Perturbation methods for differential operators on networks

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Joint work with Klaus-Jochen Engel, University of L'Aquila

K.-J. Engel, MKF, *Waves and Diffusion on Metric Graphs with General Vertex Conditions*, arXiv:1712.03030.

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- K.-J. Engel, MKF, *Flows on Metric Graphs*, preprint.

Outline

1. Dynamical processes on networks (= metric graphs)
2. C_0-semigroups in a nutshell
3. Boundary perturbation
4. Back to networks

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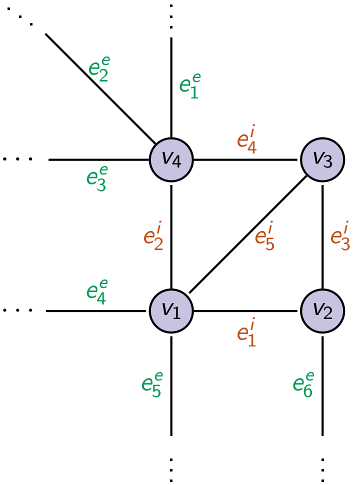

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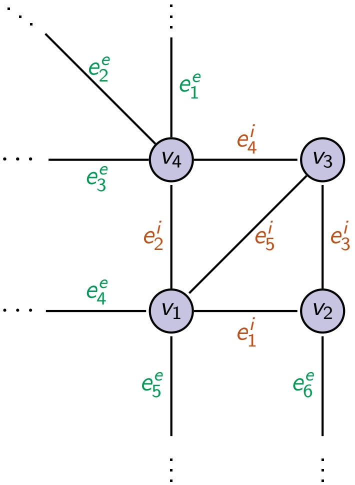

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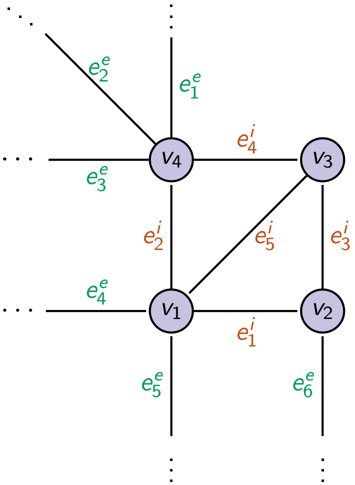

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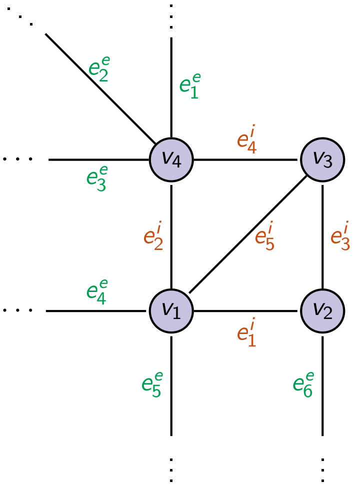

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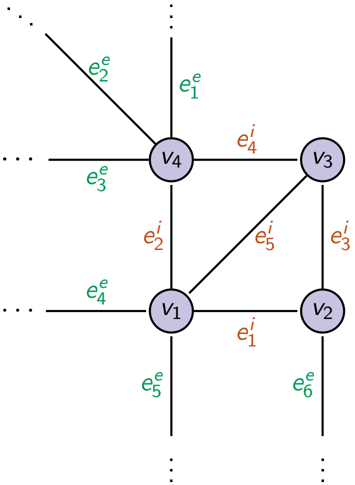

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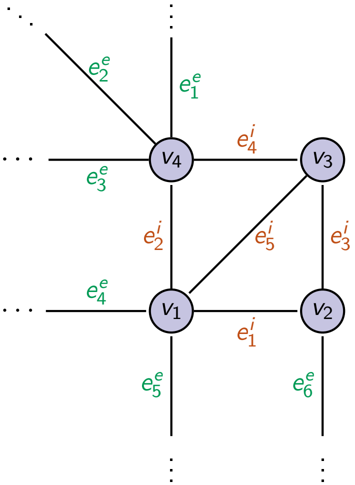

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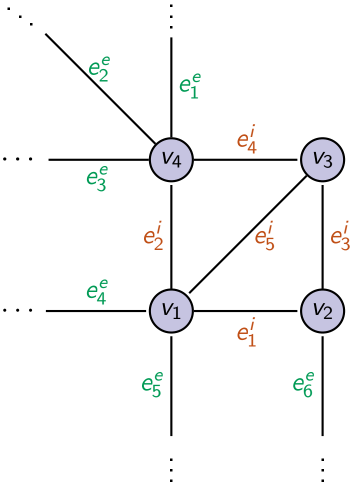

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- structure given by incidence / adjacency matrices

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Lumer, Roth, von Below, Nicaise, Ali Mehmeti, Cattaneo, Langnese, Leugering, Schmidt, Voigt, Exner, Kostrykin, Schrader, Kottos, Smilansky, Kuchment, Kurasov, KF, Sikolya, Mugnolo, Engel, Nagel, Klöss, Radl, Dorn, Keicher, Bayazit, Rhandi, Arendt, Dier, Post, Banasiak, Falkiewicz, Namayanja, Husein, Siegl, Seifert, ...

C_0-semigroups in a nutshell

X a Banach space, $T : [0, \infty) \to \mathcal{L}(X)$.

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1. Rewrite (*IBVP*) as an abstract Cauchy problem

$$(ACP_1)\begin{cases}\dot{x}(t) = Ax(t),\\ x(0) = x_0,\end{cases} \quad \textit{and} \quad (ACP_2)\begin{cases}\ddot{x}(t) = Ax(t),\\ x(0) = x_0,\\ \dot{x}(0) = x_1,\end{cases}$$

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K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, 2000.

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- K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, 2000.
- A. Batkai, MKF, A. Rhandi, Positive Operator Semigroups: from Finite to Infinite Dimensions, Birkhauser-Verlag, Basel, 2017. *(Networks: Chapter 18)*

Boundary perturbation

Assumptions

▸ skip

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Greiner, Salomon, Staffans-Weiss, Hadd-Manzo-Rhandi, **Adler-Bombieri-Engel**

skip

Theorem (Adler-Bombieri-Engel, 2014)

Assume that there exist $1 \leq p < +\infty$, $t_0 > 0$ and $M \geq 0$ such that

$$\text{(i)} \int_0^{t_0} T_{-1}(t_0 - s)L_A v(s)\, ds \in X \text{ for all } v \in \mathrm{L}^p([0,t_0],\partial X),$$

$$\text{(ii)} \int_0^{t_0} \left\|\Phi\, T(s)x\right\|_{\partial X}^p ds \leq M \cdot \|x\|_X^p \text{ for all } x \in D(A),$$

$$\text{(iii)} \int_0^{t_0} \left\|\Phi \int_0^r T_{-1}(r-s)L_A v(s)\, ds\right\|_{\partial X}^p dr \leq M \cdot \|v\|_p^p$$

for all $v \in \mathrm{W}_0^{2,p}([0,t_0],\partial X)$,

(iv) $\mathcal{Q}_{t_0}$ *is invertible, where* $\mathcal{Q}_{t_0} \in \mathcal{L}\big(\mathrm{L}^p([0,t_0],\partial X)\big)$ *is given by*

$$(\mathcal{Q}_{t_0} v)(\bullet) = \Phi \int_0^{\bullet} T_{-1}(\bullet - s)L_A v(s)\, ds \text{ for all } v \in \mathrm{W}_0^{2,p}([0,t_0],\partial X).$$

Then $(G, D(G))$ generates a C_0-semigroup on the Banach space X.

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- operator $G_1 := c(\bullet,\bullet) \cdot \frac{d}{ds}$ with

$$D(G_1) := \left\{ f \in \mathrm{W}^{1,p}(\mathbb{R}_+, \mathbb{C}^\ell) \times \mathrm{W}^{1,p}([0,1], \mathbb{C}^m) \mid \Phi f = 0 \right\}$$

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Goal

Give conditions on Φ implying that G_1 generates a C_0-semigroup on X.

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Theorem ((BC) via boundary matrices)

Let

$$\Phi = (\Phi^e, \Phi^i) := \left(V_0^e \delta_0, V_0^i \delta_1 + V_1^i \delta_0\right)$$

for some $V_0^e \in \mathcal{L}(\mathbb{C}^\ell, \partial X)$ *and* $V_0^i, V_1^i \in \mathcal{L}(\mathbb{C}^m, \partial X)$. *If*

$$\left(V_0^e, V_1^i P_+^i - V_0^i P_-^i\right) \in \mathcal{L}\left(\partial X^e \times \partial X^i, \partial X\right)$$

is invertible then the operator G_1 *generates a* C_0*-semigroup on* X.

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The compact case

If $X^e = \emptyset$ then the invertibility condition becomes

$$\det \left(V_1^i P_+^i - V_0^i P_-^i \right) \neq 0.$$

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Goal

Give conditions on Φ_0, Φ_1 implying that G_2 generates a cosine family on X.

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Again generation result under certain invertibility condition. ▸ skip proof

Ideas of the proof

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4. $\mathcal{G}^2$ generates a cosine family $\Rightarrow$ the same holds for $G = D_{\Phi_1} D_{\Phi_0}$

Boundary perturbation

Corollary ((BC) via boundary matrices)

Take $k_0, k_1 \in \mathbb{N}$ *satisfying* $k_0 + k_1 = \ell + 2m$ *and matrices* $V_0^e \in \mathrm{M}_{k_0\times\ell}(\mathbb{C})$, $W_0^e \in \mathrm{M}_{k_1\times\ell}(\mathbb{C})$, $V_0^i, V_1^i \in \mathrm{M}_{k_0\times m}(\mathbb{C})$, $W_0^i, W_1^i \in \mathrm{M}_{k_1\times m}(\mathbb{C})$. *If*

$$\det\begin{pmatrix} V_0^e & V_1^i & V_0^i \\ a^e(0)^{-1/2}W_0^e & a^i(1)^{-1/2}W_1^i & a^i(0)^{-1/2}W_0^i \end{pmatrix} \neq 0$$

then the operator G_2 *with the following (BC) in* $D(G_2)$*:*

$$V_0^e f^e(0) + V_0^i f^i(0) + V_1^i f^i(1) = 0,$$
$$W_0^e (f^e)'(0) + W_0^i (f^i)'(0) - W_1^i (f^i)'(1) + (B^e f^e)(0) + (B^i f^i)(0) = 0$$

generates a cosine family on X.

Boundary perturbation

Corollary ((BC) via boundary spaces)

If $Y_0 \oplus Y_1 = \mathbb{C}^{\ell+2m}$, then the operator G_2 with the following (BC) in $D(G_2)$:

$$\begin{pmatrix} f^e(0) \\ f^i(0) \\ f^i(1) \end{pmatrix} \in Y_1, \quad \begin{pmatrix} a^e(0)^{-1/2} \cdot (f^e)'(0) \\ a^i(0)^{-1/2} \cdot (f^i)'(0) \\ -a^i(1)^{-1/2} \cdot (f^i)'(1) \end{pmatrix} + (B^e f^e)(0) + (B^i f^i)(0) \in Y_0.$$

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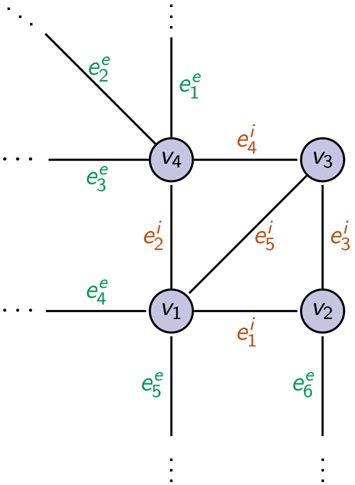
Back to networks



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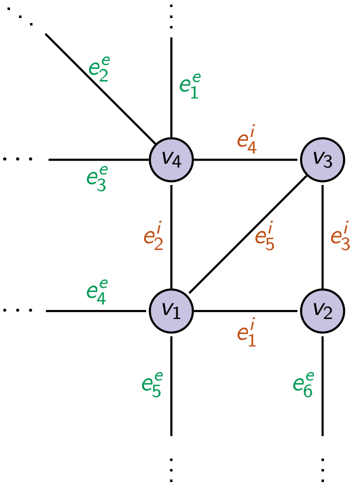

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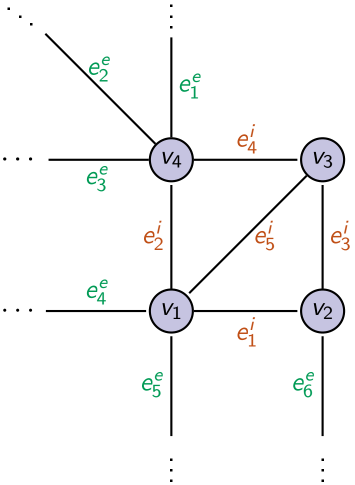

Finite connected non-compact graph:

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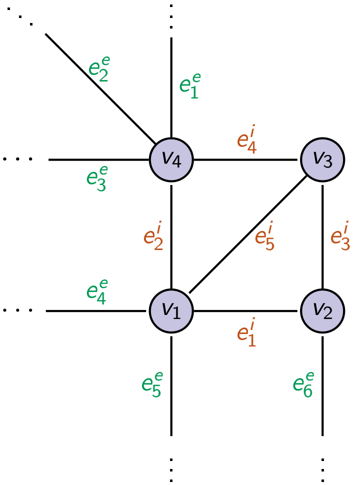

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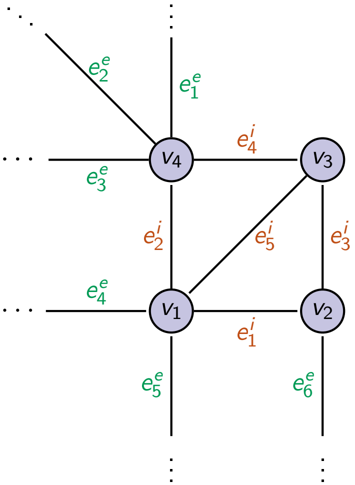

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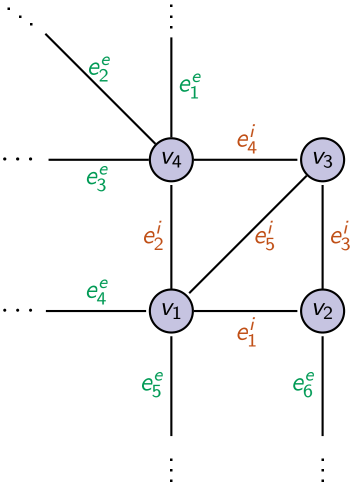

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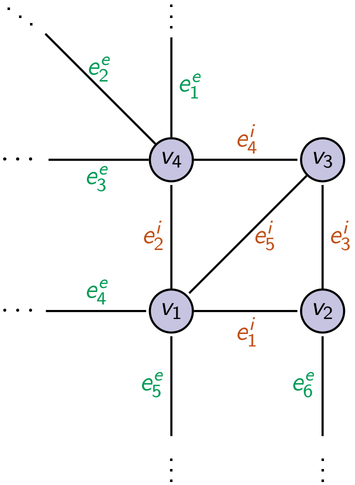

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- structure: incidence matrices $\underbrace{\Phi^{i,-}}_{n\times m}, \underbrace{\Phi^{i,+}}_{n\times m}, \underbrace{\Phi^{e,-}}_{n\times\ell}$

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(TE) with the standard (BC)

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- Invertibility condition holds $\iff$ no sinks

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- Since $Y_0 \oplus Y_1 = \mathbb{C}^{\ell+2m}$, our corollary applies.

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- use the available spectral theory directly
- apply semigroups methods to study further properties (stability, long-time behaviour, control,. . .)