Dissipative dynamics in noncommutative spaces

Bogusław Zegarliński

Imperial College London

Abstract

I will review recent progress including properties of norms and entropy functionals and discuss construction and study of dyssipative dynamics in noncommutative spaces.

Non Commutative Banach Spaces
C^***-algebra** : $\mathcal{A}$.

Positive Elements : $\mathcal{A}^+ \equiv \{a^*a : \quad a \in \mathcal{A}\}$.

A state

$$\omega(f) \equiv Tr(\rho f), \qquad with \qquad \rho \geq 0, \qquad Tr\rho = 1$$

Hamiltonian Dynamics

$$\alpha_t(f) \equiv \Delta^{it}(f) \equiv \rho^{it} f \rho^{-it}$$

Modular Operator^s

$$\Delta^s \equiv \Delta^s_\rho \equiv \rho^s f \rho^{-s}$$

Scalar Products: For $s \in [0,1]$

$$\langle f, g\rangle_{\omega,s} \equiv Tr\left(\left(\rho^{\frac{s}{2}} f \rho^{\frac{1-s}{2}}\right)^* \left(\rho^{\frac{s}{2}} g \rho^{\frac{1-s}{2}}\right)\right) = \omega\left((\Delta^s(f))^* \Delta^s(g)\right)$$

Interpolating Family of Non Commutative $\mathbb{L}_p(\omega, s)$ **-spaces** :

For $p \in [1, \infty]$, $\qquad s \in [0, 1]$

$$\|f\|_{p,\omega,s}^{p} \equiv Tr \left| \rho^{\frac{s}{p}} f \rho^{\frac{1-s}{p}} \right|^{p}$$

For $n \in \mathbb{N}$, $\qquad s \in [0, 1]$

$$\|f\|_{2n,\omega,s}^{2n} \equiv Tr \left(\rho^{\frac{s}{2n}} f^* \rho^{\frac{1-s}{n}} f \rho^{\frac{s}{2n}} \right)^{n}$$

For $f^* = f$ and $p = n \in \mathbb{N}$

$$\|f\|_{n,\omega,s}^{n} = \omega \left(\Delta^{\frac{s}{n}}(f) \Delta^{\frac{s+1}{n}}(f) ... \Delta^{\frac{s+n}{n}}(f) \right)$$

Positive elements : $\mathbb{L}_p(\omega, s)^+$

Nice fitting together of $\mathcal{A}^+$ and $\mathbb{L}_2^+(\omega, \frac{1}{2})$.

Completely Positive Maps $\equiv$ CPMs

Monotonicity of $\mathbb{L}_p$ norms (associated to weights)

Theorem $\forall \beta \in [0,1]$ $\forall r = 2^n$, $n \in \mathbb{N}$

$$Tr|\Phi(P)^{-\frac{(1-\beta)}{r}}\Phi(X)\Phi(P)^{-\frac{\beta}{r}})|^r \leq Tr|P^{-\frac{(1-\beta)}{r}}XP^{-\frac{\beta}{r}}|^r,$$

where Φ is a CPM.

Theorem For every $r \in [2,\infty)$ and $\beta \in [0,1]$, all the functionals

$$\Lambda(X) \equiv Tr\left|P^{-\frac{(1-\beta)}{r}}XP^{-\frac{\beta}{r}}\right|^r$$

are (jointly) CPM-monotone.

• For a convex function Ψ which is monotone increasing on $(0, \infty)$ with $\Psi(0) = 0$ and $\Psi(x) \to \infty$ as $x \to \infty$, define

$$\Lambda_Q(X) \equiv Tr\Psi(X^*(\eta(Q))^{-1}X)$$

with a positive operator concave function η and Q a positive operator.

Theorem The Orlicz functional

$$\Lambda_Q(X) \equiv Tr\Psi(X^*(\eta(Q))^{-1}X)$$

is (jointly) CPM-monotone.

• With $\Psi(t) = t^q(log(1 + t^q))^\alpha$, with $q \in [1, \infty)$ and $\alpha \in (0, \infty)$, we have the following (jointly) CPM-monotone functional

$$Tr\left(X^*(Q^{-1/q})X\right)^q \left(\log\left(1 + \left(X^*(Q^{-1/q})X\right)^q\right)\right)^\alpha$$

Monotone Scalar Products and Duality

A scalar product $\langle \cdot, \cdot \rangle_P$ associated to a density matrix P is called CPM-monotone, iff

$$\langle \Phi(X), \Phi(X) \rangle_{\Phi(P)} \leq \langle X, X \rangle_P.$$

Given a scalar $\langle \cdot, \cdot \rangle_P$ product and an Orlicz functional $\Lambda_P()$, we define a dual functional

$$\Xi_P(X) \equiv \sup \left(\Re \langle X, Y \rangle_P - \Lambda_P(Y) \right)).$$

Consider complementary Young functions Φ and Ψ (continuous, strictly increasing on $[0, \infty$, going to zero at the origin and at infinity faster than linearly).

Fact : The inverse functions satisfy

$$a < \Phi^{-1}(a)\Psi^{-1}(a) < 2a, \ for \ a > 0.$$

A scalar product associated to a density matrix ρ as follows:

$$\begin{aligned}(X,Y)_{\rho,\alpha} &\equiv Tr((\Phi^{-1}(\rho)\Psi^{-1}(\rho))^{\alpha}X^{*}(\Phi^{-1}(\rho)\Psi^{-1}(\rho))^{1-\alpha}Y) \\ &= Tr\left(\left((\Phi^{-1}(\rho))^{1-\alpha}X(\Phi^{-1}(\rho))^{\alpha}\right)^{*}\left((\Psi^{-1}(\rho))^{1-\alpha}Y(\Psi^{-1}(\rho))^{\alpha}\right)\right),\end{aligned}$$

Young Inequality

$$(X,Y)_{\rho,\alpha} \leq \mathbf{\Phi}_{\rho,\alpha}(X) + \mathbf{\Psi}_{\rho,\alpha}(Y)$$

where

$$\begin{aligned}\mathbf{\Phi}_{\rho,\alpha}(X) &\equiv Tr(\Phi(|(\Phi^{-1}(\rho))^{1-\alpha}X(\Phi^{-1}(\rho))^{\alpha}|)), \\ \mathbf{\Psi}_{\rho,\alpha}(Y) &\equiv Tr(\Psi(|(\Psi^{-1}(\rho))^{1-\alpha}Y(\Psi^{-1}(\rho))^{\alpha}|)),\end{aligned}$$

- We remark that for Orlicz functions Φ and Ψ the function $[0,\infty) \ni x \to \Theta(x) \equiv \Phi^{-1}(x)\Psi^{-1}(x)$ is log-concave.

If it is also operator log-concave in the sense, that for any positive operators A and B

$$\Theta(\frac{A+B}{2}) \geq \Theta(A)\sharp\Theta(B),$$

where

$$A\sharp B \equiv A^{\frac{1}{2}} \cdot \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} \cdot A^{\frac{1}{2}}$$

denotes operator geometric mean, then by Theorem 2.3 of Ref.[T.AndoF.Hiai2011], the function Θ is operator monotone. Hence by a well-known result, the function Θ is operator concave (see e.g., Theorem V.2.5 of Ref.[R. Bhatia1997]). Thus we conclude with the following property.

Proposition For $\Phi^{-1} \cdot \Psi^{-1}$ operator concave, the following scalar product is CPM-monotone

$$\langle X, Y\rangle_{\rho,\alpha} \equiv Tr\left(\frac{1}{(\Phi^{-1}(\rho))^{1-\alpha}} X \frac{1}{(\Phi^{-1}(\rho))^{\alpha}}\right)^{*} \left(\frac{1}{(\Psi^{-1}(\rho))^{1-\alpha}} Y \frac{1}{(\Psi^{-1}(\rho))^{\alpha}}\right).$$

Conjecture Suppose an Orlicz functional $\Lambda_P(\cdot)$ and a scalar product $\langle \cdot, \cdot \rangle_P$ are CPM-monotone. Then the dual functional $\Xi_P(\cdot)$ is also CPM-monotone.

Logarithmic Sobolev Inequality: Perturbation Theory

- The Classical Case

$$Ent_\mu(f^2) \le c \int |\nabla f|^2 d\mu \qquad \textbf{(LS}_2\textbf{)}$$

where

$$Ent_\mu(f^2) \equiv \mu\left(f^2 \log \frac{f^2}{\mu(f^2)}\right)$$

and a constant $c \in (0, \infty)$ independent of f.

Linearisation Formula with respect to the measure:

$$\mu\left(f^2\log\frac{f^2}{\mu(f^2)}\right) = \inf_{t>0}\left(\mu\left(f^2\log\frac{f^2}{t} - \mu f^2 + t\right)\right)$$

It was shown in Ref.[S.G.BobkovF.Götze1999] that such inequality is equivalent to the following bound

$$\|(f-\mu(f))^2\|_N \leq C\int |\nabla f|^2 d\mu$$

with $\|\cdot\|_N$ denoting the Luxemburg norm corresponding to a Young function $N(x) = |x|log(1+|x|)$ and some constant $C \in (0,\infty)$ independent of f .

• Non Commutative Case

Let $\omega(f) \equiv Tr(\rho f)$

with a density matrix $\rho > 0$, $Tr\rho = 1$. Let

$$\|f\|_p^p \equiv Tr|F_{\rho,p}|^p$$

with

$$F_{\rho,p} \equiv \rho^{\frac{1}{2p}} f \rho^{\frac{1}{2p}}.$$

Fact:

$$\frac{d}{dp}\left(\|f\|_p^p\right)_{|p=2} = Ent_{2,\rho}(f)$$

where

$$Ent_{2,\rho}(f) \equiv Tr|F_{\rho,2}|^2 \left(\log \frac{|F_{\rho,2}|^2}{Tr|F_{\rho,2}|^2} - \log \rho\right).$$

Some properties of norms and Relative Entropy

Concavity properties of the $L_p(\omega)$, for $p \in (1,2]$, norms:

$$\|f\|_p^2 \geq |\omega(f)|^2 + (p-1)\|\tilde{f}\|_p^2,$$

with $\tilde{f} \equiv f - \omega(f)$.
Rothaus Inequality:

$$Ent_{2,\rho}(f) \leq Ent_{2,\rho}(\tilde{f}) + 2\|\tilde{f}\|_2^2.$$

Let

$$\mathcal{L}_\rho(f) \equiv \sup_{a \in \mathbb{R}} Ent_{2,\rho}(f+a).$$

Properties:

– For any $\zeta \in (0, \infty)$

$$\mathcal{L}_\rho(\zeta f) = \zeta^2 \mathcal{L}_\rho(f).$$

–

$$\mathcal{L}_\rho(f) = \mathcal{L}_\rho(\tilde{f})$$

Given a constant $\gamma \geq 1$, we introduce the following functional

$$\mathcal{N}_\rho(f) \equiv Tr(|F_{\rho,2}|^2(\log(\gamma\rho + |F_{\rho,2}|^2) - \log \rho)).$$

$\mathcal{N}_\rho(f)$ is an Orlicz functional. Let

$$\|f\|_N \equiv \inf\{\xi > 0 : \mathcal{N}_\rho\left(\frac{f}{\xi}\right) \leq 1\}$$

be the corresponding Luxemburg norm (corresponding formally to the Young function $N(x^2) \equiv x^2 log(\gamma + x^2)$).

Theorem [AZ2014]
There exist constants $c_0, c_1 \in (0, \infty)$ such that

$$c_0\|\tilde{f}\|_N \leq \mathcal{L}_\rho(f) \leq c_1\|\tilde{f}\|_N.$$

Markov Semigroups

Semigroup of operators $(P_t)_{t\geq 0}$ (linear or nonlinear)

$P_t : \mathbb{B} \to \mathbb{B}$, where $(\mathbb{B}, \|\cdot\|)$ a Banach space
($(\mathcal{A}, \|\cdot\|), (\mathbb{L}_p(\omega, s), \|\cdot\|_{p,s})$, Orlicz space, ...);

- $P_t P_s = P_{t+s}$, $t, s \geq 0$;
- $P_{t=0} =$ id;
- $t \mapsto P_t f$ continuous for any $f \in \mathbb{B}$,

(strongly, (in op norm, weakly,..., in vNnn algebras)).

Positive : For a proper convex cone $\mathbb{B}^+$

$$P_t : \mathbb{B}^+ \to \mathbb{B}^+ \qquad (\qquad P_t : \mathcal{A}^+ \to \mathcal{A}^+ \& \ \pi(P_t \mathcal{A}^+) \subseteq \mathbb{B}^+ \quad ?, \ )$$

2-Positive

Schwartz Inequailty$_{[\text{Choi}'80]}$

$$P_t(f^* f) \geq P_t(f^*) P_t(f)$$

n-Positive

$$P_t^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{A})$$

$$P_t^{(n)}(f \otimes E_{\text{ij}}) = P_t(f) \otimes E_{\text{ij}}$$

(where $E_{\mathsf{ij}}, i, j = 1, ..., n$ are matrix units spanning $M_n(\mathbb{C})$), is *positive*.

Completely Positive

$\forall n \in \mathbb{N}\ P_t^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ is *positive*

Unit Preserving

- $P_t \mathbb{I} = \mathbb{I}$, $\forall t \geq 0$;

Symmetric in $\mathbb{L}_2(\omega, s)$

([SQV′84]+via Dirichlet Forms[AH-K'77],[Ci′]+Korean Grp[Pa])

E.g.'s

a) Linear$_{\text{[GoderisMaes' 91],[Matsui]}_{\text{GroundStateRepresentation}},\text{[BaKoPa' 03]}_{\text{Ext}_{\text{classicalIsi}}}}$

b) Gaussian type semigroups ([CiFaLi],[OZa],[Pa...])

c) On ∞-dim algebras$_{\text{[MZ],[MOZ],...,}}$

d) Diffusion Type (Ho'' rmander type Generators) ([LOZ' 10])

e) via Drichlet Forms([Pa′ 05] avoiding L_1 asymptotic abelianess,...)

f) No E.g.s of symmetric jump type @ ∞-dim spaces with *non-classical* interaction

g) Nonlinear([LOZ′ 13])

$$S_t(f) \equiv e^{-t} f + \int_0^t \mathrm{ds} \log \omega(\exp(e^{-s} f))$$

(nonlinear annealing algorithm to find a ground state)

Markov Semigroups on infinite dimensional algebras Construction and Ergodicity

[MOZ][MZ]$_{\text{SpinSystems}}$,[LOZ]$_{\text{HoermanderType}}$,[MaesG],[Matsui]$_{\text{GroundState}}$

Markovian Quadratic Form for a Markov Generator $\mathcal{L}$

$$\Gamma_{\mathcal{L}}(f) \equiv \frac{1}{2}(\mathcal{L}(f^*f) - \mathcal{L}(f^*)f - f^*\mathcal{L}(f))$$

Hypercontractivity in Noncommutative Spaces.

Definition of **Hypercontractivity**:

A Markov semigroup $P_t \equiv e^{t\mathcal{L}}$ is **hypercontractive** in $\mathbb{L}_q(\omega)$, $1 < q < \infty$ spaces iff

For any $1 < p_0 < p < \infty$

$$\exists T \in (0, \infty) \forall t > T \qquad ||P_t f||_p \leq ||f||_{p_0}$$

REM : Hypercontractivity in an interpolating family of Banach spaces $(\mathbb{B}_r)_{r \in I}$.

E.g. In $\mathbb{L}_p(d\lambda)$ or Orlicz spaces [BartheCatieauxRoberto].

Spectral Theory + Gaussian Bounds.

$$Hypercontractivity \iff ?\, ParticleStructure?$$

- Invariant Subspaces $\mathbb{L}_2(\mu) = \mathcal{H}_0 \oplus_{n\in\mathbb{N}} \mathcal{H}_n$

$$\mathcal{H}_j \perp \mathcal{H}_k, k \neq j, P_t(\mathcal{H}_n) \subset \mathcal{H}_n$$

- Spectrum

$$\exists \varepsilon > 0 \quad \forall n \in \mathbb{N} \qquad \sigma(\mathcal{L} \restriction \mathcal{H}_n) \subset (-\infty, -\varepsilon n),$$

- Gaussian Bounds: $\forall n \in \mathbb{N} \quad \forall f \in \mathcal{H}_n$

$$\exists C > 0 \qquad ||f||_{\mathbf{4}} \leq C^n ||f||_{\mathbf{2}},$$

E.g.s

- Free Quantum Field [Ne′ 66],[Si]...
- For Fermions [Gr′ 66],[CL′ 92]([Li′ 90])...
- 1-D Ising [B&Z'00]
- Product States on *NC* $\mathcal{A}$ & Weak Product Property [HO &Z'01],[B&Z'0

- Exotic CCR (q- OU[Biane′ 97],[Bozejko′ 99],[BozKuSp′ 97];t- OU[Krolak′ 05].)

- Quasi-Free & Fermionic [TePaKa′ 14]

Conjecture :[OH &Z'01]

$\mathcal{A}_0 = M_{k\times k}, k \geq 2$, and $\mathcal{A}^{(n)} = \mathcal{A}_0^{\otimes n}$

$\mathrm{Tr}\, f^2 \log f^2\, \mathrm{Tr}\, f^2 \log \mathrm{Tr}\, f^2 \leq c_{\mathrm{opt}}(k) \sum_{i=1}^{n} \mathrm{Tr}\, |\, \mathrm{Tr}_i\, f - f|^2$

holds for any $f \in \mathcal{A}_{\mathrm{sa}}^{(n)}$ with optimal constant

$c_{\mathrm{opt}}(k) = (k/(k2)) \log(k1)$

Hypercontractivity for product states I.

- Product state $\omega \equiv \otimes_l \quad \omega_{\Lambda_l}$, where $\omega_{\Lambda_l} \equiv \mathbf{Tr}_{\Lambda_l}(\rho_{\Lambda_l} \cdot)$

  $0 < \rho_{\Lambda_l} \leqq ||\rho_{\Lambda_l}|| < \infty$, $\Lambda_l \cap \Lambda_k = \emptyset$ for $k \neq l$.

- $\mathbb{L}_p(\omega, s)$ norms

$$||f||^p_{\mathbb{L}_p(\omega,s)} \equiv \mathbf{Tr} \left| \rho_\Lambda^{\frac{1-s}{p}} f \rho_\Lambda^{\frac{s}{p}} \right|^p ,$$

for $f \in \mathcal{A}_\Lambda$ with $\rho_\Lambda \equiv \prod_{\Lambda_l \cap \Lambda \neq \emptyset} \rho_{\Lambda_l}$.

- $\mathbb{L}_2(\omega, s)$ scalar product

$$< f \ , \ g >_{\mathbb{L}_2(\omega,s)} \equiv \mathbf{Tr} \left(\rho_\Lambda^{\frac{1-s}{2}} f^* \rho_\Lambda^{\frac{s}{2}} g \right) \quad .$$

- Markov generator symmetric in $\mathbb{L}_2(\omega, s)$, $\forall s \in [0, 1]$,

$$\mathcal{L}f \equiv \sum_{l \in \mathcal{R}} (E_{\Lambda_l}(f) - f)$$

defined with

- Generalized Conditional Expectation

$$E_X(f) \equiv \mathbf{Tr}_X(\xi^*_{\Lambda_l} f \xi_{\Lambda_l}) \quad ,$$

where for a bdd set $X \subset \mathcal{R}$,

$$\xi_X \equiv \rho_{\Lambda_l}^{\frac{1}{2}} \left(\mathbf{Tr}_X \rho_{\Lambda_l} \right)^{-\frac{1}{2}}$$

Theorem :

- **Hypercontractivity** : The Markov semigroup $P_t \equiv e^{t\mathcal{L}}$ satisfies

$$||P_t f||_{\mathbb{L}_{p(t)}(\omega,s)} \leq ||f||_{\mathbb{L}_2(\omega,s)}$$

for any $s \in [0,1]$ with $p(t) \equiv 1 + e^{\alpha t}$, with some $\alpha > 0$.

- **Weak product property**: [Bodineau & Z'00], [Hebisz, Olkiewicz & Z'01]

Therefore (for $s = \frac{1}{2}$)

$$\exists \tilde{c}_{\Lambda_0} \in (0,\infty)$$

$$QEnt_2(f) \quad \leq \quad \tilde{c}_{\Lambda_0} \, \langle f, -\sum_{i\in\mathbb{Z}^d} (E_i(f) - f)\rangle_{\mathbb{L}_2(\omega,\frac{1}{2})}$$

where

$$QEnt_p(f) \equiv \lim_{\Lambda \to \mathcal{R}} \mathbf{Tr}\, |\rho_\Lambda^{1/2p} f \rho_\Lambda^{1/2p}|^p (\log |\rho_\Lambda^{1/2p} f \rho_\Lambda^{1/2p}| - 1/2p \log \rho_\Lambda) \quad .$$

Hypercontractivity and Spectral Gap

(H) $\Rightarrow \|P_t f - \omega(f)\|_2^2 \leq e^{-\tilde{m}t} \|f - \omega(f)\|_2^2$

Equivalence Theorem

Suppose P_t is a L_2-symmetric Feller semigroup which is hypercontractive, that is we have

$$\|P_t f\|_{q(t)} \leq \exp\{2d\left(\frac{1}{2} - \frac{1}{q(t)}\right)\|f\|_2 \qquad (*)$$

with $d \in [0,\infty)$ and $q(t) = 1 + e^{2t/c}$ defined with some constant $c \in (0,\infty)$.

Then the following Logarithmic Sobolev inequality is true.

$$\langle f, \mathbf{T}_2(f)\rangle - \|f\|_2^2 \log \|f\|_2 \leq c\mathcal{E}_2(f,f) + d\|f\|_2^2 \qquad \textbf{(LS}(c,d)\textbf{)}$$

Optimal Product Property ??

Bounded Perturbation Lemma ??

LS(c) **for infinite dimensional models** ???

At least for classical interaction ?

1-D models ?

Strong Ergodicity via Hypercontractivity ???

Equivalence of Complete Analyticity and Log-Sobolev Ineq ???

[SZ′92]

Slower tails weaker functional inequallities ???

Chellanging Computational Problems :

@ Large Interacting Systems & Slow Decay to Equilibrium

(Phase transitions, Disordered systems, Ground States,......)

Can Quantum Computing Say something about Quantum

Logarithmic Sobolev Inequality: Perturbation Theory

• The Classical Case **Bounded Perturbation of the Relative Entropy**:

$$Ent_\mu(f^2) \le c \int |\nabla f|^2 d\mu \qquad (\mathbf{LS}_2)$$

where

$$Ent_\mu(f^2) \equiv \mu \left(f^2 \log \frac{f^2}{\mu(f^2)} \right)$$

and a constant $c \in (0, \infty)$ independent of f.

Linearisation Formula (with respect to the measure):

$$\mu\left(f^2 \log \frac{f^2}{\mu(f^2)}\right) = \inf_{t>0}\left(t \cdot \mu\left(\frac{f^2}{t}\log\frac{f^2}{t} - \frac{f^2}{t} + 1\right)\right)$$

with

$$\varphi(z) \equiv z \log z - z + 1$$

Proposition:
If $d\nu \equiv e^{-U} d\mu$, then

$$Ent_\nu(f^2) \leq e^{-\inf(U)} Ent_\mu(f^2)$$

Bounded Perturbation of the classical Dirichlet form:
Let

$$\mathcal{E}_\mu(f) \equiv \mu|\nabla f|^2$$

Proposition:
If $d\nu \equiv e^{-U} d\mu$, then

$$\mathcal{E}_\mu(f) \leq e^{supU} \mathcal{E}_\nu(f)$$

Bounded Perturbation of Log-Sobolev Inequality:

Theorem:

If $d\nu \equiv e^{-U} d\mu$ and

$$Ent_\mu(f^2) \leq c_\mu \int |\nabla f|^2 d\mu,$$

then

$$Ent_\nu(f^2) \leq e^{\mathrm{osc}(U)} c_\mu \mathcal{E}_\nu(f)$$

Bounded Perturbation of Poincare Inequality: Classical Case

$$m\mu\,(f-\mu f)^2 \leq \mathcal{E}_\mu(f)$$

Note that

$$\nu\,(f-\nu f)^2 = \inf_{a\in\mathbb{R}} \nu\,(f-a)^2 \leq \nu\,(f-\mu f)^2$$

$$\leq e^{-\inf U}\mu\,(f-\mu f)^2 \leq e^{-\inf U} m^{-1}\mathcal{E}_\mu(f) \leq e^{osc(U)} m^{-1}\mathcal{E}_\nu(f)$$

- **Non Commutative Case**

Dirichlet Forms in Non Commutative Setup

C^***-algebra** : $\mathcal{A}$.

Positive Elements : $\mathcal{A}^+ \equiv \{a^*a : \quad a \in \mathcal{A}\}$.

A state

$$\omega(f) \equiv Tr(\rho f), \qquad with \qquad \rho \geq 0, \qquad Tr\rho = 1$$

Hamiltonian Dynamics

$$\alpha_t(f) \equiv \Delta^{it}(f) \equiv \rho^{it} f \rho^{-it}$$

Modular Operator^s

$$\Delta^s \equiv \Delta^s_\rho \equiv \rho^s f \rho^{-s}$$

Relative Modular Operator^s

$$\Delta^s_{\tilde{\rho},\rho} \equiv \tilde{\rho}^s f \rho^{-s}$$

Scalar Products: For $s \in [0,1]$

$$\langle f, g\rangle_{\omega,s} \equiv Tr\left(\left(\rho^{\frac{s}{2}} f \rho^{\frac{1-s}{2}}\right)^* \left(\rho^{\frac{s}{2}} g \rho^{\frac{1-s}{2}}\right)\right) = \omega\left((\Delta^s(f))^* \Delta^s(g)\right)$$

Interpolating Family of Non Commutative $\mathbb{L}_p(\omega, s)$ **-spaces** :

For $p \in [1, \infty]$, $\quad s \in [0,1]$

$$\|f\|^p_{p,\omega,s} \equiv Tr\left|\rho^{\frac{s}{p}} f \rho^{\frac{1-s}{p}}\right|^p$$

For $n \in \mathbb{N}$, $\quad s \in [0,1]$

$$\|f\|^{2n}_{2n,\omega,s} \equiv Tr\left(\rho^{\frac{s}{2n}} f^* \rho^{\frac{1-s}{n}} f \rho^{\frac{s}{2n}}\right)^n$$

Positive elements : $\mathbb{L}_p(\omega, s)^+$

Nice fitting together of $\mathcal{A}^+$ and $\mathbb{L}_2^+(\omega, \frac{1}{2})$.

Dirichlet Forms

Suppose for some $a \in \mathbb{R}$

$$\Delta^{\beta}(X) = e^{a} X.$$

Then the following is a **Dirichlet form** in $\mathbb{L}(\omega, \frac{1}{2})$.

$$\mathcal{E}_X(f) \equiv \mathcal{E}_{X,\omega}(f) \equiv \langle \delta_X(f), \delta_X(f) \rangle_{\omega, \frac{1}{2}} + \langle \delta_{X^*}(f), \delta_{X^*}(f) \rangle_{\omega, \frac{1}{2}}$$

where

$$\delta_Z(f) \equiv i[Z, f]$$

Perturbation of Dirichlet Forms.

Theorem : *Suppose the following Poincaré Inequality holds*

$$\|f - \omega_{\tilde{\rho}}(f)\|^2_{\tilde{\rho},\frac{1}{2}} \leq \tilde{c}\, \mathcal{E}_{\tilde{X}}(f)$$

Suppose $\quad \tilde{X} \equiv X + B.$
If

$$\| \left| \Delta^{\frac{1}{4}}_{\tilde{\rho},\rho}(\mathbb{I}) \right|^2 \|^2_{\rho,\infty} < \infty$$

and

$$4\tilde{c} \cdot \left(\| \left| \tilde{\Delta}^{\frac{1}{4}}(B) \right|^2 \|_{\tilde{\rho},\infty} + \| \left| \tilde{\Delta}^{\frac{1}{4}}(B^*) \right|^2 \|_{\tilde{\rho},\infty} \right) < 1,$$

then

$$\exists \tilde{C} \in (0,\infty) \qquad \mathcal{E}_{\tilde{X}}(f) \leq \tilde{C} \cdot \mathcal{E}_X(f)$$

Perturbation of Poincaré Inequality.

Theorem: *Suppose* $\exists\, \tilde{c} \in (0,\infty)$

$$\|f - \omega_{\tilde{\rho}}(f)\|^2_{\tilde{\rho},\frac{1}{2}} \leq \tilde{c}\, \mathcal{E}_{\tilde{X},\tilde{\rho}}(f)$$

Suppose $X \equiv \tilde{X} - B$ *with*

$$4\tilde{c}\left(\| \left|\tilde{\Delta}^{\frac{1}{4}}(B)\right|^2 \|_{\tilde{\rho},\infty} + \| \left|\tilde{\Delta}^{\frac{1}{4}}(B^*)\right|^2 \|_{\tilde{\rho},\infty}\right) < 1$$

Assume that

$$\| \left|\Delta_{\rho,\tilde{\rho}}(\mathbb{I})\right|^2 \|^2_{\tilde{\rho},\infty} + \| \left|\Delta_{\tilde{\rho},\rho}(\mathbb{I})\right|^2 \|^2_{\rho,\infty} < \infty.$$

Then $\exists c \in (0,\infty)$

$$\|f - \omega_{\rho}(f)\|^2_{\rho,\frac{1}{2}} \leq c\, \mathcal{E}_{X,\rho}(f).$$

Example: Perturbation of Gaussian State on CCR Algebra

CCR on Infinite Dimensional Hilbert space

$$[A, A^*] = \mathbb{I} \qquad \textbf{and} \qquad N \equiv A^* A$$

With a density matrix $\rho \equiv \frac{1}{Z} e^{-\beta N}$ define

$$\omega(f) \equiv Tr(\rho f)$$

For $V \leq C N^{-1}$ define

$$\rho \equiv \frac{1}{Z} e^{-\beta(N+V)}$$

Thank you for your attention !

—— Dziekuje za uwage ! ——

—— Merci ! ——

—— Grazie Mille !!! ——

—— Muchas gracias ! ——

—— Danke schoen! ——

——Shukran! ——

—— XièXie !! ——

THE END

KONIEC
