Lecture 1

1. According to the Standard Model of particle physics, matter is constituted out of leptons and quarks. These particles have half integer spin and therefore enjoy fermionic statistics. Forces are mediated by the photon (QED), the W^{\pm} and Z (weak force), the gluons (QCD, or the strong interaction), and the Higgs field, which endows particles with mass. All the force carriers are bosons. They have integer spin. Supersymmetry is a conjectured symmetry of Nature that posits that for every bosonic degree of freedom, there is fermionic degree of freedom and vice versa. In unbroken supersymmetry, each particle has a superpartner that is mass degenerate and whose spin differs by $\frac{1}{2}$. All of the other quantum numbers are the same.

The superpartner of the electron field is the *selectron* or scalar electron field (spin-0), while the superpartner of the photon field is the *photino* field (spin- $\frac{1}{2}$). Since we do not see a massless photino or a selectron with mass 0.511 MeV, supersymmetry, if it exists, is broken in Nature.

2. We should ask ourselves: why supersymmetry? The Coleman–Mandula no go theorem demonstrates that the S-matrix of a quantum field theory only allows spacetime and internal symmetries to be combined in a trivial manner. The conserved quantities are the generators of the Poincaré group and Lorentz scalars. Supersymmetry evades the Coleman–Mandula theorem by introducing additional generators of symmetries, the *supercharges*, which are spinors. The Poincaré algebra is enhanced to a Lie superalgebra. Notably, in (3 + 1)-dimensions, we have

$$\{Q_{\alpha}, \overline{Q}_{\dot{\beta}}\} = 2 \left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} P_{\mu} , \qquad P_{\mu} = -i\partial_{\mu} , \qquad (1)$$

where Q_{α} and $\overline{Q}_{\dot{\beta}}$ are fermionic generators with indices that run from 1,2. Roughly speaking,

$$Q|\text{boson}\rangle = |\text{fermion}\rangle , \qquad Q|\text{fermion}\rangle = |\text{boson}\rangle .$$
 (2)

From the perspective of particle physics, supersymmetry addresses the gauge hierarchy problem and presents WIMP candidates for the dark matter that comprises some 27% of the energy density in the visible Universe.

The Higgs potential resembles a Mexican hat:

$$V(\phi) = -\mu(\Lambda)^2 \phi^{\dagger} \phi + \lambda(\Lambda)(\phi^{\dagger} \phi)^2 .$$
(3)

The couplings are a function of the energy scale at which we probe the theory. The field achieves its minimum not at zero but at

$$\langle \phi \rangle \approx \frac{\mu(0)}{\sqrt{2\lambda(0)}}$$
 (4)

for energies much smaller than the ultraviolet cutoff Λ . For $\lambda \sim 1$, the renormalized mass $\mu(0) \sim M_{\rm EW} \sim 10^2$ GeV. The electroweak scale is where $SU(2)_L \times U(1)_Y$ is broken to $U(1)_{\rm EM}$ by the Higgs effect. We may compute $\mu(0)$ from the Feynman diagrams in Figure 1.



Figure 1: One loop Higgs mass renormalization.

Each of the diagrams corresponding to radiative corrections is quadratically divergent. We find that

$$\mu^2(0) = \mu^2(\Lambda) + \Lambda^2(c_1\lambda(\Lambda) + c_2g^2(\Lambda) + \dots) .$$
(5)

Because the Standard Model is an effective theory, we must integrate in physics as we approach the cutoff. Let's suppose new physics enters at $\Lambda = M_X \sim 10^{16}$ GeV, corresponding, say, to the grand unification scale. Then

$$\frac{\mu^2(0)}{M_X^2} = \frac{\mu^2(M_X)}{M_X^2} + (c_1\lambda(M_X) + c_2g^2(M_X) + \ldots) \sim 10^{-28} .$$
(6)

There is a scarcely credible, highly unnatural fine tuning between the bare mass and the radiative corrections to accomplish this detailed cancellation. What physics explains why the electroweak scale is so much smaller than the fundamental scale in the theory? This is the *gauge hierarchy problem*.

We know that fermionic loops have a minus sign relative to bosonic loops. Invoking supersymmetry, we have, for example, the Feynman diagrams in Figure 2 involving top quarks and their stop superpartners. These diagrams exactly cancel! In fact, with soft supersymmetry breaking, we do not encounter ultraviolet divergences in scalar masses.



Figure 2: The top and stop one loop diagrams cancel.

While there are hundreds or perhaps thousands of putative models, the mechanism Nature actually employs remains elusive. If R-parity is a symmetry of a supersymmetric extension of the Standard Model, the lightest superpartner is stable. This provides a candidate for weakly interacting cold dark matter. To date, there is no observational evidence for supersymmetry at the scales accessible to experiments. In the absence of supersymmetry, some degree of fine tuning may be necessary.

3. Let us promote the standard coordinates to *superspace*:

$$x^{\mu} \to (x^{\mu}, \theta_{\alpha}, \overline{\theta}_{\dot{\alpha}})$$
 (7)

Note that θ and $\overline{\theta}$ are Grassmann quantities. This means, for example, that

$$\theta_{\alpha}\theta_{\beta} = -\theta_{\beta}\theta_{\alpha} \ . \tag{8}$$

Integration over θ , $\overline{\theta}$ is the same as differentiation.

We define the covariant derivatives and supercharges as follows:

$$D_{\alpha} = \frac{\partial}{\partial \theta^{\alpha}} + i \left(\sigma^{\mu}\right)_{\alpha \dot{\alpha}} \overline{\theta}^{\dot{\alpha}} \partial_{\mu} , \qquad \overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}} - i \theta^{\alpha} \left(\sigma^{\mu}\right)_{\alpha \dot{\alpha}} \partial_{\mu} , \qquad (9)$$

$$Q_{\alpha} = \frac{\partial}{\partial \theta^{\alpha}} - i \left(\sigma^{\mu}\right)_{\alpha \dot{\alpha}} \overline{\theta}^{\dot{\alpha}} \partial_{\mu} , \qquad \overline{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}} + i \theta^{\alpha} \left(\sigma^{\mu}\right)_{\alpha \dot{\alpha}} \partial_{\mu} . \tag{10}$$

The supersymmetry generators Q and \overline{Q} define the supertranslations

$$(x^{\mu}, \theta_{\alpha}, \overline{\theta}_{\dot{\alpha}}) \to (x^{\mu} + i\theta\sigma^{\mu}\overline{\xi} - i\xi\sigma^{\mu}\overline{\theta}, \theta_{\alpha} + \xi_{\alpha}, \overline{\theta}_{\dot{\alpha}} + \overline{\xi}_{\dot{\alpha}}) , \qquad (11)$$

where we have suppressed some indices in the obvious way. By acting on test functions, we deduce that

$$\{Q_{\alpha}, Q_{\beta}\} = \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0 , \qquad \{Q_{\alpha}, \overline{Q}_{\dot{\alpha}}\} = 2 (\sigma^{\mu})_{\alpha \dot{\alpha}} P_{\mu} .$$
(12)

The covariant derivatives also anticommute with the supersymmetry generators.

The minimal supersymmetry theory in four dimensions, so called $\mathcal{N} = 1$ supersymmetry, has four supercharges. Extended supersymmetry comes in different varieties: $\mathcal{N} = 2$, which has eight supercharges, and $\mathcal{N} = 4$, which has sixteen supercharges. We have the algebra

$$\{Q^{I}_{\alpha}, Q^{J}_{\beta}\} = C_{\alpha\beta}Z^{IJ} , \qquad \{\overline{Q}^{I}_{\dot{\alpha}}, \overline{Q}^{J}_{\dot{\beta}}\} = \overline{C}_{\dot{\alpha}\dot{\beta}}\overline{Z}^{IJ} , \qquad \{Q^{I}_{\alpha}, \overline{Q}^{J}_{\dot{\alpha}}\} = 2\left(\sigma^{\mu}\right)_{\alpha\dot{\alpha}}P_{\mu}\delta^{IJ} , \qquad (13)$$

where we have introduced *central charges*. In extended theories, there is a non-Abelian R-symmetry that lets us rotate the charges into each other. The maximal $\mathcal{N} = 4$ super-Yang–Mills theory has a massless vector field, four Weyl fermions, and six scalars. This is secretly a theory of gravity and represents an entry into a vast subject known as the AdS/CFT correspondence.

We should think of superfields as functions on superspace. As the Grassmann variables anticommute, the Taylor expansion terminates, and we can write a general superfield as

$$\Phi(x,\theta,\overline{\theta}) = \phi + \theta\psi + \overline{\theta}\overline{\chi} + \theta^2 F + \overline{\theta}^2 G + \theta\sigma^{\mu}\overline{\theta}A_{\mu} + \theta^2\overline{\theta}\overline{\lambda} + \theta\overline{\theta}^2\zeta + \theta^2\overline{\theta}^2 D .$$
(14)

Counting parameters, we have bosonic variables $\phi(x)$, F(x), G(x), $A_{\mu}(x)$, and D(x) along with fermionic variables $\psi(x)$, $\overline{\chi}(x)$, $\overline{\lambda}(x)$, and $\zeta(x)$. The latter are two component Weyl spinors, so in total there are eight bosonic degrees of freedom and eight fermionic degrees of freedom.

A chiral superfield is one for which

$$\overline{D}_{\dot{\alpha}}\Phi = 0. \tag{15}$$

We notice that $\overline{D}_{\dot{\alpha}}$ kills θ^{α} and also $y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\overline{\theta}$. Thus,

$$\Phi = \Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y)$$

= $\phi + i\theta\sigma^{\mu}\overline{\theta}\partial_{\mu}\phi + \frac{1}{4}\theta^2\overline{\theta}^2\partial^2\phi + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\partial_{\mu}\psi\sigma^{\mu}\overline{\theta} + \theta^2 F$. (16)

Similarly, we define *antichiral superfields* as those for which $D_{\alpha}\overline{\Phi} = 0$.

4. Consider a functional $\mathcal{K}(\Phi, \overline{\Phi})$. Suppose we define a metric by taking derivatives with respect to the scalar components of the chiral and antichiral superfields:

$$g_{m\overline{n}} = \partial_m \partial_{\overline{n}} \mathcal{K}(\phi^m, \overline{\phi}^{\overline{n}}) , \qquad \partial_m = \frac{\partial}{\partial \phi^m} , \quad \partial_{\overline{n}} = \frac{\partial}{\partial \overline{\phi}^{\overline{n}}} .$$
 (17)

Given this metric, we may construct a Levi-Civita connection and from there the Riemann tensor. Then, on shell,

$$\mathcal{L} \supset \int d^4\theta \ \mathcal{K} = -g_{m\overline{n}}\partial_\mu\phi^m\partial^\mu\overline{\phi}^{\overline{n}} - ig_{m\overline{n}}\overline{\psi}^{\overline{n}}\overline{\sigma}^\mu D_\mu\psi^m + \frac{1}{4}R_{m\overline{n}p\overline{q}}\psi^m\psi^p\overline{\psi}^{\overline{n}}\overline{\psi}^{\overline{q}} \ , \tag{18}$$

where

$$D_{\mu}\psi^{m} = \partial_{\mu}\psi^{m} + \Gamma^{m}_{np}\partial_{\mu}\phi^{p}\psi^{n} .$$
⁽¹⁹⁾

The functional $\mathcal{K}(\Phi, \overline{\Phi})$ is the *Kähler potential*. It yields the kinetic terms for the component superfields. Note that there is no kinetic term for F; this is an *auxiliary* field that we can eliminate using the equations of motion. For renormalizable theories, we take $\mathcal{K}(\Phi, \overline{\Phi}) = \overline{\Phi}_i \Phi^i$. 5. A vector superfield V is real: $V^{\dagger} = V$. In Wess-Zumino gauge, we may write

$$V(x,\theta,\overline{\theta}) = -\theta\sigma^{\mu}\overline{\theta}A_{\mu}(x) + i\theta^{2}\overline{\theta\lambda}(x) - i\overline{\theta}^{2}\theta\lambda(x) + \frac{1}{2}\theta^{2}\overline{\theta}^{2}D(x) .$$
⁽²⁰⁾

The vector field $A_{\mu}(x) = A^{a}_{\mu}(x)T^{a}$ is an element of the Lie algebra \mathfrak{g} corresponding to the gauge group G. Gauge transformations act as

$$V \to V + \Lambda + \overline{\Lambda}$$
, (21)

where Λ is any chiral superfield. Chiral and antichiral superfields with charge q transform as

$$\Phi \to e^{q\Lambda} \Phi , \qquad \overline{\Phi} \to e^{q\overline{\Lambda}} \overline{\Phi} .$$
 (22)

Using V, we may define

$$\mathcal{W}_{\alpha} = -\frac{1}{4}\overline{D}^2(e^{-V}D_{\alpha}e^V) , \qquad \overline{\mathcal{W}}_{\dot{\alpha}} = \frac{1}{4}D^2(e^{V}\overline{D}_{\dot{\alpha}}e^{-V}) .$$
(23)

The expression

$$\mathcal{L} \supset \operatorname{Im}\left[\tau\left(\int d^2\theta \operatorname{tr}(\mathcal{W}_{\alpha}\mathcal{W}^{\alpha}) + \int d^2\overline{\theta} \operatorname{tr}(\overline{\mathcal{W}}_{\dot{\alpha}}\overline{\mathcal{W}}^{\dot{\alpha}})\right)\right] , \qquad \tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g_{\mathrm{YM}}^2}$$
(24)

recovers tr F^2 , the gauge kinetic term, and tr $F\widetilde{F}$, the term proportional to the ϑ -angle.

6. Interactions arise from the *superpotential*. This is a holomorphic function of the chiral superfields. We have

$$\mathcal{L} \supset \int d^2 \theta \ W(\Phi) + \text{h.c.}$$
 (25)

Due to the power of holomorphy, we can apply our complex analysis toolkit to work with supersymmetric quantum field theories. Crucially, the superpotential W is not perturbatively renormalized. Non-perturbative effects such as instanton corrections may enter.

According to the Coleman–Mandula theorem, any global symmetry must commute with the Poincaré algebra. Global symmetries need not commute with the supersymmetry generators, however. We can have an R-symmetry:

$$[R, Q_{\alpha}] = -Q_{\alpha} , \qquad [R, \overline{Q}_{\dot{\alpha}}] = \overline{Q}_{\dot{\alpha}} . \tag{26}$$

Because the *R*-symmetry does not commute with Q and \overline{Q} , the component fields have different *R*-charges. Thus, for a chiral superfield

$$R(\phi) = r$$
, $R(\psi) = r - 1$, $R(F) = r - 2$. (27)

We label the *R*-charge of Φ with the *R*-charge of the lowest component of the superfield. We as well assign an *R*-charge +1 to θ_{α} and -1 to $d\theta_{\alpha}$. We require that terms in the superpotential have *R*-charge +2.

7. The Wess-Zumino model is an example of an interacting four dimensional supersymmetric quantum field theory. It has the Lagrangian density

$$\mathcal{L} = \int d^4\theta \ \overline{\Phi}_i \Phi^i + \int d^2\theta \ \left(\nu_i \Phi^i + \frac{1}{2}m_{ij}\Phi^i \Phi^j + \frac{1}{3!}\lambda_{ijk}\Phi^i \Phi^j \Phi^k\right) + \text{h.c.}$$
(28)

For convenience, we will assume that we can shift the fields so that we remove the tadpoles. Notice that if the fields Φ^i have *R*-charge +1, the coupling λ has *R*-charge -1.

Let us restrict to the case where there is a single complex scalar field. In components, we can write the Lagrangian in terms of a scalar A, a pseudoscalar B, and a four component Majorana (real) spinor ψ :

$$\mathcal{L}_{WZ} = \mathcal{L}_{kin} + \mathcal{L}_{mass} + \mathcal{L}_{int} , \qquad (29)$$

$$\mathcal{L}_{\rm kin} = \frac{1}{2} \partial_{\mu} A \partial^{\mu} A + \frac{1}{2} \partial_{\mu} B \partial^{\mu} B + \frac{1}{2} \overline{\psi} \partial \psi , \qquad (30)$$

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}m^2 A^2 + \frac{1}{2}m^2 B^2 + m\overline{\psi}\psi , \qquad (31)$$

$$\mathcal{L}_{\text{int}} = \lambda \left(\overline{\psi} (A - B\gamma^5) \psi + \frac{1}{2} \lambda (A^2 + B^2)^2 + m A (A^2 + B^2) \right) . \tag{32}$$

Exercises

- Starting from the supersymmetry algebra, show that the energy of any state is nonnegative. Usually, we can set the zero point of energy where we like and measure the energies of states relative to this reference. Why is this not what we do in a theory with supersymmetry?
- What are the scaling dimensions of the various component superfields in the Wess–Zumino model?
- Suppose

$$\mathcal{L} = \partial_{\mu}\phi^*\partial^{\mu}\phi + i\psi^{\dagger}\overline{\sigma}^{\mu}\partial_{\mu}\psi .$$
(33)

Show that this changes only by a total derivative under the transformation:

$$\delta_{\varepsilon}\phi = \varepsilon^{\alpha}\psi_{\alpha} = \varepsilon\psi ,$$

$$\delta_{\varepsilon}\psi_{\alpha} = -i(\sigma^{\mu}\varepsilon^{\dagger})_{\alpha}\partial_{\mu}\phi ,$$

$$\delta_{\varepsilon}\psi_{\dot{\alpha}}^{\dagger} = i(\varepsilon\sigma^{\mu})_{\dot{\alpha}}\partial_{\mu}\phi^{*} ,$$

(34)

where ε is an infinitesimal spinor.

• Show that the action $S_{WZ} = \int d^4x \, \mathcal{L}_{WZ}$ is invariant under supersymmetry transformations

$$\delta_{\varepsilon}A = \overline{\varepsilon}\psi ,$$

$$\delta_{\varepsilon}B = \overline{\varepsilon}\gamma^{5}\psi ,$$

$$\delta_{\varepsilon}\psi = \left[\partial \!\!\!/ - m - \lambda(A + B\gamma^{5})\right](A + B\gamma^{5})\varepsilon ,$$
(35)

with ε a constant Majorana spinor.

• Show that the superpotential $W = \frac{1}{2}m\Phi^2 + \frac{1}{3!}\lambda\Phi^3$ is not renormalized.

References

 L. Susskind, "The gauge hierarchy problem, technicolor, supersymmetry, and all that," Phys. Rept. 104, 181 (1984).

This forms the basis of 2 above.

- [2] P. C. Argyres, "An introduction to global supersymmetry," http://homepages.uc.edu/~argyrepc/cu661-gr-SUSY/susy2001.pdf.
- [3] J. M. Figueroa-O'Farrill, "BUSSTEPP lectures on supersymmetry," hep-th/0109172. These are good references on global supersymmetry in four dimensions.

Lecture 2

8. Since $[P_{\mu}, Q_{\alpha}] = 0$, it follows that $[P^2, Q_{\alpha}] = 0$. This means that any two states $|\psi\rangle$ and $Q_{\alpha}|\psi\rangle = |\chi\rangle$ are mass degenerate.

From

$$\{Q_{\alpha}, \overline{Q}_{\dot{\beta}}\} = 2(\sigma^{\mu})_{\alpha\dot{\beta}}P_{\mu} , \qquad (36)$$

we conclude that

$$\delta^{\alpha\dot{\beta}}\langle\psi|\{Q_{\alpha},\overline{Q}_{\dot{\beta}}\}|\psi\rangle = \sum_{\dot{\alpha}}\langle\psi|(\overline{Q}_{\dot{\alpha}})^{\dagger}\overline{Q}_{\dot{\alpha}}|\psi\rangle + \sum_{\alpha}\langle\psi|(Q_{\alpha})^{\dagger}Q_{\alpha}|\psi\rangle$$
$$= \sum_{\dot{\alpha}}||\overline{Q}_{\dot{\alpha}}|\psi\rangle||^{2} + \sum_{\alpha}||Q_{\alpha}|\psi\rangle||^{2} \ge 0$$
(37)

$$= 2\delta^{\alpha\dot{\beta}}(\sigma^{\mu})_{\alpha\dot{\beta}}\langle\psi|P_{\mu}|\psi\rangle = 4\langle\psi|P_{0}|\psi\rangle = 4E||\psi\rangle||^{2}.$$
(38)

We realize that $E \ge 0$ for any state $|\psi\rangle$ in a supersymmetric theory. The inequality saturates if and only if $Q_{\alpha}|\psi\rangle = \overline{Q}_{\dot{\alpha}}|\psi\rangle = 0$.

The supersymmetric ground state is invariant under the action of any of the supersymmetry generators. In global supersymmetry, the expectation value of the energy of the ground state therefore serves as an order parameter for the existence of supersymmetry.

9. Recall from last time that a chiral superfield has the expansion

$$\Phi(y) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) , \qquad (39)$$

with components that shift in terms of a constant Weyl spinor ϵ as

$$\delta\phi = \sqrt{2}\epsilon\psi \ , \qquad \delta\psi = \sqrt{2}\epsilon F + i\sqrt{2}\sigma^{\mu}\overline{\epsilon}\partial_{\mu}\phi \ , \qquad \delta F = i\sqrt{2}\overline{\epsilon\sigma}^{\mu}\partial_{\mu}\psi \ . \tag{40}$$

The kinetic term for the chiral fields is

$$\mathcal{L}_{\rm kin} = \sum_{i} \left(\partial_{\mu} \phi^{*i} \partial^{\mu} \phi^{i} - \frac{i}{2} \overline{\psi}^{i} \overline{\sigma}^{\mu} \partial_{\mu} \psi^{i} + F^{*i} F^{i} \right) \,. \tag{41}$$

The superpotential is integrated over half of superspace. Isolating the terms that have θ^2 , we find

$$\mathcal{L}_W = \sum_i \frac{\partial W}{\partial \Phi^i} F^i + \sum_{i,j} \frac{\partial^2 W}{\partial \Phi^i \partial \Phi^j} \psi^i \psi^j .$$
(42)

The Euler–Lagrange equation tells us that

$$F^{*i} = -\frac{\partial W}{\partial \Phi^i} \ . \tag{43}$$

Substituting the equation of motion for the auxiliary field F back into the Lagrangian, we see that the potential contains the F-terms,

$$V \supset \sum_{i} |F^{i}|^{2} = \sum_{i} \left| \frac{\partial W}{\partial \Phi^{i}} \right|^{2} .$$

$$\tag{44}$$

There is an F-term for each chiral superfield Φ^i that appears in the superpotential.

Similarly, from the vector superfield, the Lagrangian incorporates the following structure:

$$\mathcal{L} \supset \sum_{a} \left(\frac{1}{2g^2} (D^a)^2 + D^a \sum_{i} \phi^{*i} T^a \phi^i \right) \,. \tag{45}$$

From here, we derive an equation of motion for the auxiliary field D:

$$D^a = -g^2 \sum_i \phi^{*i} T^a \phi^i .$$
⁽⁴⁶⁾

The potential becomes

$$V(\phi) = \sum_{i} |F^{i}|^{2} + \sum_{a} (D^{a})^{2} .$$
(47)

The second term in (47) is the *D*-term. Notice that there is a *D*-term for each generator of the gauge group G. In case the gauge group contains a U(1) factor, we can include the Fayet-fliopoulos term

$$L_{\rm FI} = \xi \int d^4\theta \ V \ . \tag{48}$$

We have the D-term

$$D = \xi + \sum_{i} q_i \phi^{*i} \phi^i .$$
⁽⁴⁹⁾

Since the energy is zero within a supersymmetric vacuum, we require that

$$F^i = 0, \quad \forall i, \tag{50}$$

$$D^a = 0, \quad \forall a. \tag{51}$$

In general, there is more than one solution to these equations. The set of field configurations $\{\langle \phi^i \rangle\}$ that satisfy the *F*-term equation (50) and *D*-term equation (51) defines the *moduli space* of vacua of the $\mathcal{N} = 1$ supersymmetric gauge theory.

10. An orbifold of a space is an identification of points related by the linear action of a discrete group. As a first example, the complex number z = x + iy defines a point on the plane with coordinates (x, y). Let us identify the points z and -z. This is the orbifold \mathbb{C}/\mathbb{Z}_2 . The only point that is invariant under the action of the orbifold is z = 0. Another \mathbb{Z}_2 identification may be $z \sim \overline{z}$. Here, the fixed points of the orbifold are the real numbers.

As a slightly more complicated example, consider the torus \mathbb{T}^2 . We start with a lattice defined by the unit vector \hat{x} and the vector $\vec{\tau}$ in the complex plane. The torus identifies the points

$$z \sim z + m + n\tau$$
, $m, n \in \mathbb{Z}$. (52)

The fundamental domain is a unit cell as shown in Figure 3.

Let us choose the complex structure so that we tesselate the complex plane \mathbb{C} by a fundamental domain defined by $\tau = e^{\frac{2\pi i}{3}}$. Take three copies of this torus to construct a \mathbb{T}^6 . A point on \mathbb{T}^6 is then specified by three complex numbers (z_1, z_2, z_3) which reside in the fundamental domain.

The group \mathbb{Z}_3 consists of the cubed roots of unity: $\mathbb{Z}_3 = \{1, \omega_3, \omega_3^{-1}\}$. We take the orbifold $\mathbb{T}^6/\mathbb{Z}_3$ by defining the map

$$\gamma: (z_1, z_2, z_3) \mapsto (\omega_3 z_1, \omega_3 z_2, \omega_3^{-2} z_3)$$
(53)



Figure 3: A torus with complex structure τ and area A. The complex structure describes the shape of the torus while the Kähler modulus describes its size.

and identifying a point with its image: $p \sim \gamma(p)$. In order to emphasize that the product of the phases acting on the coordinates is unity, we write the third phase in (53) as ω_3^{-2} instead of ω_3 . There are $3^3 = 27$ fixed points under the action of the orbifold group. To be explicit,

$$\omega_{3} \cdot 0 = 0,
\omega_{3} \cdot \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}} = \frac{1}{\sqrt{3}} e^{\frac{5\pi i}{6}} \sim \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}} - 1 \sim \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}},
\omega_{3} \cdot \frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}} = \frac{2}{\sqrt{3}} e^{\frac{5\pi i}{6}} \sim \frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}} - 2 \sim \frac{2}{\sqrt{3}} e^{\frac{\pi i}{6}}.$$
(54)

Obviously, these same points are invariant on multiplication by $\omega_3^2 = \omega_3^{-1}$ and $\omega_3^3 = 1$. Locally, each fixed point is of the form $\mathbb{C}^3/\mathbb{Z}_3$.

Orbifolds are not smooth spaces. There are *conical singularities* at fixed points of the \mathbb{Z}_3 orbifold action. This is drawn in Figure 4.



Figure 4: The space near a fixed point of a \mathbb{Z}_3 action on \mathbb{T}^2 .

The Hilbert space of states for point particles on X_6 consists of those states that are invariant under γ . That is to say, we restrict to wavefunctions that enjoy the property that

$$\psi(z_i) = \psi(\omega_3 z_i) . \tag{55}$$

The operator

$$P = \frac{1}{3}(1 + \gamma + \gamma^2) \tag{56}$$

projects the Hilbert space of states on \mathbb{T}^6 to the Hilbert space of states on X_6 . If we consider the sigma model on the closed string worldsheet, to account for the orbifold identification, we should augment the spectrum with *twisted sector* states:

$$(x^{\mu}(\sigma+2\pi), z_i(\sigma+2\pi)) = (x^{\mu}(\sigma), \omega_3^{k_i} z_i(\sigma)) , \qquad k_i = 0, 1, 2 , \qquad (57)$$

where the coordinates (x^{μ}, z_i) describe the embedding in spacetime.

We may generalize the construction to $X_6 = \mathbb{T}^6/\mathbb{Z}_n$. The *n*-th roots of unity are $\omega_n^k = e^{\frac{2\pi i k}{n}}$, $k = 0, \ldots, n-1$. A \mathbb{Z}_n action on the coordinates of \mathbb{T}^6 is

$$(z_1, z_2, z_3) \sim (\omega_n^a z_1, \omega_n^b z_2, \omega_n^c z_3)$$
 (58)

Except at the singularities associated to the fixed points, X_6 is a flat six dimensional geometry. It is an interesting open problem to classify all of the inequivalent orbifold actions on the coordinates of \mathbb{C}^m by elements of a discrete group.

Superstring theory is a consistent theory of quantum gravity in 9+1 dimensions. The spectrum of the superstring includes solitionic objects called D-branes. A D*p*-brane extends in *p* spatial dimensions and defines a submanifold in spacetime on which open strings end with Dirichlet boundary conditions. Suppose we place D3-branes at a singular point on the orbifold X_6 . There is a low energy effective theory on the worldvolume of the brane. The theory preserves $\mathcal{N} = 1$ supersymmetry if we enforce the constraint in (58) that $a + b + c = 0 \mod n$. This is the *Calabi–Yau* condition for a local Abelian orbifold.

11. The cone over a Sasaki–Einstein base \mathcal{B} is a Calabi–Yau geometry. The worldvolume gauge theory for N D3-branes at the tip of the cone is an $\mathcal{N} = 1$ theory that is dual to string theory on $\mathrm{AdS}_5 \times \mathcal{B}$. The gauge theories corresponding to these singularities have bifundamental matter and a product gauge group

$$G = \prod_{i=1}^{n} G_i . \tag{59}$$

Each of the factors G_i is a U(N) group. A chiral superfield Φ_{jk}^i transforms in the fundamental representation \Box of G_i , in the antifundamental representation $\overline{\Box}$ of G_j , and as a singlet under all of the other factors. In the case i = j, we think of this field as transforming in the adjoint representation of G_i . (Recall that the fundamental and antifundamental representations of U(N) have dimension N; the adjoint representation has dimension N^2 , equal to the dimension of the group.) The k is a multiplicity index in case there is more than one superfield with these charges.

The quiver — once upon a time, called a moose — provides a graphical language to express the matter content of $\mathcal{N} = 1$ quantum field theories with only bifundamental and adjoint matter. The quiver consists of a collection of nodes and a set of arrows that connect pairs of nodes. There is a node for each factor in (59) and an arrow for each chiral superfield. In our conventions, the superfield transforms in the antifundamental representation of the group corresponding to the node it departs and the fundamental representation of the group corresponding to the node it enters. For anomaly cancellation, the number of arrows coming into a node equals the number of arrows exiting the node. We draw the quiver for the suspended pinched point (SPP) in Figure 5 and summarize the matter content.

We must supply both the quiver and the superpotential to describe the theory. The superpotential is invariant under gauge transformations. We must therefore contract the gauge indices.

\bigcap	Field	$U(N)_1$	$U(N)_2$	$U(N)_3$
\ ¥	Φ_1^2			1
3	Φ_1^3		1	
Л	Φ^1_2			1
	Φ_2^3	1		
\neq χ	Φ^1_3		1	
	Φ_3^2	1		
(2) ↓ (1)	Φ_3^3	1	1	Adj

Figure 5: The quiver for the suspended pinched point.

The terms in the superpotential are a subset of the closed loops in the quiver. For the SPP theory, we have

$$W = \operatorname{tr} \left[\Phi_3^3 \Phi_3^1 \Phi_1^3 - \Phi_3^3 \Phi_3^2 \Phi_2^3 + \Phi_3^2 \Phi_2^1 \Phi_1^2 \Phi_2^3 - \Phi_2^1 \Phi_1^3 \Phi_3^1 \Phi_1^2 \right] .$$
 (60)

Notice that following the arrows, each term in the superpotential begins and ends at the same node. Explicitly, the first term in the superpotential is

$$\operatorname{tr} \Phi_3^3 \Phi_3^1 \Phi_1^3 = (\Phi_3^3)_a^b (\Phi_3^1)_b^c (\Phi_1^3)_c^a . \tag{61}$$

Here, for Φ_3^3 , *a* is an antifundamental index of $U(N)_3$ and *b* is a fundamental index of $U(N)_3$; for Φ_3^1 , *b* is an antifundamental index of $U(N)_3$ and *c* is a fundamental index of $U(N)_1$; for Φ_3^3 , *c* is an antifundamental index of $U(N)_1$ and *a* is a fundamental index of $U(N)_3$. Fundamental and antifundamental indices of the same gauge group contract to give a color singlet.

12. Another class of models arises from considering D-branes at asymptotically locally Euclidean (ALE) singularities. In the simplest case, consider the quotient spaces $\mathbb{C} \times \mathbb{C}^2/\Gamma$, where Γ is a discrete subgroup of SU(2). We have seen examples of these already in the \mathbb{Z}_n orbifolds. These are $\mathcal{N} = 2$ theories whose quivers correlate to the affine Dynkin diagrams associated to the ADE groups. (This is the content of the *McKay correspondence*.) The field theories are dual to string theory on $\mathrm{AdS}_5 \times S^5/\Gamma$. Douglas and Moore showed that the resolution of the singularity is identical to the equations describing the moduli space of vacua of the worldvolume theory on the D-branes.

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Lecture 3

13. As a first example, let us consider $\mathcal{N} = 4$ super-Yang–Mills theory with gauge group U(N). In $\mathcal{N} = 1$ language, the theory contains three chiral multiplets transforming in the adjoint representation. The quiver is given in Figure 6.



Figure 6: The quiver for $\mathcal{N} = 4$ super-Yang–Mills theory.

The superpotential for this theory is

$$W = \operatorname{tr} \phi_1[\phi_2, \phi_3] . \tag{62}$$

As the ϕ_i transform in the adjoint representation, we can regarded them as complex valued $N \times N$ matrices. From differentiation of the superpotential, we obtain the *F*-term equations:

$$0 = \frac{\partial W}{\partial \Phi_1} = [\phi_2, \phi_3] , \qquad 0 = \frac{\partial W}{\partial \Phi_2} = [\phi_3, \phi_1] , \qquad 0 = \frac{\partial W}{\partial \Phi_3} = [\phi_1, \phi_2] . \tag{63}$$

This means that the ϕ_i are matrices which are simultaneously diagonalizable. Each of the matrices have N complex eigenvalues, so the moduli space of vacua is 6N real dimensional.

We can write the potential in terms of six real scalars:

$$V \propto g_{\rm YM}^2 \operatorname{tr} \sum_{i,j} [X_i, X_j]^2 .$$
(64)

At a generic point in moduli space, the eigenvalues of the matrices are unequal. The gauge group is broken from U(N) to $U(1)^N$. This is the *Coulomb branch*.

14. We want a systematic procedure for solving for the vacuum moduli space of $\mathcal{N} = 1$ theories. In this discussion, we follow Luty–Taylor, but antecedents for these results exist in the literature. The action for the $\mathcal{N} = 1$ theory

$$S = \int d^4x \left[\int d^4\theta \ \Phi_i^{\dagger} e^V \Phi_i + \left(\frac{1}{4g^2} \int d^2\theta \ \mathrm{tr} \mathcal{W}_{\alpha} \mathcal{W}^{\alpha} + \int d^2\theta \ W(\Phi) + \mathrm{h.c.} \right) \right]$$
(65)

enjoys an enormous gauge redundancy. It is invariant under

$$\Phi \mapsto g\Phi , \qquad e^V \mapsto (g^{-1})^{\dagger} e^V g^{-1} ,$$
(66)

where $g = e^{i\Lambda}$ and Λ is a chiral superfield. We work with G^c the complexification of the gauge group. The *complexification* of a real Lie group G is a complex Lie group G^c containing G as a real subgroup such that there is a Lie algebra isomorphism between the two. The complexification of SU(N) is $SL(N, \mathbb{C})$; the complexification of U(N) is $GL(N, \mathbb{C})$.

Writing

$$V_A = C - \theta \sigma^{\mu} \overline{\theta} v_{\mu A} + i \theta^2 \overline{\theta} \overline{\lambda}_A - i \overline{\theta}^2 \theta \lambda_A + \frac{1}{2} \theta^2 \overline{\theta}^2 D_A , \qquad (67)$$

where A is an adjoint index, the D-term equations can be written as

$$\frac{\partial}{\partial C_A} (\phi^{\dagger} e^C \phi) = 0 , \qquad (68)$$

where ϕ is the scalar component of the chiral superfield Φ . One can then establish the following properties:

- The F-flatness conditions are holomorphic and invariant under G^c .
- The *D*-flatness conditions fix the gauge.
- For every solution to the *F*-terms, there is a solution to the *D*-terms in the completion of the orbit of the complexified gauge group.
- The set of gauge invariant operators (GIOs) provides a basis for the *D*-orbits.
- The *vacuum moduli space* is the symplectic quotient of the master space, which is the manifold of scalar field vevs that satisfy the *F*-term equations:

$$\mathcal{M} = \mathcal{F} / / G = \mathcal{F} / G^c . \tag{69}$$

15. Let's recall a few facts from algebra. A *ring* is a set with two binary operations, + and \times such that it an Abelian group under addition and a monoid under multiplication, an operation which distributes with respect to addition. We have:

$$a + b = b + a , \qquad (a + b) + c = a + (b + c) ,$$

$$a + 0 = a , \qquad a + (-a) = 0 ,$$

$$(a \times b) \times c = a \times (b \times c) , \qquad a \times 1 = 1 \times a = a ,$$

$$a \times (b + c) = (a \times b) + (a \times c) , \qquad (a + b) \times c = (a \times c) + (b \times c) .$$

(70)

The inverse under multiplication need not exist in the ring. We will often omit the \times .

A left ideal I is a non-empty subset of R such that for all $x, y \in I$, $r \in R$, the compositions x + y and rx are in I. That is to say,

$$r_1 x_1 + \dots r_n x_n \in I , \qquad \forall r_i \in R , \quad \forall x_i \in I .$$

$$(71)$$

We say $RI \subseteq I$. Similarly, a *right ideal* I stems from the property that $IR \subseteq I$. An *ideal* is both a left ideal and a right ideal.

A quotient ring R/I is the set of equivalent classes of elements in R modulo elements of an ideal:

$$a \sim b \qquad \Longleftrightarrow \qquad a - b \in I$$
. (72)

Examples illustrate these definitions.

- Consider the ring of integers \mathbb{Z} . There is an addition and a multiplication operation. Note that $n^{-1} \notin \mathbb{Z}$ for $n \neq \pm 1$. The integer multiples of k, where k is a positive integer, form an ideal $I = k\mathbb{Z}$. The quotient ring $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z} = \{0, 1, \dots, k-1\}$.
- Consider the ring of polynomials in the variable x with real valued coefficients: $R = \mathbb{R}[x]$. Take the ideal $I = (x^2 + 1)$, consisting of all polynomials with a factor $(x^2 + 1)$. In the quotient ring R/I, we have polynomials that look like a + bx. This is isomorphic to \mathbb{C} with the rôle of $i = \sqrt{-1}$ played by the class [x].

• Consider the algebraic variety $V = \{(x, y) | x^m = y^n\} \subset \mathbb{R}^2$. The ring of real valued polynomial functions on V is identified with the quotient ring $\mathbb{R}[x, y]/(x^m - y^n)$.

The last example provides a natural language for expressing the vacuum moduli space of a quantum field theory. We first define the master space \mathcal{F} as a quotient ring:

$$\mathcal{F} = \mathbb{C}[\Phi_i] / \{F_i\} , \qquad (73)$$

where the ideal is defined for us by the F-terms.

- 16. To construct the vacuum moduli space we employ the following algorithm.
 - We define the polynomial ring $\mathbb{C}[\Phi_i, y_j]$. The y_j are new variables, one for each element of the minimal basis of GIOs in the theory.
 - We define the ideal $I = \{F_i(\Phi), y_j r_j(\Phi)\}$. The r_j are the GIOs in the minimal list.
 - We eliminate the variables Φ_i from the ideal.
 - This gives an ideal $\mathcal{M} \subset \mathbb{C}[y_i]$ in terms of the y variables.
- 17. This is a lot of formalism. Let's do an example. Consider the conifold gauge theory. Its quiver is shown in Figure 7.



Figure 7: The quiver for the conifold.

The superpotential for the theory is

$$W = \operatorname{tr} \left[\varphi_1 \chi_1 \varphi_2 \chi_2 - \varphi_1 \chi_2 \varphi_2 \chi_1 \right] . \tag{74}$$

For simplicity, let us consider a $U(1) \times U(1)$ theory. The superpotential vanishes as the fields commute. Thus, there are no *F*-terms to consider. From examination of the quiver, we see that the minimal list of gauge invariant operators consists of the closed loops:

$$z_1 = \varphi_1 \chi_1 , \qquad z_2 = \varphi_1 \chi_2 , \qquad z_3 = \varphi_2 \chi_1 , \qquad z_4 = \varphi_2 \chi_2 .$$
 (75)

We see that there is a relation between the gauge invariant operators:

$$z_1 z_4 = z_2 z_3 . (76)$$

Taking the z_i as coordinates in \mathbb{C}^4 , the vacuum moduli space consists of the points that satisfy this relation. In fact, Klebanov–Witten showed that if we place D3-branes at the singularity at the origin, the worldvolume gauge theory is precisely the quiver model that we have considered. The vacuum moduli space recapitulates the string realization of the field theory. This is the lesson of Douglas–Moore. 18. A class of examples of considerable interest to us is supersymmetric QCD (SQCD). We can have two theories in the same universality class, meaning that while the field content and interactions are different in the ultraviolet, they flow to the same infrared fixed point. This is *Seiberg duality*. The duality occurs in a window where

$$\frac{3}{2}N_c < N_f < 3N_c . (77)$$

The number of flavors of quarks is N_f and N_c sets the rank of the gauge group of the electric theory. The matter content and interactions of the two theories is summarized in Table 1.



Figure 8: The electric and magnetic theories belong to the same universality class. The physics is the same in the infrared.

	electric	magnetic
gauge group	$SU(N_c)$	$SU(N_f - N_c)$
global symmetries	$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$	$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$
chiral fields	$q: (\Box, N_f, 1, \frac{1}{N_c}, 1 - \frac{N_c}{N_f})$ $\overline{a}: (\overline{\Box} \ 1 \ \overline{N}_c \ - \frac{1}{2} \ 1 - \frac{N_c}{N_c})$	$Q: (\Box, 1, N_f, -\frac{1}{N_f - N_c}, \frac{N_c}{N_f})$ $\overline{O}: (\overline{\Box} \ \overline{N}_c, 1, \frac{1}{N_c}, \frac{N_c}{N_c})$
chinal heids	$q \cdot (\Box, 1, \mathbf{N}_f, -\frac{1}{N_c}, 1 - \frac{1}{N_f})$	$M: (1, N_f, \overline{N}_f, 0, 2(1-rac{N_c}{N_f}))$
superpotential	W = 0	$W_{\rm dual} = \lambda M \overline{Q} Q$

Table 1: Seiberg dual theories.

The quarks and antiquarks in SQCD transform in defining representations of $SU(N_c)$ whereas the quarks and antiquarks in the dual transform in defining representations of $SU(N_f - N_c)$. The SQCD theory has superpotential W = 0. Its dual, however, has a nonvanishing superpotential:

$$W_{\rm dual} = \lambda M \overline{Q} Q \ . \tag{78}$$

The mesons and baryons are gauge invariant operators. They are the same on both sides of the duality, however in the magnetic theory, we regard the mesons as fundamental fields. Moreover, the number of quarks in a baryon is fixed by the rank of the gauge group, which is different for the electric and magnetic theories.

The duality exemplifies a strong coupling/weak coupling correspondence (an S-duality). One of the checks of Seiberg duality is that the vacuum moduli space on the two sides is the same.

19. We are developing tools to compute the vacuum moduli spaces of semi-realistic quantum field theories relevant to particle physics.

Exercises

- Write the *F*-term equations for the SPP theory.
- Work out the GIOs of the electric and magnetic theories.
- Suppose $N_f = N_c$. How many GIOs are there in the electric theory? What is the dimension of the vacuum moduli space?
- Consider the electroweak theory with gauge group $SU(2)_L \times U(1)_Y$. We have $L^i_{\alpha} : (\mathbf{2}, -\frac{1}{2}), \overline{H}_{\alpha} : (\mathbf{2}, -\frac{1}{2}), e^i : (\mathbf{1}, 1)$. What is a minimal list of GIOs?
- Take a quiver with n nodes arrayed on a circle with arrows in both directions between adjacent nodes. Let the gauge group be $U(1)^n$ so that the superpotential vanishes. Compute the vacuum moduli space.

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A classic in quantum field theory.