let $\Gamma := (V, E, \gamma, a, b)$ be a finite directed metric graph, i.e.

- V finite set of vertices of Γ,
- E finite set of edges of Γ,
- $\gamma = (\gamma_0, \gamma_1)\colon E \to V \times V$, $\gamma_0(e)$ "starting vertex" and $\gamma_1(e)$ "end vertex",
- $a, b\colon E \to \mathbb{R}$ such that $a_e < b_e$ for all $e \in E$; identify $e \in E$ with $[a_e, b_e] \subseteq \mathbb{R}$

For $e \in E$ let μ_e be a finite Borel measure on $[a_e, b_e]$ satisfying either

- $a_e, b_e \in \operatorname{spt} \mu_e$ and $\mu_e(\{a_e, b_e\}) = 0$, or
- $\mu_e = 0$.

Denote

$$E_0 := \{e \in E;\ \mu_e = 0\}, \qquad E_1 := E \setminus E_0.$$

For $v \in V$ let $\mu_v \geqslant 0$ be a weight of v and

$$V_0 := \{v \in V;\ \mu_v = 0\}, \qquad V_1 := V \setminus V_0.$$

The form

Hilbert space

$$\mathcal{H}_\Gamma := \bigoplus_{e\in E_1} L_2([a_e,b_e],\mu_e) \oplus \mathbb{K}^{V_1}$$

with inner product given by

$$(f\,|\,g)_{\mathcal{H}_\Gamma} := \sum_{e\in E_1}\int_{a_e}^{b_e} f_e(x)\overline{g_e(x)}\,d\mu_e(x) + \sum_{v\in V_1} f_v\overline{g_v}\mu_v.$$

Define ι from $\mathcal{H}_\Gamma$ to $\prod_{e\in E_1} W^1_{2,\mu_e}(a_e,b_e)\times\mathbb{K}^{V_1}$, by

$$\begin{aligned} D(\iota) &:= \{f\in\mathcal{H}_\Gamma;\ f_e\in D(\iota_e)\ (e\in E_1)\},\\ (\iota f)_e &:= \iota_e f_e \qquad (e\in E_1),\\ (\iota f)_v &:= f_v \qquad (v\in V_1). \end{aligned}$$

We define the trace mapping (or boundary value mapping) $\operatorname{tr}\colon \prod_{e\in E_1} C[a_e,b_e]\times\mathbb{K}^{V_1}\to\mathbb{K}^{E_1'\cup V_1}$, where $E_1' := E_1\times\{0,1\}$, by

$$\operatorname{tr} f(e,j) := \begin{cases} f_e(a_e) & \text{if } e\in E_1,\ j=0,\\ f_e(b_e) & \text{if } e\in E_1,\ j=1,\end{cases}$$

$$\operatorname{tr} f(v) := f_v \qquad (v\in V_1).$$

The form

Let $X \subseteq \mathbb{K}^{E_1' \cup V_1}$ by a subspace, L self-adjoint in X. Then we define the form τ by

$$D(\tau) := \{f \in D(\iota);\ \operatorname{tr}(\iota f) \in X\},$$

$$\tau(f,g) := \sum_{e \in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x)\overline{(\iota_e g_e)'(x)}\, dx + (L \operatorname{tr}(\iota f) \,|\, \operatorname{tr}(\iota g)) \,.$$

Example

- $X = \{0\}$ yields Dirichlet boundary conditions.
- $X = \mathbb{K}^{E_1' \cup V_1}$ and $L = 0$ yields Neumann boundary conditions.
- $\mu_e = 0$ for all e and $L = \Delta_{\mathrm{disc}}$ yields discrete Laplacian.

X and L couple boundary values from the edges and values in the vertices.

Properties of τ

Lemma

τ is symmetric. Moreover, $D(\tau)$ is dense if and only if

$$\mathrm{pr}_{V_1}(X) = \mathbb{K}^{V_1}, \tag{1}$$

where $\mathrm{pr}_{V_1} : \mathbb{K}^{E_1' \cup V_1} \to \mathbb{K}^{V_1}$ is the projection.

Theorem

τ is bounded from below and closed.

The associated operator

Define the signed trace (or signed boundary values)

$$\operatorname{str}\colon \prod_{e\in E_1} BV(a_e, b_e) \to \mathbb{K}^{E_1'} \subseteq \mathbb{K}^{E_1' \cup V_1}$$

by

$$\operatorname{str} g(e,j) := \begin{cases} g_e(a_e+) & \text{if } e \in E_1,\ j = 0, \\ -g_e(b_e-) & \text{if } e \in E_1,\ j = 1. \end{cases}$$

Define the maximal operator $\hat{H}$ on the edges by

$$\begin{aligned} D(\hat{H}) := \{ f \in \prod_{e\in E_1} D(\iota_e);\ & \partial_{\mu_e}(\iota_e f_e)' \text{ exists}, \\ & \partial_{\mu_e}(\iota_e f_e)' \in L_2([a_e, b_e], \mu_e)\ (e \in E_1) \}, \\ \hat{H} f := (-\partial_{\mu_e}(\iota_e f_e)')_{e\in E_1} \qquad & (f \in D(\hat{H})). \end{aligned}$$

The associated operator

Let

$$X_0 := \{x \in X;\ \operatorname{pr}_{V_1} x = 0\} = X \cap \mathbb{K}^{E_1'},$$

and let Q_0 be the orthogonal projection from $\mathbb{K}^{E_1' \cup V_1}$ onto X_0. Also, for $v \in V_1$, let $\xi^v \in X$ be such that $\xi^v|_{V_1} = \mathbb{1}_{\{v\}}$.
For $f \in D(\iota)$ write $(\iota f)' := \big((\iota_e f_e)'\big)_{e \in E_1}$.

Theorem

The operator H associated with the form τ is given by

$$\begin{aligned} D(H) = \big\{f \in \mathcal{H}_\Gamma;\ (f_e)_{e \in E_1} \in D(\hat{H}),\ \operatorname{tr}(\iota f) \in X, \\ Q_0 \operatorname{str}(\iota f)' = Q_0 L \operatorname{tr}(\iota f)\big\}, \end{aligned}$$

$$\begin{aligned} ((Hf)_e)_{e \in E_1} &= \hat{H}(f_e)_{e \in E_1}, \\ (Hf)_v &= \frac{1}{\mu_v}\big(L \operatorname{tr}(\iota f) - \operatorname{str}(\iota f)' \,|\, \xi^v\big) \qquad (v \in V_1). \end{aligned}$$

Proof.

(i) For $f \in D(\hat{H})$, $g \in D(\tau)$:

$$\sum_{e \in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x) \overline{(\iota_e g_e)'(x)} \, dx = \left(\hat{H} f \,\middle|\, (g_e)_{e \in E_1} \right) - (\mathrm{str}((\iota f)') \,|\, \mathrm{tr}\,(\iota g)) \, .$$

(ii) Use (i) to establish description of H.

Sublattices

Consider $\mathbb{K}^n$ as $C(\{1,\ldots,n\})$. Write $|x| = (|x_1|,\ldots,|x_n|)$ for $x \in \mathbb{K}^n$, and $x \wedge y = (x_1 \wedge y_1,\ldots,x_n \wedge y_n)$ for $x, y \in \mathbb{R}^n$.

Definition

Let $X \subseteq \mathbb{K}^n$. Then X is called sublattice if X is a subspace and $x \in X$ implies $|x| \in X$. A sublattice X is called Stonean if $x \wedge \mathbb{1} \in X$ for all real $x \in X$.

Lemma

Let $X \subseteq \mathbb{K}^n$ be a subspace, $m := \dim X$.

(a) *X is a sublattice $\iff$ there exist $x^1,\ldots x^m \in X_+$, $x^j \wedge x^k = 0$ for $j \neq k$ such that $X = \mathrm{lin}\{x^j;\ j = 1,\ldots,m\}$.*

(b) *X is a Stonean sublattice $\iff$ there exists a partition $C_1,\ldots,C_m$ of a subset of $\{1,\ldots,n\}$ such that $X = \mathrm{lin}\{\mathbb{1}_{C^j};\ j = 1,\ldots m\}$.*

Positivity and Contractivity

Definition

Let $Y := L_2(\nu)$ for some measure ν, $-A$ the generator of a C_0-semigroup on Y. Then $(e^{-tA})_{t\geqslant 0}$ is called positivity preserving if $e^{-tA}y \geqslant 0$ for all $y \in Y_+$, $t \geqslant 0$. It is called submarkovian if it is positivity preserving and (e^{-tA}) is L_∞-contractive.

Lemma

Let X be a m-dimensional subspace and a sublattice of $\mathbb{K}^n$, $L = (l_{jk})$ self-adjoint in X.

(a) *(e^{-tL}) is positivity preserving $\Longleftrightarrow$ $l_{jk} \leqslant 0$ for all $j, k \in \{1, \ldots, m\}$, $j \neq k$.*

(b) *Let X be Stonean. Then (e^{-tL}) is submarkovian $\Longleftrightarrow$ $l_{jk} \leqslant 0$ for all $j, k \in \{1, \ldots, m\}$, $j \neq k$ and $\sum_{j=1}^{m} \sqrt{|C_j|}\, l_{jk} \geqslant 0$ for all $k = 1, \ldots, m$.*

Positivity and Contractivity

Theorem

(a) *Assume that X is a sublattice of $\mathbb{K}^{E_1' \cup V_1}$ and that the semigroup $(e^{-tL})_{t \geqslant 0}$ is positivity preserving. Then $(e^{-tH})_{t \geqslant 0}$ is positivity preserving.*
(b) *Assume that X is a Stonean sublattice of $\mathbb{K}^{E_1' \cup V_1}$ and that the semigroup $(e^{-tL})_{t \geqslant 0}$ is submarkovian. Then $(e^{-tH})_{t \geqslant 0}$ is submarkovian.*

Remark

(b) says that τ is a Dirichlet form.

Local Boundary Conditions

Up to now: no graph structure needed!

For $v \in V$ let

$$X_v \subseteq \mathbb{K}^{E'_{\mathbf{1},v}} \quad \text{if } v \in V_0, \qquad X_v \subseteq \mathbb{K}^{E'_{\mathbf{1},v} \cup \{v\}} \quad \text{if } v \in V_1.$$

a subspace, L_v self-adjoint in X_v such that

$$X = \bigoplus_{v \in V} X_v, \qquad L = \bigoplus_{v \in V} L_v.$$

For τ to be densely defined: For $v \in V_1$:

$$X_{v,0} := \{\xi \in X_v;\ \xi(v) = 0\}.$$

Then τ densely defined if $X_{v,0} \neq X_v$ for all $v \in V_1$, i.e. for all $v \in V_1$ there exists $\xi^v \in X_v$ such that $\xi^v(v) = 1$.