Weierstrass Institute for
Applied Analysis and Stochastics

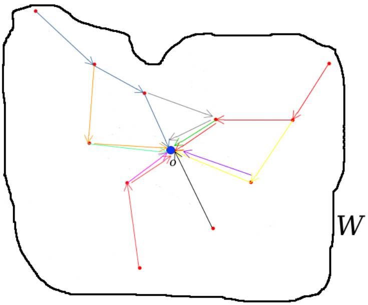
Leibniz
Association

A Gibbsian model for message routeing in a highly dense telecommunication network

Wolfgang König
TU Berlin and WIAS

joint work with András Tóbiás, Berlin

- We consider a **spatial wireless ad-hoc network**.
- Compact communication area, one base station, many users (randomly) distributed.
- **Messages travel** simultaneously from each user to the base station in a number of hops; time is not considered.
- Message trajectories are random and a priori uniform, but subject to some distribution that **punishes high total interference** in the spirit of a **Gibbs measure**. (Also high congestions is punished $\Longrightarrow$ dropped in this talk.)
- $\Longrightarrow$ competition between **entropy** (i.e., probability) and **energy** (i.e., interference and congestion).
- Main objective: description of the **effective main flow of message trajectories** in simple geometric terms.
- Answers only in the simplified regime of a **high spatial density** of users.
- Main mathematical strategy: **large deviations** and analysis of the arising characteristic **variational formula**.



- Communication **area** $W \subset \mathbb{R}^d$ (compact).
- **Base station** at o inside W.
- **Users** X_i located according to a (e.g., Poisson) point process $X = (X_i)_{i \in I}$ in W.
- Each user sends one message to the base station, possibly using other users as **relays**.
- The **trajectory** of each message is chosen randomly.

- **Trajectory** S^i of the message $X_i \to o$ is random and has the form

$$s^i = (k_i; s_0^i = X_i, s_1^i, \dots, s_{k_i-1}^i, s_{k_i}^i = o).$$

- $k_i \in \{1, \dots, k_{\max}\}$ number of hops, $s_l^i \in X$ used relays.
- **path-loss function** ℓ: $[0, \infty) \to (0, \infty)$ (continuous, decreasing) describes propagation of the signal strength over distance. (E.g., $\ell(x) = \min\{x^{-\alpha}, 1\}$, $\alpha > d$, $\Longrightarrow$ ideal Hertzian propagation.)
- **Signal-to-interference ratio (SIR)** of a single hop $X_i \to x$:

$$\mathrm{SIR}(X_i,\ x,\ X) = \frac{\ell(|X_i - x|)}{\sum_{j \in I} \ell(|X_j - x|)}.$$

  The denominator is called *interference* at x.
- For a trajectory collection $s = (s^i)_{i \in I}$ we define

$$\mathfrak{S}(s) = \sum_{i \in I} \sum_{l=1}^{k_i} \frac{1}{\mathrm{SIR}(s_{l-1}^i,\ s_l^i,\ X)}.$$

We work conditional on the point process $X = (X_i)_{i\in I}$ of user locations. Fix $\gamma > 0$.

- Distribution of the message trajectories $X_i \to o$ using the users $\in X$ as relays with $\leq k_{\max}$ hops: uniform *a priori* measure, weighted by a penalization term preferring low interference.
- **Gibbs distribution** of a message trajectory family $s = (s^i)_{i\in I}$:

$$\mathrm{P}_X^\gamma(s) = \frac{1}{Z_X^\gamma} \frac{1}{\prod_{i\in I} \# I^{k_i - 1}} \mathrm{e}^{-\gamma \mathfrak{S}(s)}.$$

- **Partition function** (normalizing constant):

$$Z_X^\gamma = \sum_{r=(r^i)_{i\in I}} \frac{1}{\prod_{i\in I} \# I^{k_i - 1}} \mathrm{e}^{-\gamma \mathfrak{S}(r)}$$

We work conditional on the point process $X = (X_i)_{i \in I}$ of user locations. Fix $\gamma > 0$.

- Distribution of the message trajectories $X_i \to o$ using the users $\in X$ as relays with $\leq k_{\max}$ hops: uniform *a priori* measure, weighted by a penalization term preferring low interference.
- **Gibbs distribution** of a message trajectory family $s = (s^i)_{i \in I}$:

$$\mathrm{P}_X^\gamma(s) = \frac{1}{Z_X^\gamma} \frac{1}{\prod_{i \in I} \# I^{k_i - 1}} \mathrm{e}^{-\gamma \mathfrak{S}(s)}.$$

- **Partition function** (normalizing constant):

$$Z_X^\gamma = \sum_{r = (r^i)_{i \in I}} \frac{1}{\prod_{i \in I} \# I^{k_i - 1}} \mathrm{e}^{-\gamma \mathfrak{S}(r)}$$

- Non-selfish routing, random mechanism for a common welfare.
- Competition between entropy and energy. Relation is controlled by γ.
- Interference only w.r.t. initial transmissions from all the X_j's, not from the hops.
- No time in the model, all hops simultaneously.

Considered regime: quenched high-density limit

- Want easily accessible assertions on **typical flow of the message trajectories**.
- Good chances in the limit where very many **users form a densely packed cloud** in W.
- Additional **density parameter** $\lambda \asymp \#I = \#I_\lambda =$ number of users in W.
- Assume $L_\lambda := \frac{1}{\lambda}\sum_{i\in I_\lambda} \delta_{X_i} \Longrightarrow \mu(\mathrm{d}x)$ as $\lambda \to \infty$, for some absolutely continuous measure μ on W. (E.g., $X \sim \mathrm{PPP}(W, \lambda\mu)$.)
- Since $\mathfrak{S}(s) \asymp \lambda^2$, we **replace** γ **by** γ/λ.
- Combinatorics $= \mathrm{e}^{O(\lambda)} \Longrightarrow$ **large deviations** on the scale $\lambda \to \infty$.

- Want easily accessible assertions on **typical flow of the message trajectories**.
- Good chances in the limit where very many **users form a densely packed cloud** in W.
- Additional **density parameter** $\lambda \asymp \#I = \#I_\lambda =$ number of users in W.
- Assume $L_\lambda := \frac{1}{\lambda}\sum_{i\in I_\lambda}\delta_{X_i} \Longrightarrow \mu(\mathrm{d}x)$ as $\lambda\to\infty$, for some absolutely continuous measure μ on W. (E.g., $X\sim\mathrm{PPP}(W,\lambda\mu)$.)
- Since $\mathfrak{S}(s)\asymp\lambda^2$, we **replace** γ **by** γ/λ.
- Combinatorics $=\mathrm{e}^{O(\lambda)} \Longrightarrow$ **large deviations** on the scale $\lambda\to\infty$.
- The family $s=(s^i)_{i\in I}$ forms a cloud and is registered via its **empirical measure**

$$R_{\lambda,k}(s)=\frac{1}{\lambda}\sum_{i\in I^\lambda}\delta_{(s_0^i,\dots,s_{k-1}^i)}\mathbb{1}\{k_i=k\},\quad k=1,\dots,k_{\max}.$$

- Write π_l for the projection on the l-th marginal. Since each user sends exactly one message to o, we have for all s

$$\sum_{k=1}^{k_{\max}}\pi_0 R_{\lambda,k}(s)=L_\lambda=\frac{1}{\lambda}\sum_{i\in I^\lambda}\delta_{X_i}\overset{\lambda\to\infty}{\Longrightarrow}\mu$$

Theorem 1: A law of large numbers

Let $\gamma > 0$. Almost surely w.r.t. $(X^\lambda)_{\lambda>0}$, the distribution of the empirical measures $(R_{\lambda,k}((S^i)_{i\in I^\lambda}))_{k=1}^{k_{\max}}$ under $\mathrm{P}^{\gamma/\lambda}_{X^\lambda}$ converges, as $\lambda\to\infty$, weakly to the unique minimizer of the **variational formula**

$$\inf\Big\{\mathrm{J}(\Sigma)+\gamma\mathrm{S}(\Sigma)\colon \Sigma=(\nu_k)_{k=1}^{k_{\max}},\ \sum_{k=1}^{k_{\max}}\pi_0\nu_k=\mu\Big\},$$

where

$$\mathrm{S}(\Sigma)=\sum_{k=1}^{k_{\max}}\int_{W^k}\nu_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1})\sum_{l=1}^{k}g(x_{l-1},x_l),\quad x_k=o,$$

where

$$g(x,y)=\frac{\int_W \ell(|z-y|)\mu(\mathrm{d}z)}{\ell(|x-y|)},\quad x,y\in W.$$

and

$$\mathrm{J}(\Sigma)=\sum_{k=1}^{k_{\max}}\int_{W^k}\mathrm{d}\nu_k\log\frac{\mathrm{d}\nu_k}{\mathrm{d}\mu^{\otimes k}}+\log\mu(W)\sum_{k=1}^{k_{\max}}(k-1)\nu_k(W).$$

Proposition 1: Formula for limiting trajectory measure

The minimizing $\Sigma = (\nu_k)_{k=1}^{k_{\max}}$ is given as

$$\nu_k(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1}) = \mu(\mathrm{d}x_0) A(x_0) \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma \sum_{l=1}^{k} g(x_{l-1}, x_l)},$$

where $A\colon W \to (0, \infty)$ is such that $\sum_{k=1}^{k_{\max}} \pi_0 \nu_k = \mu$ holds, i.e.,

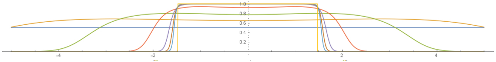
$$\frac{1}{A(x_0)} = \sum_{k=1}^{k_{\max}} \int_{W^{k-1}} \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma \sum_{l=1}^{k} g(x_{l-1}, x_l)}.$$

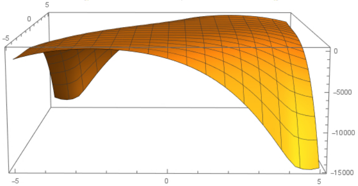
- The minimizers ν_k are the main objectives of our next analysis.
- ν_k is the distribution of the “typical” flow of message k-hop trajectories from a starting site $\sim \mu$ to the base station o.
- Note that the hop number k is also “random”.

$$d = 1, \qquad W = [-5, 5], \qquad \ell(r) = \min\{1, r^{-4}\}, \qquad \mu = \mathrm{Leb}|_W, \qquad k_{\max} = 2.$$

Left: $x_0 \mapsto \nu_1(\mathrm{d}x_0)/\mu(\mathrm{d}x_0)$**.** Blue line: $\gamma = 1$. Orange line: $\gamma = 1.5$. Green line: $\gamma = \infty$. Note the strong effect of SIR penalization already for small values of γ. $\Longrightarrow$ sharp transition from one hop (distance ≤ 1.5) to two hops (distance ≥ 1.45).

Right: $(x_0, x_1) \mapsto \log \nu_2(\mathrm{d}x_0, \mathrm{d}x_1)/\mu(\mathrm{d}x_0)\mu(\mathrm{d}x_1)$ **for** $\gamma = 1$**.**
Noisy for $|x_0| \in (1.45, 1.5)$. Concentration on $|x_1| \approx \frac{1}{2}|x_0|$ for $|x_0| \in (1.5, 2.5)$ and on $|x_1| \approx c|x_0|$ for $|x_0| \in (2.5, 5]$ with some $c \in (0, 1)$.





We will analyse typical message trajectory distribution $(\nu_k)_{k=1}^{k_{\max}}$ in two regimes:

Large distances, many hops.

That is, W is large, and $k_{\max} = \infty$, and the starting site x_0 is far from o.

- How many hops will be taken?
- How large are the hops on an average? Different lengths at the beginning and the end of the trajectory?
- Does the long trajectory approach a straight line?

We will analyse typical message trajectory distribution $(\nu_k)_{k=1}^{k_{\max}}$ in two regimes:

Large distances, many hops.

That is, W is large, and $k_{\max} = \infty$, and the starting site x_0 is far from o.

- How many hops will be taken?
- How large are the hops on an average? Different lengths at the beginning and the end of the trajectory?
- Does the long trajectory approach a straight line?

High interference punishment

Hence, $\gamma \to \infty$.

- Do the trajectories approach a straight line?
- How costly are deviations from the straight line?

Consider the following **long-distance limit**:

- We choose W as a closed ball $\overline{B_r(o)}$ with $r \gg 1$.
- A typical user x_0 at distance $r_0 = |x_0| \gg 1$ from the base station o.
- Intensity measure m of users: Lebesgue measure on W
- Regime: $r > r_0 \asymp r \to \infty$
- Integrable Hertzian path-loss function: $\ell(r) = \min\{1, r^{-\alpha}\}$ for some $\alpha > d$ (so that $\int_{\mathbb{R}^d} \ell(|x|)\,\mathrm{d}x < \infty$, and $1/\ell(|\cdot|)$ is convex).

Consider the following **long-distance limit**:

- We choose W as a closed ball $\overline{B_r(o)}$ with $r \gg 1$.
- A typical user x_0 at distance $r_0 = |x_0| \gg 1$ from the base station o.
- Intensity measure m of users: Lebesgue measure on W
- Regime: $r > r_0 \asymp r \to \infty$
- Integrable Hertzian path-loss function: $\ell(r) = \min\{1, r^{-\alpha}\}$ for some $\alpha > d$ (so that $\int_{\mathbb{R}^d} \ell(|x|)\,\mathrm{d}x < \infty$, and $1/\ell(|\cdot|)$ is convex).

Recall the normalization term of the minimizing measures $\Sigma = (\nu_k)_k$:

$$\frac{1}{A(x_0)} = \sum_{k=1}^{k_{\max}} a_k(x_0) \quad \text{with } a_k(x_0) = \int_{\overline{B_r(o)}^{k-1}} \prod_{l=1}^{k-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(B_r(o))} \mathrm{e}^{-\gamma \sum_{l=1}^{k} g(x_{l-1}, x_l)}.$$

Then $a_k(x_0)$ is the weight for all the k-step paths $x_0 \to o$. It has no reference to $k_{\max}$, but to r and r_0.

We first determine $a_k(x_0)$ for $k \to \infty$, coupled with r and r_0. In particular, we want to find for what k the weight $a_k(x_0)$ is maximal.

Large deviations for the hop number

Fix $t \in (0, \infty)$. Then, in our limit $r > r_0 \asymp r \to \infty$, for any choice of $r_0 \mapsto k(r_0) \in \mathbb{N}$,

$$\text{if } \frac{k(r_0)}{r_0 \log^{-\frac{1}{\alpha}} r_0} \begin{cases} \to t, \\ \leq t, \\ \geq t, \end{cases} \quad \text{then } \frac{\log a_{k(r_0)}(x_0)}{r_0 \log^{1-\frac{1}{\alpha}} r_0} \begin{cases} = -\big(dt + b\gamma t^{1-\alpha}\big) + o(1), \\ \leq -b\gamma t^{1-\alpha} + o(1), \\ \leq -dt + o(1), \end{cases}$$

where $b = \int_{\mathbb{R}^d} \mathrm{d}y \, \ell(|y|)$.

Large deviations for the hop number

Fix $t \in (0,\infty)$. Then, in our limit $r > r_0 \asymp r \to \infty$, for any choice of $r_0 \mapsto k(r_0) \in \mathbb{N}$,

$$\text{if } \frac{k(r_0)}{r_0 \log^{-\frac{1}{\alpha}} r_0} \begin{cases} \to t, \\ \leq t, \\ \geq t, \end{cases} \quad \text{then } \frac{\log a_{k(r_0)}(x_0)}{r_0 \log^{1-\frac{1}{\alpha}} r_0} \begin{cases} = -(dt + b\gamma t^{1-\alpha}) + o(1), \\ \leq -b\gamma t^{1-\alpha} + o(1), \\ \leq -dt + o(1), \end{cases}$$

where $b = \int_{\mathbb{R}^d} \mathrm{d}y\, \ell(|y|)$.

- Hence, on the scale $k(r_0) \asymp r_0 \log^{1-\frac{1}{\alpha}} r_0$, the weights $a_{k(r_0)}(x_0)$ satisfy a large-deviations principle.
- The rate function is $(0,\infty) \ni t \mapsto dt + b\gamma t^{1-\alpha}$, which is strictly minimal in some t^*.
- Hence, in our limit, $k \mapsto a_k(x_0)$ is maximal at $k = k^*(r_0) \sim t^* r_0 \log^{1-\frac{1}{\alpha}} r_0$.
- As a further consequence, the behaviour of the sum $\frac{1}{A(x_0)} = \sum_{k=1}^{k_{\max}} a_k(x_0)$ is the same as the one of $a_{k^*(r_0)}(x_0)$, if the restriction $k \leq k_{\max}$ is neglected.

We expected that the hop lengths stay bounded in long trajectories. But, surprisingly, we discover instead.

> The typical size of a hop tends to infinity like $c\log^{1/\alpha} r_0$,
>
> and hence the number of hops is $c' r_0 / \log^{1/\alpha} r_0$ (sublinear in the distance).

We think the reason is the large mass $\mathrm{Leb}(B_r(o))$ of the area that is available for any of the hops.

Furthermore (without details): the message trajectory becomes concentrated along the **straight line** between x_0 and o with **equal hops**. Macroscopic deviations from the straight line have exponentially fast decaying probability.

- How does the typical trajectory distribution $\Sigma = (\nu_k)_{k=1}^{k_{\max}}$ behave if the interference-punishment parameter γ **is large**?
- **Preliminary answer** is easy. Indeed, recall:

$$\nu_k(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1}) = \mu(\mathrm{d}x_0) A(x_0) \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma \sum_{l=1}^{k} g(x_{l-1}, x_l)},$$

  and $A(x_0)$ normalizes such that $\sum_k \nu_k(\mathrm{d}x - 0, W^k) = 1$.
- Hence, the **Laplace method** shows that, as $\gamma \to \infty$, $\nu_k(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1})/A(x_0)$ concentrates on the minimizer(s) $(x_1, \dots, x_{k-1})$ of $\sum_{l=1}^{k} g(x_{l-1}, x_l)$.
- **Geometric shapes:** Under natural extra assumptions (ℓ strictly monotone decreasing, W a ball, μ rotationally invariant) the minimizing trajectories lie on the **straight line** $x_0 \to o$.
- **Deviations:** For $\varepsilon > 0$, the probability of having at least 1 hop deviating from the straight line by at least $\varepsilon > 0$ decays exponentially fast as $\gamma \to \infty$.