Interdependent decision making: A view from statistical mechanical window

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- An individual's beliefs, preferences and opportunities are shaped by peer groups, role models and social networks.
- The focus here is on group determinants of behaviour, i.e. how an individual's decision/choice is determined by one's social context.
- One can become a member of group through neighbourhood residence, ethnicity, enrolment at a particular school, employment at a particular firm, etc.
- Binary Choice: Choosing from two alternatives e.g
  - entry into illegal activity
  - use of drugs
  - use of treated mosquito nets
  - attendance of antenatal care etc.

Non-cooperative Binary Choice

"An object can have no value unless it has a utility. No one will give anything for an article unless it yields him satisfaction. Doubtless people are sometimes foolish, and buy things, as children do, to please a moment's fancy; but at least they think at the moment that there is a wish to be gratified"
– F.M. Taussig, Principles of Economics, 1912

Non-cooperative Binary Choice

The Model:

- Consider a group of N-agents who must choose a binary action at some common time.
- The choice of the i^{th} agent is coded by $\sigma_i \in \{-1, 1\}$.
- Choices of N-agents: N-tuple $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$
- Agent i chooses σ_i to maximizes his/her utility function

$$V(\sigma_i) = h\sigma_i + J\bar{m}_i^e\sigma_i + \epsilon(\sigma_i)$$

Non-cooperative Binary Choice

The Model:

1. Private Utility: $h\sigma_i, \quad h \in \mathbb{R}$
2. Social Utility: $J\bar{m}_i^e\sigma_i, \quad J > 0$. $\bar{m}_i^e$ is agent i's belief about the mean choice level of the rest of the agents.
3. Random Utility: $\epsilon(\sigma_i)$–part of the utility that is unobserved by the experimenter. $\epsilon(\sigma_i)$'s are assumed to be independent and extreme-value distributed with

$$\mathbb{P}\left(\epsilon(-\sigma_i) - \epsilon(\sigma_i) \leq x\right) = \frac{1}{1 + \exp(-\beta x)}, \qquad \beta > 0.$$

Non-cooperative Binary Choice

Note that if agent i chooses σ_i then $V(\sigma_i) > V(-\sigma_i)$. Therefore, the probability that agent i chooses σ_i takes the form

$$\begin{aligned} P(\sigma_i) &= P\left(V(\sigma_i) > V(-\sigma_i)\right) \\ &= P\left(h\sigma_i + J\bar{m}_i^e\sigma_i + \epsilon(\sigma_i) > -h\sigma_i - J\bar{m}_i^e\sigma_i + \epsilon(-\sigma_i)\right) \\ &= P\left(\epsilon(-\sigma_i) - \epsilon(\sigma_i) < 2h\sigma_i + 2J\bar{m}_i^e\sigma_i\right) \\ &= \frac{\exp\left(\beta\left[h\sigma_i + J\sigma_i\bar{m}_i^e\right]\right)}{\sum_{\eta_i\in\{-1,1\}}\exp\left(\beta\left[h\eta_i + J\eta_i\bar{m}_i^e\right]\right)}. \end{aligned}$$

β– describes the extent to which the deterministic part of the utility determines the actual choice.

Non-cooperative Binary Choice

- Suppose agents in the reference group do not coordinate their decisions, i.e. σ_i's are independent. Then at equilibrium, the probability that the reference group chooses $\sigma = (\sigma_1, \ldots, \sigma_N)$ is given by

$$P(\sigma) = \prod_{i=1}^{N} \frac{\exp\left(\beta\left[h\sigma_i + J\sigma_i\bar{m}_i^e\right]\right)}{\sum_{\eta_i\in\{-1,1\}} \exp\left(\beta\left[h\eta_i + J\eta_i\bar{m}_i^e\right]\right)}$$

- Assume that each agents' belief of the mean choice level is set to a common value $\bar{m}$, i.e.

$$\bar{m}_i^e = \bar{m}, \qquad \text{for every} \qquad i = 1, \ldots, N.$$

Then, $\sigma_1, \ldots, \sigma_N$ are independent and identically distributed random variables with common mean

$$E(\sigma_1) = \tanh\left(\beta h + \beta J\bar{m}\right).$$

Non-cooperative Binary Choice

- Rational Expectation: Under the equilibrium state expectations are rational [C. Manski, 1993], i.e.

  $$E(\sigma_1) = \bar{m}.$$

  Thus

  $$\bar{m} = \tanh\left(\beta h + \beta J \bar{m}\right).$$

- Equilibrium mean choice level of an agent in the reference group is a solution to the above fixed-point equation.
- There could be multiple solutions due to the properties of the tanh function.
- The multiplicity of average equilibrium levels of choice depends sensitively on the strength of the private and social utilities.

Non-cooperative Binary Choice

In particular, the following result is found in [W.A. Brock and S.N. Durlauf, 1995].

Proposition

The following holds for the solutions to the above fixed-point equation:

1. *Suppose $h = 0$, then;*
   a. *There is a unique solution provided $0 < \beta J \leq 1$.*
   b. *There are three solutions in the $\beta J > 1$ regime.*
2. *For the case $h \neq 0$, there is a threshold H, depending on β and h, such that*
   a. *There is a unique solution with the same sign as h if $|\beta h| > H$.*
   b. *There are three solutions, one of them has the same sign as h, and the others have opposite sign, if $|\beta h| \leq H$.*

Curie-Weiss Model

The Curie-Weiss model is given by an energy function/Hamiltonian H_N on $\{-1,1\}^N$, given by

$$H_N(\sigma) = \frac{J}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j + h\sum_{i=1}^{N}\sigma_i$$
$$= N\left[\frac{J}{2}m_N^2 + hm_N\right] + \frac{1}{2},$$

where

$$m_N = \frac{1}{N}\sum_{i=1}^{N}\sigma_i.$$

- $h \in \mathbb{R}$– interaction bias (external field)
- $J > 0$– interaction strength
- First term in H_N aligns pairs of spins, while the second aligns spins in the direction of h.

Curie-Weiss Model

- Let α be probability measure on $\{-1, 1\}$, such that $\alpha(-1) = \alpha(1) = \frac{1}{2}$ and $P_N = \alpha^{\otimes N}$ be the corresponding product measure on $\{-1, 1\}^N$
- The equilibrium state of the system is given by the probability measure

$$\mu_N(\sigma) = \frac{1}{Z_N} \exp\left(\beta H_N(\sigma)\right) P_N(\sigma),$$

where partition function Z_N is given by

$$Z_N = \sum_{\tilde{\sigma} \in \{-1,1\}^N} \exp\left(\beta H_N(\tilde{\sigma})\right) P_N(\tilde{\sigma}).$$

Curie-Weiss Model

- Consider the specific empirical magnetization

$$m_N(\beta, h) = \frac{1}{N} \int_{\Omega_N} \left(\sum_{i=1}^{N} \sigma_i \right) d\mu_N(\sigma)$$
$$= \frac{1}{\beta N} \frac{\partial}{\partial h} \log Z_N.$$

- The specific magnetization

$$m(\beta, h) = \lim_{N \to \infty} m_N(\beta, h)$$

is dependent on the exponential growth rate of the partition function, i.e.

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N$$

Curie-Weiss Model

Existence of he exponential growth rate of the partition function follows from sub-additivity.

Theorem

Suppose $z(\beta, h)$ *is the global minimizer of the function*

$$i_{\beta,h}(z) = -\frac{1}{2}\beta J z^2 - \beta h z + \frac{1-z}{2}\log(1-z) + \frac{1+z}{2}\log(1+z)$$

on $|z| \leq 1$ *and* $z(\beta, \pm) = \lim_{h\to 0^{\pm}} z(\beta, h)$*.*
Then the specific magnetization $m(\beta, h)$ *of the Curie-Weiss model equals* $z(\beta, h)$ *for* $\beta > 0, \quad h \neq 0$ *and* $0 < \beta J \leq 1,$ $h = 0$*. In particular,*

$$m(\beta, \pm) = \lim_{h\to 0^{\pm}} m(\beta, h) = \begin{cases} z(\beta, 0) = 0 & \text{for } \; 0 < \beta J \leq 1, \\ z(\beta, \pm) \neq 0 & \text{for } \; \beta J > 1. \end{cases}$$

Curie-Weiss Model

- For $\beta J > 1$, any small change in h around $h = 0$ will lead to a dramatic change in the macroscopic behaviour of the system.
- The specific magnetization $m(\beta, h)$ satisfies the self-consistent equation

$$m = \tanh(\beta h + J\beta m).$$

- Some of the fixed-point solutions may be local minimum and maximum points of $i_{\beta,h}(z)$.

Coordinated Binary Choice

. . . the purposefulness of the objects of analysis in social science contexts also means that issues of the endogeneity of neighbourhoods and the potential for the existence of institutions which coordinate collective action will naturally arise These issues have no analogue in physical contexts and are suggestive of the limitations in importing methods from physics into socio-economic studies.

[W.A. Brock and S.N. Durlauf, Handbook of Econometrics, Vol. 5, Elsevier Science B.V., 2001]

Coordinated Binary Choice

- Reference group of is made up of N individuals labelled by $I_N = \{1, 2, \ldots, N\}$.
- Group is divided into three non-overlapping subgroups with N_s, N_r and N_t agents labelled by $I_{N,s}$, $I_{N,r}$ and $I_{N,t}$ respectively.
- N_s, N_r and N_t are such that

$$s = \frac{N_s}{(N-1)}, \quad r = \frac{N_r}{(N-1)}, \quad \text{and} \quad t = \frac{(N_t - 1)}{(N-1)}.$$

- Note that $N = N_s + N_r + N_t \quad \text{and} \quad s + r + t = 1.$

Coordinated Binary Choice

- $I_{N,s}$-labelled agents have all chosen 1
- $I_{N,r}$-labelled agents have chosen -1
- $I_{N,t}$-labelled agents are to choose between 1 and -1 subject to maximization of the utility function

$$V_c(\sigma_i) = h\sigma_i + J\bar{m}_i^e\sigma_i + \epsilon(\sigma_i).$$

Note that for any $i \in I_{N,t}$

$$\bar{m}_i^e = \frac{1}{N-1}\sum_{j\in I_N\setminus\{i\}}\sigma_j = s - r + \frac{t}{N_t - 1}\sum_{j\in I_{N,t}\setminus\{i\}}\sigma_j$$

$$= s - r + t\, m_i^e$$

Coordinated Binary Choice

- Therefore

$$V_c(\sigma_i) = (h + J[s - r])\sigma_i + Jtm_i^e\sigma_i + \epsilon(\sigma_i).$$

- Using the form of the distribution of $\epsilon(-\sigma_i) - \epsilon(\sigma_i)$ we get that for any $i \in I_{N,t}$

$$P(\sigma_i) = \frac{\exp\left(\beta\left[\{h + J(s - r)\}\sigma_i + Jtm_i^e\sigma_i\right]\right)}{\sum_{\eta_i \in \{-1,1\}} \exp\left(\beta\left[\{h + J(s - r)\}\eta_i + Jtm_i^e\eta_i\right]\right)}.$$

- The equilibrium distribution for the choices made by the agents in $I_{N,t}$, if they deciding non-cooperatively is given by

$$P(\sigma) = \prod_{i \in I_{N,t}} \frac{\exp\left(\beta\left[\{h + J(s - r)\}\sigma_i + Jtm_i^e\sigma_i\right]\right)}{\sum_{\eta_i \in \{-1,1\}} \exp\left(\beta\left[\{h + J(s - r)\}\eta_i + Jtm_i^e\eta_i\right]\right)}$$

Coordinated Binary Choice

- Suppose $m_i^e = m$, for every $i \in I_{N,t}$, then

$$\begin{aligned} E(\sigma_i) &= \tanh\left(\beta[h + J(s-r)] + \beta J t m\right) \\ &= \tanh\left(\beta[h + J(s-r)] + \beta J[1-s-r]m\right) \end{aligned}$$

- It follows from the rational expectation condition that

$$E(\sigma_i) = m^* = \tanh\left(\beta[h + J(s-r)] + \beta J[1-s-r]m^*\right)$$

- Therefore, the mean choice level $\bar{m}$ of the group equals to

$$\bar{m}_i^e = s - r + tm^* = s - r + (1-s-r)m^*.$$

Conditional Curie-Weiss Model

- A group of N-individuals is partitioned into three non-overlapping groups labelled by I_{N,r_N}, I_{N,s_N} and I_{N,t_N}, with

$$r_N = \frac{|I_{N,r_N}|}{N}, \quad s_N = \frac{|I_{N,s_N}|}{N} \quad \text{and} \quad t_N = \frac{|I_{N,t_N}|}{N}.$$

- Note that $r_N + s_N + t_N = 1$. Assume that

$$r_N \to r, \quad s_N \to s \quad \text{and} \quad t_N \to t \quad \text{as} \quad N \to \infty.$$

  Then $s + r + t = 1$.

- In the sequel we assume that $0 < s + r < 1$.

Conditional Curie-Weiss Model

- Consider the subset $\Omega_{N,s,r}$ of $\{-1,1\}^N$ consisting of configurations $\sigma = (\sigma_1, \ldots, \sigma_N)$ such that

$$\sigma_i = \begin{cases} 1 & \text{if } i \in I_{N,s_N}; \\ -1 & \text{if } i \in I_{N,r_N}; \\ \eta_i & \text{if } i \in I_{N,t_N}, \end{cases}$$

  where $\eta_i \in \{+1, -1\}$.

- Conditional Curie-Weiss model $\mu_{N,s,r}$ is defined as

$$\mu_{N,s,r}(\sigma) = \mu_N(\sigma|\Omega_{N,s,r}) = \frac{\mu_N(\sigma)}{\mu_N(\Omega_{N,s,r})} = \frac{e^{\beta H_{N,s,r}(\sigma)}}{Z_{N,s,r}}$$

Conditional Curie-Weiss Model

Here

$$H_{N,s,r}(\sigma) = \frac{Jt_N}{2|I_{N,t_N}|} \sum_{i,j \in I_{N,t_N}} \sigma_i \sigma_j + [J(s_N - r_N) + h] \sum_{i \in I_{N,t_N}} \sigma_i.$$

- Note that $\mu_{N,s,r}$ is a probability measure on $\Omega_{N,s,r}$.
- The specific magnetization is given by

$$m(\beta, J, s, r, h) = \lim_{N \to \infty} \int_{\Omega_{N,s,r}} \left(\frac{1}{N} \sum_{i=1}^{N} \sigma_i \right) d\mu_{N,s,r}(\sigma)$$

Conditional Curie-Weiss Model

Define

$$i_{\beta,J,s,r,h}(z) = -\frac{1}{2}\beta J(1-s-r)z^2 - \beta[J(s-r)+h]z + \frac{1-z}{2}\log(1-z) + \frac{1+z}{2}\log(1+z).$$

Let

- $z(\beta, J, s, r, h)$ be a global minimizer of $i_{\beta,J,s,r,h}(z)$ on $-1 \leq z \leq 1$.
- and

$$z^{\pm} = \lim_{h \to J(r-s)^{\pm}} z(\beta, J, s, r, h).$$

Conditional Curie-Weiss Model

Theorem

For $\beta, J > 0$, $h \neq J(r-s)$ *and* $0 < \beta J \leq (1-s-r)^{-1}$, $h = J(r-s)$, *the specific magnetization* $m(\beta, J, s, r, h)$ *is given by*

$$m(\beta, J, s, r, h) = s - r + (1-s-r)z(\beta, J, s, r, h).$$

In particular,

$$m(\beta, J, s, r, (r-s)^{\pm}) = \lim_{h \to J(r-s)^{\pm}} m(\beta, J, s, r, h)$$

$$= \begin{cases} s - r, & 0 < \beta J \leq (1-s-r)^{-1}, \\ s - r + (1-s-r)z^{\pm} \neq s - r, & \beta J > (1-s-r)^{-1}. \end{cases}$$

Conditional Curie-Weiss Model

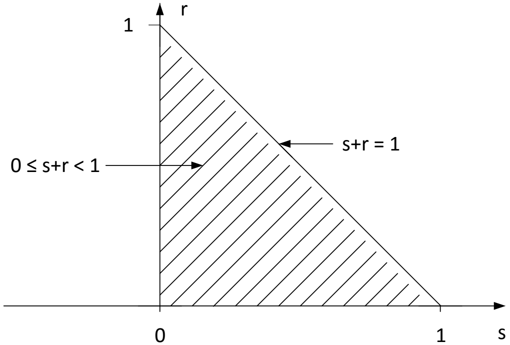


Figure: *The shaded region in the sr– plane is the region of interest.*

Conditional Curie-Weiss Model

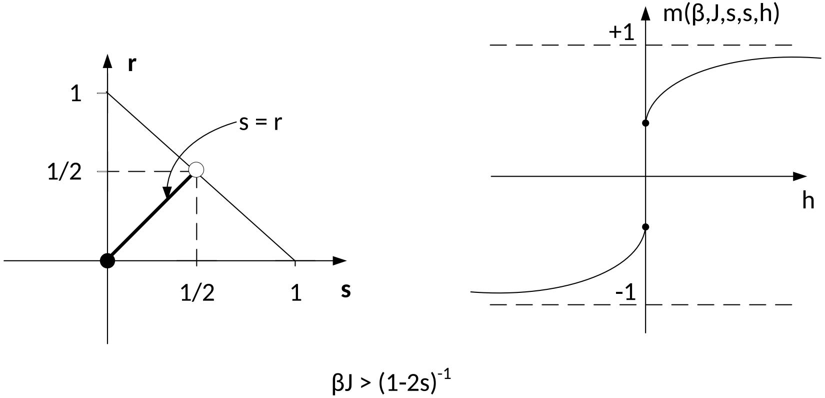


Figure: *Discontinuity in the map* $h \mapsto m(\beta, J, s, r, h)$ *at* $h = 0$ *for* $\beta J > (1-2s)^{-1}$ *and* $s = r$.

Conditional Curie-Weiss Model

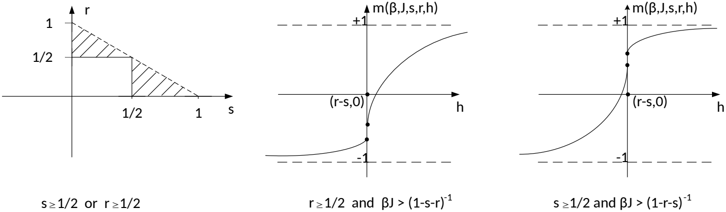
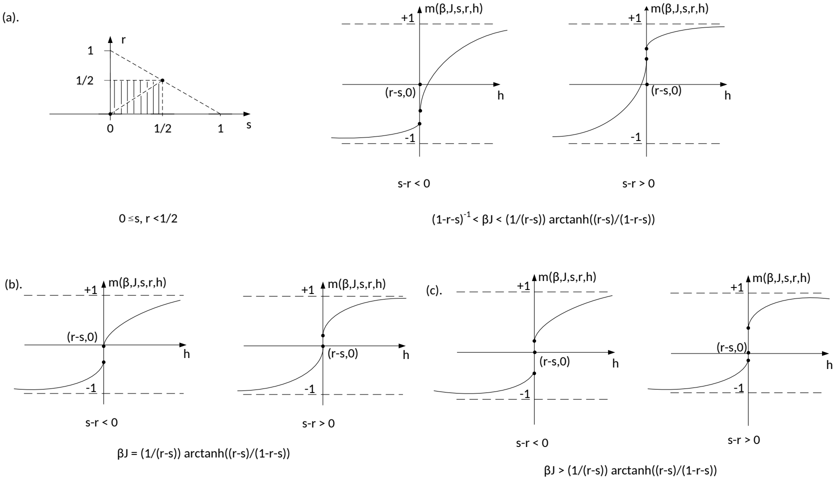


Figure: *Discontinuity in the map $h \mapsto m(\beta, J, s, r, h)$ at $h = J(r - s)$ for $\beta J > (1 - s - r)^{-1}$ and for choices of (s, r) from the shaded region in the sr-plane. The shaded region corresponds to $\frac{1}{2} \leq r < 1$ such that $s - r < 0$ or $\frac{1}{2} \leq s < 1$ with $s - r > 0$.*

Conditional Curie-Weiss Model



THANK YOU