LARGE DEVIATION PRINCIPLES FOR EMPIRICAL MEASURES OF MULTITYPE RANDOM NETWORKS

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OUTLINE

- Multitype Random Network(MRN) Model;
  Fienberg, Meyer and Wasserman(1985).
- Empirical Measures on Multitype Random Networks;
  Doku-Amponsah (2006).
- Statement and Discussion of Results.
- Further Work
  Hellman and Staudigl(2014)

MULTITYPE RANDOM NETWORK MODEL

Functions: $\kappa_n, \ell_n : \Omega \times \Omega \to [0, \infty]$ and $\kappa, \ell : \Omega \times \Omega \to [0, \infty]$
Type Law: $\eta : \Omega \to (0, 1)$
Define **Multitype Random Network** X with n sites as follows:

- Partition the set of sites, $[n] = \{1, 2, 3, ..., n\}$, into finitely many types $\Omega = \{a_1, a_2, a_3, ..., a_m\}$ independently according to the type law η.
- For a couple of sites (i, j) take the intensities of link creation and link destruction to be $w_{ij}^{(n)}(a, b) = (1 - A_{ij})\kappa_n(a, b)$, $\nu_{ij}^{(n)}(a, b) = A_{ij}\ell_n(a, b)$,
- $W^{(n)}$ is called the *attachment mechanism* while $V^{(n)}$ is called the *volatility mechanism*.

TYPES OF MULTITYPE RANDOM NETWORK

- $X = X(\kappa_n, \ell_n) := \{(X(i), \in \Omega), E\}$ under the joint law of the type and network,
- $X(i)$: the type of site i and X as MRN.
- X is called *symmetric MRN if both functions κ_n and ℓ_n are symmetric*
- Otherwise we call it *asymmetric MRN.*
- MRN = coloured random graph with edge probability is

$$p_n(a,b) = \frac{\kappa_n(a,b)}{\kappa_n(a,b) + \ell_n(a,b)}.$$

See Hellmann and Staudigl(2014).

EMPIRICAL TYPE AND CO-OPERATIVE MEASURES OF THE MRN

- **Empirical type measure:**

$$L^1(a) := \frac{1}{n} \sum_{i \in [n]} \delta_{X(i)}(a), \quad a \in \Omega.$$

- **Empirical cooperative measure** on $\Omega \times \Omega$
Symmetric X**:**

$$L^2(a,b) = \frac{1}{n} \sum_{(i,j) \in E} [\delta_{(X(i),X(j))} + \delta_{((X(j),X(i))}](a,b)$$

Asymmetric X**:**

$$L^2(a,b) = \frac{2}{n} \sum_{(i,j) \in E} \delta_{(X(i),X(j))}(a,b).$$

EMPIRICAL LOCALITY MEASURES OF THE MRN

- $M^1(a,l) := \frac{1}{n}\sum_{i\in[n]} \delta_{(X(i),L(i))}(a,l),$
- $L(i) = \left(l^i(b), b\in\Omega\right),\ l^i(b) = \sum_{j\sim i} 1_{\{X(j)=b\}}.$
- $Deg_Z(k) = \sum_{(a,\ell)\in\Omega\times N(\Omega)} \delta_k(\|\ell\|)M^1(a,\ell)$
- $L^2(a,b) = \sum_{\ell\in N(\Omega)} \ell(b)M^1(a,\ell)$
- $W_0 = \sum_{a\in\Omega} M^1(a,0).$ O'Connell(1998), Doku-Amponsah (2016)

NOTATION

- $N(\Omega)$ is the space of counting measures on Ω, i.e. those measures taking values in $N \cup \{0\}$, endowed with the discrete topology.
- $M(\Omega)$ is the space of probability measures on Ω endowed with the weak topology.
- $\tilde{M}_*(\Omega \times \Omega)$ is the subspace of symmetric measures in $\tilde{M}(\Omega \times \Omega)$ (the space of finite measures on $\Omega \times \Omega$) also equipped with the weak topology.
- $M(\Omega \times N(\Omega))$ is the space of probability measures on Ω endowed with the weak topology.

RELATIVE ENTROPY FOR EMPIRICAL PAIR MEASURES

- For a measure $\pi \in \tilde{M}(\Omega \times \Omega)$ and a measure $\rho \in M(\Omega)$, define (a relative entropy) $\mathfrak{H}_{\kappa/\ell}$ by

$$\mathfrak{H}_{\kappa/\ell}(\pi \,\|\, \rho) := H\Big(\pi \,\|\, \frac{\kappa}{\ell}\rho \otimes \rho\Big) + \|\frac{\kappa}{\ell}\rho \otimes \rho\| - \|\pi\|\,,$$

- $\frac{\kappa}{\ell}\rho \otimes \rho(a, b) = \frac{\kappa(a,b)}{\ell(a,b)}\rho(a)\rho(b)$, for $a, b \in \Omega$
- Note $\mathfrak{H}_{\kappa/\ell}(\pi \,\|\, \rho) \geq 0$ with equality iff $\pi = \frac{\kappa}{\ell}\rho \otimes \rho$.

CONSISTENCY OF EMPIRICAL MEASURES

- We call a pair of measures $(\pi, \mu) \in \tilde{M}(\Omega \times \Omega) \times M(\Omega \times N(\Omega)))$ **sub-consistent** if

$$\pi(a, b) \le \sum_{\ell \in N(\Omega)} \ell(b)\mu(a, \ell), \qquad \forall a, b \in \Omega.$$

- We call a pair of measures $(\pi, \mu) \in \tilde{M}(\Omega \times \Omega) \times M(\Omega \times N(\Omega))$ **consistent** if $\pi(a, b) = \sum_{\ell \in N(\Omega)} \ell(b)\mu(a, \ell), \qquad \forall a, b \in \Omega.$
- Write

$$\Gamma := \Big\{ (\pi, \mu) : (\pi, \mu) \text{ is consistent } \Big\}$$

LDP AND ASYMPTOTIC DISTRIBUTIONS

- **Large deviation principle**
  $U\colon G(\Omega)\to M$ is said to satisfy a *large deviation principle* with rate function I if, for all Borel sets $B\subset M$,

$$-\inf_{m\in B^o} I(m) \le \liminf_{n\to\infty}\frac{1}{n}\log P_n\big\{U(X)\in B\big\} \le \limsup_{n\to\infty}\frac{1}{n}\log P_n\big\{U(X)\in B\big\} \le -\inf_{m\in B^c} I(m).$$

- **Limiting multivariate poisson distribution**
  For every $(\pi,\nu)\in\Omega$, we define a probability measure $Q=Q[\pi,\mu]$ on $\Omega\times N(\Omega)$ by

$$Q(a\,,\,l)=\mu_1(a)\prod_{b\in\Omega}\frac{e^{-\left[\pi(a,b)/\mu_1(a)\right]}\left[\pi(a,b)/\mu_1(a)\right]^{l(b)}}{l(b)!}.$$

ASUMPTIONS

As $n \to \infty$ we assume that κ_n, ℓ_n satisfy

- $n\frac{\kappa_n(a,b)}{\ell_n(a,b)} \to \frac{\kappa(a,b)}{\ell(a,b)}$
- $\frac{\kappa_n(a,b)}{\ell_n(a,b)} \to 0$

Theorem A

(L^1, L^2) obeys an LDP in $M(\Omega) \times M(\Omega \times \Omega)$ with good rate function,

$$I(\rho, \pi) = H(\rho \,\|\, \eta) + \frac{1}{2}\mathfrak{H}_{\kappa/\ell}(\pi \,\|\, \rho)\,.$$

Proof of Theorem A

- Exponential change of measure; See Doku-Amponsah (2006).
- Local Large deviation principle;See Doku-Amponsah (2017).
- Large deviation for mixtures. See Biggins (2004).

Theorem B

(L^2, M^1) obeys an LDP in $M(\Omega \times \Omega) \times M(\Omega \times N(\Omega))$ with good rate function,

$$J_1(\pi, \mu) = \left\{ \begin{array}{ll} H(\mu \,\|\, Q) + H(\mu_1 \,\|\, \eta) + \frac{1}{2}\,\mathfrak{H}_{\kappa/\ell}(\pi \,\|\, \mu_1) & \text{if } (\pi, \mu) \in \Gamma \\ \infty & \text{otherwise.} \end{array} \right\}.$$

Proof of Theorem B

- Random Allocation of Coloured Balls into Coloured Bins; See Doku-Amponsah and Moerters (2010).
- Theorem A
- Large deviations for mixtures; See Biggins (2004)

DISCUSSION OF MAIN RESULTS

- Asymptotically, the empirical cooperative measure behave as

$$\frac{\kappa}{\ell}\eta \otimes \eta.$$

- Asymptotically, the empirical locality measure behave as

$$Q(a\,,\,l) = \eta(a)\prod_{b\in\Omega}\frac{e^{-\left[\frac{\kappa(a,b)}{\ell(a,b)}\eta(b)\right]}\left[\frac{\kappa(a,b)}{\ell(a,b)}\eta\big(b\big)\right]^{l(b)}}{l(b)!}.$$

EVOLUTION AND CO-EVOLUTION PROCESSES ON NETWORKS

- Evolution processes defines an irreducible Markov Chains on the set of networks, say $G[n]$, with a unique invariant distributions which is the law of an inhomogeneous random network. See, Jackson and Watts (2002a).
- Co-Evolution Processes has all the properties of the evolution processes plus additional parameter which models the action profile of a player. ie. its has Action Adjustment rule, Link creation rule and Link destruction rule.See, Jackson and Watts (2002a).
- Indeed, co-evolution processes may seen as extensions of random networks.See, Hellman and Staudigl(2014).

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