On mixing behaviour of chaotic linear dynamical systems.

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Evolutionary Processes on Networks
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Papers on the subject

1. S. EL M, K. Latrach, *On the ergodic approach for the study of chaotic linear infinite-dimensional systems*, **Diff. Integ. Eq**. **2013**
2. M. Chakir, S. EL M, *Strong mixing Gaussian measures for chaotic semigroups*, **J. Math. Ana. App**. **2018**.
3. F. Bayart, E. Matheron, *Mixing operators and small subsets of the circle*, **J. Reine angew. Math**. **2016**.
4. Z. Brzeźniak, H. Long, *A note on gamma-radonifying and summing operators*, **Stochastic Analysis, Banach Centre Publications 105**, IMPAS, Warsaw **2015**.
5. R. Rudnicki, *An ergodic approach to chaos*, **Discrete. Contin. Dyn. Syst**, 35, **2015**

Outline

1. Problematic
   - Contributions
2. Perspetives

Deterministic but umpredictable

$$\begin{aligned} \frac{dX_t}{dt} &= F(X_t),\ t \geq 0. \\ X_{n+1} &= F(X_n),\ n \geq 0. \end{aligned}$$

Deterministic : Given initial state $X_0 \Longrightarrow$ Complete knowledge about all the future states.

Umpredictable : Due to sensitive dependence on initial states X_0.

Small changes in the initial data $X_0 \Longrightarrow$ **huge** changes in the solution X_t, $(X_n.)$

Consequence : **Disorder**, **erratic solutions** and **strange behaviour**.

Readable references :

1. R.L. Devaney, *An Introduction to Chaotic Dynamical Systems* (2nd edn), *Addison–Wesley Studies in Nonlinearity,* 1989.
2. R.L. Devaney, *Chaos, Fractals and Dynamics : Computer experiments in Mathematics,* 1992.

Problem : Find a hidden order under the apparent disorder.

What does order mean ?

A measurable quantity that becomes constant : Let us say that an equilibrium will be achieved after an increasing disorder.

At the equilibrium

The system is predictable in terms of averages :(The idea goes back to Ludwig Boltzmann in statistical mechanics)

$$\textit{Temporal average} : \quad \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} f(F^k(x_0)), \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x(t,x_0)).$$

$$\textit{Spatial average} : \quad \langle f \rangle = \frac{1}{\mu(X)}\int_X f \, d\mu.$$

Ergodic Hypothesis :

$$\textit{Temporal average} \; = \; \textit{Spatial average}.$$

Birkhoff Ergodic Theorem : When Temporal and spatial averages coincide ?

$$F: X \to X, \quad \mu(F^{-1}(A)) = \mu(A), \text{ for every Borel set } A$$

$$F_t: X \to X, \quad \mu(F_t^{-1}(A)) = \mu(A), \text{ for every Borel set } A \text{ and } t \geq 0$$

μ is called invariant measure, and a Borel subset A is invariant with respect to (F_t), if $F_t^{-1}(A) = A$, $\forall t \geq 0$.

(F_t) is ergodic with respect to μ, if every invariant set A has measure 0 or 1. (or simply μ is ergodic.)

Theorem : *If μ is ergodic then for every, $f \in L^1(X, d\mu)$, and for μ a.e $x_0 \in X$,*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(F_t(x_0)) = \int_X f(x)\, \mu(dx).$$

Formulation of the problem for chaotic linear systems*

X separable Banach space.

$$\frac{dx(t)}{dt} = A\,x(t),\;\; x(0) \in X.$$

Problem 1 : *Assume that the semigroup solution (T_t) is chaotic. Does there exist (or when exists) an invariant measure with respect to (T_t) having interesting ergodic properties ?*

Problem 2 : *Characterize the invariance of the measure using the generator A.*

* R. Rudnicki, *Chaos for some infinite-dimensional dynamical systems*, Appl. Sci. 27 (2004) 723-738.

Linear chaos : Definition and comments

Definition

(T_t) *is chaotic if :*

1. *Transitive :* $\forall\, G_1, G_2$ *nonempty opens,* $\exists t > 0,\ T_t(G_1) \cap G_2 \neq \emptyset$
2. $\{p \in X,\ \exists t > 0,\ T_t\, p = p\}$ *is dense.*

Result1 : TOO MUCH eigenvalues of A around $i\mathbb{R}$, $\Rightarrow$ Chaos.
Result2 : TOO MUCH imaginary eigenvalues of A, $\Rightarrow$ Chaos.

* W. Desch, W. Schappacher, and G. F. Webb, *Hypercyclic and chaotic semigroups of linear operators*, Ergodic Theory Dynam. Systems **17** (1997), 793–819.
**J. Banasiak, M Moszynski, *A generalization of Desch-Schappacher-Webb criteria for chaos* Discrete Contin. Dyn. Syst 12 (5), 959-972
*** S. EL Mourchid, *The imaginary point spectrum and hypercyclicity*, Semigroup Forum. **73** (2006), 313-316.

Preliminaries

Definition

A centered Gaussian measure μ, on a separable Banach E is a probability measure such that each continuous linear functional $x^* \in E^*$ is a centered Gaussian variable when considered as a random variable on the probability space (E, μ).

There is ξ, an $E-$valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\mu(F) = \mathbb{P}(\xi \in F),\ F \in \mathcal{F}.$$

E^* can be embedded into a subspace of $L^2(E, \mu)$. The covariance operator of μ is then the unique bounded conjugate linear operator $Q:\ E^* \to E$, such that for every $(x^*, y^*) \in E^* \times E^*$,

$$\langle y^*, Qx^* \rangle = \int_E \langle y^*, x \rangle \overline{\langle x^*, x \rangle}\ \mu(dx) = \langle y^*, x^* \rangle_{L^2(d\mu)}.$$

Preliminaries

Theorem

$\hat{\mu}$ is the complex-valued function defined on the dual space E^* by

$$\hat{\mu}(x^*) = \int_X e^{i\Re\langle x^*, x\rangle} d\mu(x) = e^{\frac{-\langle x^*, Qx^*\rangle}{4}}.$$

Definition

(T_t) is called a strongly mixing semigroup if for all Borelian sets A, B

$$\lim_{t\to\infty} \mu(T_t^{-1}A \cap B) = \mu(A)\mu(B).$$

μ will be called a strong mixing measure.

Invariance and strong mixing via Q. (*)

Theorem

Let (T_t) be a C_0-semigroup on a separable Banach space E. If μ is a centered Gaussian measure with covariance operator Q, then

$$\mu \text{ is invariant if and only if } T_t Q T_t^* = Q, \text{ for all } t > 0,$$

$$\mu \text{ is strong mixing if and only if } \lim_{t\to\infty} \langle x^*, T_t Q y^* \rangle = 0, \text{ for all } x^*, y^* \in E^*.$$

(*)R. Rudnicki, *Gaussian measure- Preserving linear transformations*, Univ. Iagell. Ac. Math., **30** (1993).

Partial answer : $X = L^p(T, d\sigma)$: EL M- Latrach 2013

Theorem

Assume that $\sigma_p(A) \cap i\mathbb{R}$ is contained in $i(\omega_1, \omega_2)$ for some ω_1 and ω_2 such that $-\infty \leq \omega_1 < \omega_2 \leq +\infty$, and there is a measurable function $u : (\omega_1, \omega_2) \mapsto X$ satisfying the following conditions :

1. $u_s := u(s) \in \ker(is - A)$ *for a.e.* $s \in (\omega_1, \omega_2)$,
2. $(\int_{\omega_1}^{\omega_2} |u_s(\cdot)|^2 ds)^{\frac{1}{2}} = v(\cdot) \in L^p(\Omega)$,
3. *span*$\{u_s,\ s \in (\omega_1, \omega_2) \setminus N\}$ *is dense in* X *for every subset* N *with zero Lebesgue measure.*

Then there exists an invariant Gaussian measure ν, such that supp$(\nu) = X$ with respect to which $T(\cdot)$ is strong mixing.

Proof based on gamma radonifying operators theory(*)

$$K : \begin{array}{l} L^2 \longrightarrow X \\ f \longmapsto \int_{\omega_1}^{\omega_2} f(s)\, u_s\, ds \end{array}$$

$Q = KK^*$ covariance operator : Every Hilbertian basis (f_n) of L^2,

$$\Big(\sum_{n=0}^{\infty} \mid Kf_n(x) \mid^2\Big)^{\frac{1}{2}} = \Big(\int_{\omega_1}^{\omega_2} \mid u_s(x) \mid^2 ds\Big)^{\frac{1}{2}} \in L^p(\Omega). \quad (1)$$

To conclude the invariance of the measure

$$G_t : \begin{array}{l} L^2 \longrightarrow L^2, \\ f \longmapsto G_t f(s) = e^{ist} f(s). \end{array}$$

$$\begin{aligned} T(t)QT^*(t) &= T(t)KK^*T^*(t), \\ &= KG_tG_t^*K^*, \\ &= KK^*. \end{aligned}$$

(*) Vakhania, Tarieladze, Chobanyan, *Probability distributions on Banach spaces* (1987).

Proof : Mixing behaviour

$$\begin{aligned}\langle x^*, T(t)Qy^*\rangle &= \langle x^*, T(t)KK^*y^*\rangle,\\ &= \langle x^*, KG_tK^*y^*\rangle,\\ &= \langle K^*x^*, G_tK^*y^*\rangle_{L^2},\\ &= \int_{\omega_1}^{\omega_2} e^{-ist}(K^*x^*)(s)\overline{(K^*y^*)}(s)ds. \quad \to 0, \text{ as } t\to\infty.\end{aligned}$$

Application to an abnormal cell population dynamic

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = -\frac{\partial (xu(t,x))}{\partial x} + \gamma(x)u(t,x) - \beta(x)u(t,x) + 4\beta(2x)\, u(t,2x)\, \chi_{(0,\frac{1}{2})}(x \\ u(0,\cdot) = \phi \in L^1(0,1). \end{cases}$$

$$\begin{cases} \frac{\partial v(t,y)}{\partial t} = e^y \frac{\partial (e^{-y} v(t,y))}{\partial y} + \gamma v(t,y) - \beta v(t,y) + 4\beta\, v(t,y-\ln 2)\, \chi_{(\ln 2,\infty)}(y), \\ v(0,\cdot) = \psi \in L^1((0,\infty), e^{-y}dy), \end{cases}$$

with $\psi(y) := \phi(e^{-y})$. Under suitable conditions, there exists a nondegenrate Gaussian measure for which $T(\cdot)$ is strong mixing. (Similar result has been obtained by different methods in *)

(*)R. Rudnicki,*Chaoticity and invariant measures for a cell population model*, J. Math. Anal. Appl. **393** (2012)

Alterative proof with solution to Prob2. (Chakir-EL M 2018)

Theorem

Assume that $\sigma_p(A) \cap i\mathbb{R}$ is contained in $(i\omega_1, i\omega_2)$ for some ω_1 and ω_2 such that $-\infty \leq \omega_1 < \omega_2 \leq +\infty$, and there is a countable family of measurable functions $(u^j)_{j\in J}$ $u^j : (\omega_1, \omega_2) \times T \to \mathbb{C}$ for every $j \in J$, satisfying the following conditions :

1. *For all $j \in J$, $u^j_s := u^j(s, \cdot) \in ker(is - A)$ for a.e $s \in (\omega_1, \omega_2)$,*
2. $\left(\sum_{j\in J} \int_{\omega_1}^{\omega_2} |u^j_s(\cdot)|^2 ds\right)^{\frac{1}{2}} = v(\cdot) \in E$,
3. *$span\{u^j_s,\ s \in (\omega_1, \omega_2) \setminus N,\ j \in J\}$ is dense in E, for every subset N with zero Lebesgue measure.*

Then (T_t) is strongly mixing with respect to a full support invariant Gaussian measure.

Solution to Prob2. (Chakir-EL M 2018)

Theorem

A centred Gaussian measure μ*, on* X *with covariance operator* R_μ*, is invariant if and only if* $AR_\mu + R_\mu A^* = 0$ *holds on* $D(A^*)$*.*

Plan of the proof :

1. Differentiation of $t \mapsto T_t R_\mu T_t^* x^*$, for $x^* \in D(A^*)$,
2. Compactness of R_μ,
3. weak $*$ denseness of $D(A^*)$ in X^*.

Proof

Démonstration.

Assume μ invariant, $R_\mu = T(t)R_\mu T^*(t)$, for all $t \geq 0$, and let $x^* \in D(A^*)$.

$$\lim_{t\to 0}\langle \frac{1}{t}(T^*(t) - I)x^*, x\rangle = \langle A^*x^*, x\rangle, \text{ for all } x \in X.$$

By uniform boundednes principle $(\frac{1}{t}(T^*(t) - I)x^*)_{0<t<1}$ is bounded in X^*.

$$\begin{aligned}(\forall (t_n) \to 0)\ \frac{1}{t_n}(T(t_n)R_\mu x^* - R_\mu x^*) &= \frac{1}{t_n}(T(t_n)R_\mu x^* - T(t_n)R_\mu T^*(t_n)x^*), \\ &= \frac{1}{t_n}T(t_n)R_\mu(x^* - T^*(t_n)x^*).\end{aligned}$$

□

Proof

$R_\mu : X^* \to X$ compact :
$\exists (t_{n_k})\ (R_\mu \frac{1}{t_{n_k}}(x^* - T^*(t_{n_k})x^*))_k \to w$.
Claim : $w = -R_\mu A^* x^*$,
For y^* arbitrarily in X^*. Write

$$\begin{aligned}\langle y^*, w\rangle &= \lim_{k\to\infty} \langle y^*, R_\mu \frac{1}{t_{n_k}}(x^* - T^*(t_{n_k})x^*)\rangle,\\ &= \lim_{k\to\infty} \langle \frac{1}{t_{n_k}}(x^* - T^*(t_{n_k})x^*), R_\mu y^*\rangle,\\ &= \langle -A^*x^*, R_\mu y^*\rangle,\\ &= \langle y^*, -R_\mu A^* x^*\rangle.\end{aligned}$$

We deduce that,

$$\forall (t_n) \to 0,\ \exists (t_{n_k})\ \lim_k \frac{1}{t_{n_k}}(T(t_{n_k})R_\mu x^* - R_\mu x^*) = -R_\mu A^* x^*$$

Finally, $R_\mu x^* \in D(A)$ and $AR_\mu x^* = -R_\mu A^* x^*$.

The Converse's Proof

Let $t > 0$, $h \neq 0$ and $x^* \in D(A^*)$. We have

$$\frac{1}{h}(T_{t+h}R_\mu T^*_{t+h}x^* - T_t R_\mu T^*_t x^*) = T_{t+h}R_\mu \frac{1}{h}[T^*_{t+h} - T^*_t]x^*,$$
$$+\frac{1}{h}[T_{t+h} - T_t]R_\mu T^*_t x^*.$$

The first term converges to $T_t R_\mu A^* T^*_t x^*$ (Compactness again !.)
The second term converges to $T_t A R_\mu T^*_t x^*$.
we obtain

$$\lim_{h\to 0}\frac{1}{h}(T(t+h)R_\mu T^*(t+h)x^* - T(t)R_\mu T^*(t)x^*) = T(t)[R_\mu A^* + AR_\mu]T^*x^* = 0.$$

Hence for all $t > 0$,

$$T(t)R_\mu T^*(t) = R_\mu, \text{ on}, D(A^*).$$

Converse's Proof

To conclude :
For all $y^* \in X^*$,

$$\langle \cdot, R_\mu y^* \rangle = \langle \cdot, T(t) R_\mu T^*(t) y^* \rangle,$$

on the *weak**-dense subspace $D(A^*)$.
The two mapping are *weak**-continuous
Thus,

$$R_\mu y^* = T(t) R_\mu T^*(t) y^*$$

holds for all $y^* \in X^*$.

Proof of the Main result where (T_t) is a chaotic translation

Recall (T_t) is chaotic in $L^p(I,\rho)$ if and only if $\int_I \rho(x)\,dx < \infty$.
$u_s(x) := \frac{\sqrt{2}\,e^{isx}}{\sqrt{\pi(1+s^2)}}$, $s \in \mathbb{R}$, $x \in I$ $Z_x = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \frac{\sqrt{2}\,e^{isx}}{\sqrt{\pi(1+s^2)}} dw_s$.

$$\begin{aligned} cov(Z_x, Z_y) &= \int_{-\infty}^{+\infty} u_s(x)\overline{u_s(y)}ds, \\ &= \int_{-\infty}^{+\infty} \frac{2}{\pi(1+s^2)} e^{is(x-y)} ds, \\ &= 2\Big(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{\frac{2}{\pi}} \frac{1}{(1+s^2)} e^{is(x-y)} ds\Big), \\ &= 2e^{-|x-y|}. \end{aligned}$$

(Z_x) is the complex Ornstein-Uhlenbeck process. The induced measure is invariant under translation since (Z_x) is stationary.

Skech of the proof : general case

$\{w^j, j \in J\}$ mutually independent two sided complex Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. For a.e. $x \in T$, define

$$Z_x := \frac{1}{\sqrt{2}} \sum_{j \in J} \int_{\omega_1}^{\omega_2} u_s^j(x)\, dw_s^j.$$

One has

$$\mathbb{E}(Z_x \overline{Z_y}) = \sum_{j \in J} \int_{\omega_1}^{\omega_2} u_s^j(x) \overline{u_s^j(y)}\, ds.$$

$$(\mathbb{E}|Z_x|^2)^{\frac{1}{2}} = \left(\sum_{j \in J} \int_{\omega_1}^{\omega_2} |u_s^j(x)|^2 ds \right)^{\frac{1}{2}}.$$

We know, $\exists c$, such that $(\mathbb{E}|Z_x|^p)^{\frac{1}{p}} \leq c(\mathbb{E}|Z_x|^2)^{\frac{1}{2}}$. We deduce by (2), $\int_T |Z_x|^p\, \sigma(dx) < \infty$, $\mathbb{P}$ almost surely.

Proof...

Define $\mu(B) = \mathbb{P}\{\omega \in \Omega,\ Z_{\cdot}(\omega) \in B\}$, where B is a Borel set in E.

$$
\begin{aligned}
\langle g, Qf \rangle &= \int_E \langle g, u \rangle \, \overline{\langle f, u \rangle} \, \mu(du), \\
&= \int_\Omega \int_T g(x) Z_x(\omega)\, \sigma(dx) \, \overline{\int_T f(y) Z_y(\omega)\, \sigma(dy)} \, \mathbb{P}(d\omega), \\
&= \int_T \int_T \int_\Omega Z_x(\omega) \, \overline{Z_y(\omega)} \, \mathbb{P}(d\omega) \, g(x) \overline{f(y)} \, \sigma(dx) \, \sigma(dy), \\
&= \int_T \int_T \mathbb{E}(Z_x \overline{Z_y}) \, g(x) \overline{f(y)} \, \sigma(dx) \, \sigma(dy), \\
&= \int_T \int_T \sum_{j \in J} \int_{\omega_1}^{\omega_2} u_s^j(x) \overline{u_s^j(y)} \, ds \, g(x) \overline{f(y)} \, \sigma(dx) \, \sigma(dy), \\
&= \sum_{j \in J} \int_{\omega_1}^{\omega_2} \int_T u_s^j(x) g(x) \, \sigma(dx) \, \int_T \overline{u_s^j(y) f(y)} \, \sigma(dy) \, ds, \\
&= \sum_{j \in J} \int_{\omega_1}^{\omega_2} \langle g, u_s^j(\cdot) \rangle \, \overline{\langle f, u_s^j(\cdot) \rangle} \, ds.
\end{aligned}
$$

μ is invariant

$$
\begin{aligned}
\langle g,(AQ+QA^*)f\rangle &= \langle g,AQf\rangle+\langle g,QA^*f\rangle,\\
&= \langle A^*g,Qf\rangle+\langle g,QA^*f\rangle,\\
&= \sum_{j\in J}\int_{\omega_1}^{\omega_2}\langle A^*g,u_s^j\rangle\overline{\langle f,u_s^j\rangle}ds+\sum_{j\in J}\int_{\omega_1}^{\omega_2}\langle g,u_s^j\rangle\overline{\langle A^*f,u_s^j\rangle}ds,\\
&= \sum_{j\in J}\int_{\omega_1}^{\omega_2}\langle g,Au_s^j\rangle\overline{\langle f,u_s^j\rangle}ds+\sum_{j\in J}\int_{\omega_1}^{\omega_2}\langle g,u_s^j\rangle\overline{\langle f,Au_s^j\rangle}ds,\\
&= \sum_{j\in J}\int_{\omega_1}^{\omega_2}\langle g,isu_s^j\rangle\overline{\langle f,u_s^j\rangle}ds+\sum_{j\in J}\int_{\omega_1}^{\omega_2}\langle g,u_s^j\rangle\overline{\langle f,isu_s^j\rangle}ds,\\
&= 0.
\end{aligned}
$$

We conclude that $(AQ+QA^*)f=0$, for all $f\in D(A^*)$.

μ has a full support and strong mixing

For all $f \in E^*$, one has

$$\langle f, Qf\rangle = \sum_{j\in J}\int_{\omega_1}^{\omega_2} |\langle f, u_s^j(\cdot)\rangle|^2\, ds \geq 0.$$

Moreover, $\langle f, Qf\rangle = 0$ implies that $f = 0$.

$$\begin{aligned}
\langle g, T_t Qf\rangle &= \langle T_t^* g, Qf\rangle,\\
&= \sum_{j\in J}\int_{\omega_1}^{\omega_2} \langle T_t^* g, u_s^j\rangle \overline{< f, u_s^j >} ds,\\
&= \sum_{j\in J}\int_{\omega_1}^{\omega_2} \langle g, T_t u_s^j\rangle \overline{\langle f, u_s^j\rangle} ds,\\
&= \sum_{j\in J}\int_{\omega_1}^{\omega_2} e^{its}\langle g, u_s^j\rangle \overline{\langle f, u_s^j\rangle} ds, \longrightarrow 0.
\end{aligned}$$

Outline

1. Problematic
   - Contributions

2. Perspetives

Perspectives

To establish a central limit theorem for a chaotic semigroup on a Banach space X.

$$\frac{1}{\sqrt{T}}\int_0^T f(T_t(\cdot))dt \to \mathcal{N}, \quad T \to \infty$$

for a class of function $f : X \to \mathbb{C}$. The inspiring paper :

1. F. Bayart, *Central limit theorems in linear dynamics*, Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques (to appear).