Transport processes in networks with scattering ramification nodes

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Graphs

$$G = (\{v_1, \ldots, v_n\}, \{e_1, \ldots, e_m\})$$

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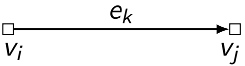

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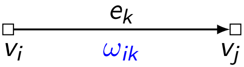

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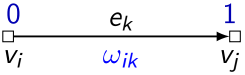

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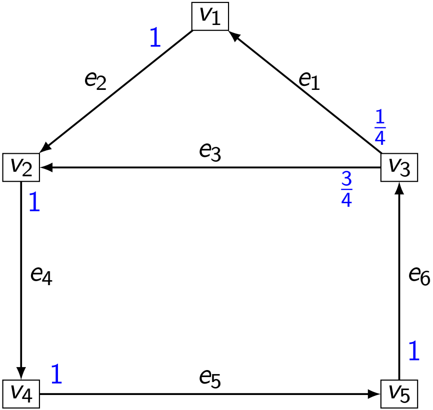

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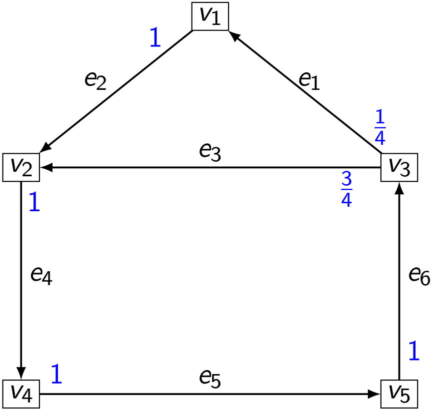

weighted adjacency matrix

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & \frac{1}{4} & 0 & 0 \\ 1 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Transport in networks

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$v \equiv const. \longrightarrow$ M. Kramar and E. Sikolya (2004)

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(F) equivalent to

$$(ACP)\begin{cases}\dot{u}(t) = Au(t), \qquad t \ge 0,\\ u(0) = g.\end{cases}$$

Wellposedness

Theorem

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Other properties of the semigroup?

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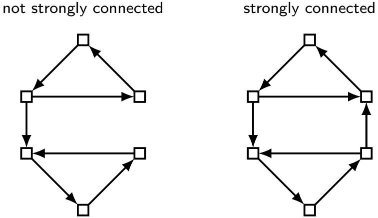

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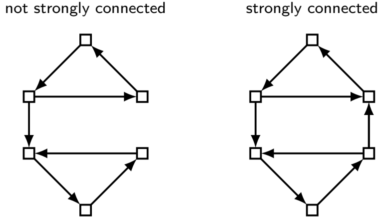

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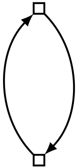

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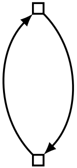

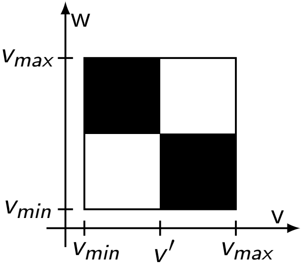

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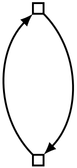

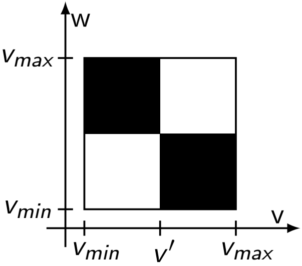

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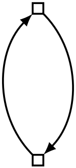

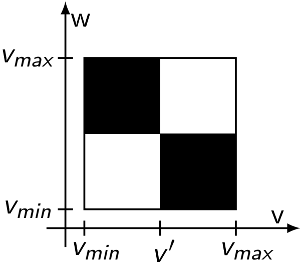

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If $u_1(x, v) = 0$ if $0 \leq x \leq 1$ and $v_{min} \leq v \leq v' < v_{max}$,

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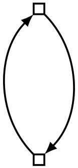

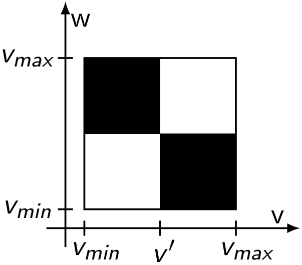

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Conclusion

G strongly connected and J irreducible $\overset{i.g.}{\not\Rightarrow}$ $(T(t))_{t \geq 0}$ irreducible

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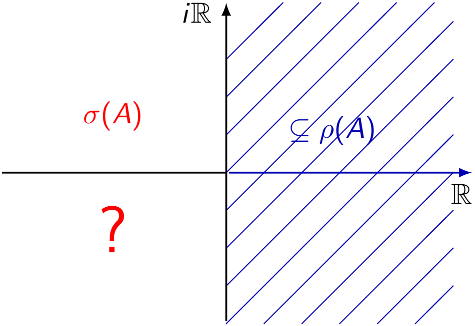

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- $0 \in \sigma(A)$

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Characteristic equation

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Schaefer 1974, Prop. V.7.4

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Spectrum of A

Theorem

$$J \textit{ as above and } k \gg 0 \quad \Rightarrow \quad \sigma(A) \cap \mathrm{i}\,\mathbb{R} = \{0\}.$$

Proof.

$\square$

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contradiction

□

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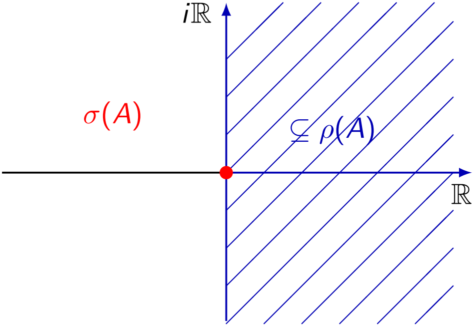

Asymptotics

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"strong convergence to 1-dim equilibrium"

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"strong convergence to 1-dim equilibrium"

Theorem

G strongly connected and $k \gg 0$

$$\Downarrow$$

$X = X_1 \oplus X_2$ *where*

- $X_1 = \mathrm{fix}(T(t))_{t \geq 0} = \ker A, \quad X_1 = \langle u \rangle$ *for some* $u \gg 0$,
- $\forall f \in X_2 \qquad T(t)f \stackrel{t \to \infty}{\longrightarrow} 0.$

Strong convergence to 1-dim equilibrium

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Proof.

- $(T(t))_{t\geq 0}$ irreducible $\Rightarrow$ $(T(t))_{t\geq 0}$ mean ergodic $\Rightarrow$ $X = \ker A \oplus \overline{\mathrm{rg} A} =: X_1 \oplus X_2$

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- Theory of positive semigroups:

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- Consider $(T(t)|_{X_2})_{t\geq 0}$ with generator $(A_2, D(A_2))$:

  $\sigma_p(A_2') \cap \mathrm{i}\,\mathbb{R} = \emptyset \Rightarrow (T_2(t))_{t\geq 0}$ strongly stable (Arendt-Batty-Lyubich-Vũ Theorem)

□

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