Abstract boundary systems and application to flow in networks with memory

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Flows in networks with delay in the vertices

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(1)\begin{cases}
\frac{d}{dt}u_j(x,t) &=& \frac{d}{dx}u_j(x,t),\ t\geq 0, x\in(0,1),\\
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\phi^-_{ij}u_j(1,t) &=& \omega_{ij}\sum_{k\in J}\phi^+_{ik}[u_k(0,t)+L_k(u_k(\cdot,t+\cdot))],\ t\geq 0,\\
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for $i \in I := \{1, \dots, p\}$, $j \in J := \{1, \dots, n\}$. Here, $\Phi^- = (\phi^-_{ij})_{p \times n}$ and $\Phi^+ = (\phi^+_{ij})_{p \times n}$ are the *outgoing* and *incoming incidence matrix* of the directed graph G, respectively.

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for $i \in I := \{1,\dots,p\}$, $j \in J := \{1,\dots,n\}$. Here, $\Phi^- = (\phi^-_{ij})_{p\times n}$ and $\Phi^+ = (\phi^+_{ij})_{p\times n}$ are the *outgoing* and *incoming incidence matrix* of the directed graph G, respectively. The weights $\omega_{ij} \neq 0$ satisfies $\sum_{j\in J}\omega_{ij} = 1$, $\forall i \in I$, and $L_k \in \mathcal{L}(L^1([-1,0]\times[0,1]),\mathbb{C})$, $k \in J$.

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for all $f = (f_k)_{k=1}^n \in W := W^{1,1}([0,1], \mathbb{C}^n)$ and $g = (g_k)_{k=1}^n \in W^{1,1}([-1,0], W)$,

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and on $X \times Y$

$$D(\mathcal{A}_{\mathbb{B}}) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in D(A_m) \times D(B_m) : f(1) - \mathbb{B} f(0) = \mathbb{B} L(g), f = g(0) \right\}$$

$$\mathcal{A}_{\mathbb{B}} = \begin{pmatrix} A_m & 0 \\ 0 & B_m \end{pmatrix}.$$

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Using semigroup theory and perturbation techniques one proves the wellposedness of (ACP) and hence of (1).

Spectrum

Define the matrix

$$\mathbb{B}_\lambda := e^{-\lambda}\mathbb{B}(I + L_\lambda), \quad \lambda \in \mathbb{C},$$

where $L_\lambda c := L((\sigma, x) \mapsto e^{\lambda\sigma} e^{\lambda x} c)$, $c \in \mathbb{C}^n$.

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Then,

$$\lambda \in \sigma(\mathcal{A}_\mathbb{B}) \Longleftrightarrow 1 \in \sigma(\mathbb{B}_\lambda) \Longleftrightarrow \lambda \in P\sigma(\mathcal{A}_\mathbb{B}).$$

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where P is the spectral projection of rank one corresponding to the spectral value $s(\mathcal{A}_{\mathbb{B}})$, and $(\mathcal{T}_{\mathbb{B}}(t))_{t\geq 0}$ is the semigroup solution to (ACP).

Flow in networks with memory

$$(2)\begin{cases} \frac{d}{dt}u_j(x,t) &= \frac{d}{dx}u_j(x,t) + \sum_{k\in J} D_{jk}(u_k(x,\cdot+t)),\ t\geq 0, x\in(0,1) \\ u_j(x,0) &= f_j(x),\ x\in(0,1), \\ \phi_{ij}^- u_j(1,t) &= \omega_{ij}\sum_{k\in J}\phi_{ik}^+[u_k(0,t)+L_k(u_k(\cdot,t+\cdot))],\ t\geq 0, \\ u_j(x,\tau) &= g_j(x,\tau),\ x\in(0,1),\tau\in[-1,0], \end{cases}$$

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for $i\in I := \{1,\dots,p\}$, $j\in J := \{1,\dots,n\}$.
Here

$$L_k(g_k) = \int_{-1}^0 d\nu_k(\theta)g_k(\theta),\ \ D_{jk}(g_k) = \int_{-1}^0 d\mu_{jk}(\theta)g_k(\theta)$$

for $g=(g_k)\in W^{1,1}([-1,0],X)$, where
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for $g = (g_k) \in W^{1,1}([-1,0],X)$, where $\mu = (\mu_{jk}) : [-1,0] \to \mathcal{L}(X)$ and $\nu = (\nu_k) : [-1,0] \to \mathcal{L}(X,\mathbb{C}^n)$ are BV. Note that the above operators are no more bounded on $L^1([-1,0],X)$.

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$$\begin{cases} \dot{u}(t) = A_m u(t) + D u_t, & t \geq 0, \\ G u(t) = M u(t) + L u_t, & t \geq 0, \\ u(0) = x, \ u(s) = \varphi(s) & s \in [-1, 0]. \end{cases}$$

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Here

$$G,\ M : W^{1,1}([0,1], \mathbb{C}^n) \to \mathbb{C}^n;$$
$$Gf = f(1),\ Mf = \mathbb{B}f(0), \quad f \in W^{1,1}([0,1], \mathbb{C}^n).$$

Abstract boundary delay equations

As in the first part (2) can be transformed as

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where

$$\begin{aligned} D(\mathcal{A}_{D,L}) &= \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in D(A_m) \times D(B_m) : \right. \\ &\qquad (G-M)f = \mathbb{B}L(g), f = g(0) \} \end{aligned}$$

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So, the hard job to do is to prove that $\mathcal{A}_{0,L}$ generates a C_0-semigroup on $X \times Y$.
The classical perturbation theory doesn't work for $\mathcal{A}_{0,L}$, since the delay operator L is unbounded. One needs a so-called Staffans-Weiss closed loop systems.

Staffans-Weiss perturbations

Assumptions:

(H1) the operator $A \subset A_m$ with domain $D(A) = ker(G)$ generates a C_0-semigroup $(T(t))_{t \geq 0}$ on X;

(H2) $G : Z \to \partial X$ is surjective;

Using **(H1)** and **(H2)**, we can define the "Dirichlet" operator

$$\begin{aligned} \mathbb{D}_\lambda &= G^{-1}_{|ker(\lambda - A_m)} \in \mathcal{L}(\partial X, Z), \\ B &= (\lambda - A_{-1})\mathbb{D}_\lambda \in \mathcal{L}(\partial X, X_{-1}), \ \lambda \in \rho(A), \end{aligned}$$

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We also assume that

(H3) the triple operator (A, B, C) generates a regular linear system on $X, \partial X, \partial X$ with the identity operator $I_{\partial X}$ as admissible feedback.

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$$\mathcal{A}_{0,L} \text{ generates } C_0\text{-semigroup on } X \times Y.$$

References

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