A W^*-correspondence approach to multivariable Schur classes

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The classical Schur class

Schur class

Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces. The operator-valued Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$ can be defined as the closed unit ball of $H^\infty(\mathcal{U},\mathcal{Y})$ over the open unit disk $\mathbb{D}$: functions $F:\mathbb{D}\to\mathcal{B}(\mathcal{U},\mathcal{Y})$ analytic on $\mathbb{D}$ with $\|F\|_\infty=\sup_{z\in\mathbb{D}}\|F(z)\|\leq 1$.

Transfer function realization

Schur class functions appear as transfer functions of dissipative systems:

$$\Sigma:\left\{\begin{array}{rl} x(n+1) &= Ax(n)+Bu(n)\\ y(n) &= Cx(n)+Du(n)\end{array}\right. \qquad (n\in\mathbb{N})$$

with contractive system matrix

$$\begin{bmatrix} A & B\\ C & D\end{bmatrix}:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}\to\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$$

and transfer function $F_\Sigma\in\mathcal{S}(\mathcal{U},\mathcal{Y})$:

$$F_\Sigma(z)=D+zC(I-zA)^{-1}B \quad (z\in\mathbb{D}).$$

Discrete Lax-Phillips scattering

Discrete-time Lax-Phillips scattering system

U on $\mathcal{K}$ unitary, $\mathcal{G},\mathcal{G}_*\subset\mathcal{K}$, $\mathcal{G}\perp\mathcal{G}_*$, such that

- $U\mathcal{G}\subset\mathcal{G}$, $\cap_{n=0}^\infty U^n\mathcal{G}=\{0\}$;
- $U^*\mathcal{G}_*\subset\mathcal{G}_*$, $\cap_{n=0}^\infty U^{*n}\mathcal{G}_*=\{0\}$;

Then $\mathcal{E}:=\mathcal{G}\ominus U\mathcal{G}$ and $\mathcal{E}_*:=U\mathcal{G}_*\ominus\mathcal{G}_*$ are wandering for U:

$$\mathcal{K}=\mathcal{G}_*\oplus\mathcal{H}\oplus\mathcal{G}=\bigoplus_{n=-\infty}^{-1}U^n\mathcal{E}_*\oplus\mathcal{H}\oplus\bigoplus_{n=0}^{\infty}U^n\mathcal{E}$$

The scattering operator $S\in\mathcal{B}(\ell^2(\mathcal{E}),\ell^2(\mathcal{E}_*))$ given by $S=\Phi_*\Phi^*$, with

$$\Phi:k\in\mathcal{K}\mapsto(P_\mathcal{E}U^{*n}k)_{n=-\infty}^\infty\in\ell^2(\mathcal{E}),\quad \Phi_*:k\in\mathcal{K}\mapsto(P_{\mathcal{E}_*}U^{*n}k)_{n=-\infty}^\infty\in\ell^2(\mathcal{E}_*),$$

is a contractive analytic Laurent operator, with symbol $F\in\mathcal{S}(\mathcal{E},\mathcal{E}_*)$ given by

$$F(z)=D+zC(I_\mathcal{H}-zA)^{-1}B,\quad \begin{bmatrix} A & B\\ C & D\end{bmatrix}=\begin{bmatrix} P_\mathcal{H}U\\ P_\mathcal{E}\end{bmatrix}\begin{bmatrix} P_\mathcal{H} & P_{\mathcal{E}_*}\end{bmatrix}$$

Characterizations of $\mathcal{S}(\mathcal{U},\mathcal{Y})$

(1) Unit ball of $H^\infty(\mathcal{U},\mathcal{Y})$: F analytic on $\mathbb{F}$ and $\|F\|_\infty\leq 1$;

(2) Contractive multiplier: The multiplication operator

$$(M_Fg)(\lambda)=F(\lambda)g(\lambda)$$

defines an operator $M_F\in\mathcal{B}(H^2_\mathcal{U},H^2_\mathcal{Y})$ with $\|M_F\|\leq 1$.

(3) von Neumann inequality for $\mathbb{D}$: F analytic and

$$\|F(T)\|\leq 1 \quad (T\in\mathcal{B}(\mathcal{H}),\ \|T\|<1)$$

(4) Positive kernel characterization: The de Branges-Rovnyak kernel

$$K_F:\mathbb{D}\times\mathbb{D}\to\mathcal{B}(\mathcal{Y}),\quad K_F(z,w)=\frac{I_\mathcal{Y}-F(z)F(w)^*}{1-z\overline{w}}$$

is a positive kernel: $[F(z_i,z_j)]_{i,j=0}^N\geq 0$ for any $z_0,\ldots,z_N$.

(5) Transfer function realization: There exists a unitary colligation

$$U=\begin{bmatrix} A & B\\ C & D\end{bmatrix}:\begin{bmatrix}\mathcal{X}\\ \mathcal{U}\end{bmatrix}\to\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}$$

such that $F(z)=D+zC(I-zA)^{-1}B$.

Sketch of proof I: Easy direction (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1)

(5) Transfer function $\Rightarrow$ (4) positive kernel

Since $F(z) = D + zC(I - zA)B$ with $U = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ unitary we can compute:

$$I - F(z)F(w)^* = (1 - z\overline{w})C(I - zA)^{-1}(I - wA)^{-*}C^*$$

Hence K_F factors as $K_F(z, w) = H(z)H(w)^*$ with $H(z) = C(I - zA)^{-1}$, making it a positive kernel.

(4) positive kernel $\Rightarrow$ (3) operator points

Via a factorization $K_F(z, w) = H(z)H(w)^*$ we find

$$I - F(z)F(w)^* = H(z)(1 - z\overline{w})H(w)^*,$$

which yields for any strict contraction T:

$$I - F(T)F(T)^* = H(T)(1 - TT^*)H(T)^* \geq 0.$$

Hence $\|F(T)\| \leq 1$.

Sketch of proof I: Easy direction (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1)

(3) operator points $\Rightarrow$ (2) multiplication operator on H^2

Let $S = M_z : H^2 \to H^2$ be the forward shift operator. Then for $r \in (0, 1)$:

$$\|rS\| = r < 1 \quad \text{and} \quad F(rS) \to M_F \text{ strongly as } r \to 1.$$

Thus

$$1 \geq \|F(rS)\| \to \|M_F\| \quad \text{as } r \to 1.$$

(2) multiplication operator on H^2 $\Rightarrow$ (1) unit ball H^∞

Let k_w be de reproducing kernel elements of H^2: $k_w(z) = \frac{1}{1 - z\overline{w}}$. Then

$$M_F^*(k_w \otimes y) = k_w \otimes (F(w)^* y) \quad (w \in \mathbb{D},\ y \in \mathcal{Y}).$$

Since $\|M_F\| \leq 1$, we have

$$\|k_w \otimes (F(w)^* y)\| \leq \|k_w \otimes y\|, \quad \text{hence} \quad \|F(w)^* y\| \leq \|y\| \quad (y \in \mathcal{Y}).$$

Sketch of proof II: Harder direction (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5)

(1) unit ball H^∞ $\Rightarrow$ (2) multiplication operator on H^2

View $H^2 \subset L^2$ and $H^\infty \subset L^\infty$. Then $F \in H^\infty \subset L^\infty$ gives

$$L_F : L^2_{\mathcal{U}} \to L^2_{\mathcal{Y}}, \quad (L_F g)(e^{it}) = F(e^{it})g(e^{it})$$

has $\|L_F\| = \|F\|_\infty \leq 1$.

$$F \text{ analytic: } L_F : H^2_{\mathcal{U}} \to H^2_{\mathcal{Y}} \text{ and } M_F = L_F|_{H^2_{\mathcal{U}}}.$$

So $\|M_F\| \leq \|L_F\| \leq 1$.

(2) multiplication operator $\Rightarrow$ (4) positive kernel

Again use $M_F^*(k_w \otimes y) = k_w \otimes (F(w)^* y)$ and $\langle k_w \otimes y, k_{w'} \otimes y' \rangle = \frac{1}{1 - w'\overline{w}}$:

$$\begin{aligned} &\langle (I - M_F M_F^*)(k_w \otimes y), (k_z \otimes y') \rangle \\ &\quad = \langle k_w \otimes y, k_z \otimes y' \rangle - \langle k_w \otimes (F(w)^* y), k_z \otimes (F(z)^* y') \rangle \\ &\quad = \langle K_F(z, w) y, y' \rangle. \end{aligned}$$

$\|M_F\| \leq 0$ gives $I - M_F M_F^* \geq 0$ and, by linearity, K_F factors as

$$K_F(z, w) = H(z)H(w)^* \text{ with } H(z) = (k_z \otimes I_{\mathcal{Y}})^*(I - M_F M_F^*)^{\frac{1}{2}}.$$

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) via SzNF dilation, GNS construction and HB separation.

Sketch of proof II: Harder direction (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5)

(4) positive kernel $\Rightarrow$ (5) Transfer function

Option (1): Construct the unitary *canonical model colligation* via the de Branges-Rovnyak reproducing kernel Hilbert space associated with K_F.

Option (2): *Lurking isometry argument.* Via factorization $K_F(z, w) = H(z)H(w)^*$ we find

$$I - F(z)F(w)^* = (1 - z\overline{w})H(z)H(w)^*.$$

Reorder: $I + H(z)H(w)^* = F(z)F(w)^* + z\overline{w}H(z)H(w)^*$ and define a partial isometry

$$V \begin{bmatrix} H(w)^* y \\ y \end{bmatrix} = \begin{bmatrix} \overline{w}H(w)^* y \\ F(w)^* y \end{bmatrix} \qquad (w \in \mathbb{D},\ y \in \mathcal{Y}).$$

Extend V^* to a unitary colligation $U = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ s.t. U^* satisfies the same identity. Solve for H and F:

$$H(z) = C(I - zA)^{-1}, \quad F(z) = D + zC(I - zA)^{-1}B.$$

Multivariable generalizations I

Drury-Arveson space (Drury, Arveson, ...)

Variation on H^2 given by the RKHS $\mathcal{H}(K_d)$ of the Szegö kernel

$$K_d(z,w) = \frac{1}{1-\langle z, w\rangle} \quad (z, w \in \mathbb{B}_d = \{z \in \mathbb{C}^d \colon \|z\| < 1\})$$

Schur class functions: $F : \mathbb{B}_d \to \mathcal{B}(\mathcal{U}, \mathcal{Y})$ analytic such that

$$M_F : \mathcal{H}(K_d) \otimes \mathcal{U} \to \mathcal{H}(K_d) \otimes \mathcal{Y} \text{ contractively.}$$

What remains (for $d > 2$):

$$(1) \Longleftarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5)$$

Comments

- (1) $\not\Rightarrow$ (2): $\|F(z)\| \leq 1$, $z \in \mathbb{B}_d$ not enough; $d = 2$ Ando's dilation theorem
- (3) evaluation in commutative row contractions.
- (4) $\Rightarrow$ (5): Canonical model not unique; via solutions of Gleason problem.
- (5) Transfer function form

$$F(z) = D + C(I - Z(z)A)Z(z)B$$

with $Z(z) = \begin{bmatrix} z_1 I_{\mathcal{X}} & \cdots & z_d I_{\mathcal{X}} \end{bmatrix}$, $A : \mathcal{X} \to \mathcal{X}^d$, $B : \mathcal{U} \to \mathcal{X}^d$.

Multivariable generalizations II

Free semigroup algebras (Popescu, Davidson, Kribs, Pitts, ...)

Functions: formal power series in d noncommutative indeterminates; powers indexed by the free semigroup $\mathcal{F}_d$ in d letters $\{1, \ldots, d\}$. The Hardy space

$$H^2(\mathcal{F}_d) = \{f(z) = \sum_{\alpha \in \mathcal{F}_d} f_\alpha z^\alpha \colon \sum_{\alpha \in \mathcal{F}_d} |f_\alpha|^2 < \infty\},$$

is a NCFRKHS (noncommutative formal RKHS) with Szegö kernel

$$K_{d,nc}(z,w) = \sum_{\alpha \in \mathcal{F}_d} z^\alpha w^{\alpha^T}.$$

Schur class: Formal powers series F with $\mathcal{B}(\mathcal{U}, \mathcal{Y})$ coefficients that define contractive multipliers $M_F : H^2(\mathcal{F}_d) \otimes \mathcal{U} \to H^2(\mathcal{F}_d) \otimes \mathcal{Y}$. Then ($d > 2$):

$$(2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5)$$

Comments

- (3) evaluation in noncommutative row contractions.
- (5) Transfer function same form but with NC indeterminates.
- Drury-Arveson setting reappears when restricting to commutative row-contractions (abelianization).

W^*-correspondence approach to H^∞ (Muhly-Solel)

W^* correspondence

A W^*-correspondence w.r.t. a pair $(\mathfrak{A}, \mathfrak{B})$ of von Neumann algebras is a bimodule E with left $\mathfrak{A}$-action and right $\mathfrak{B}$-action with $\mathfrak{B}$-valued inner product $\langle\ ,\ \rangle : E \times E \to \mathfrak{B}$ satisfying for $\lambda \in \mathbb{C}$, $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, $\eta, \eta', \eta'' \in E$:

- $\langle \lambda\eta + \eta', \eta''\rangle = \lambda\langle\eta, \eta''\rangle + \mu\langle\eta', \eta''\rangle$;
- $\langle \eta \cdot b, \eta'\rangle = \langle\eta, \eta'\rangle b, \qquad \langle a \cdot \eta, \eta'\rangle = \langle\eta, a^* \cdot \eta'\rangle$;
- $\langle\eta', \eta\rangle^* = \langle\eta, \eta'\rangle$;
- $\langle\eta, \eta\rangle_E \geq 0$; (with equality iff $\eta = 0$)

such that E is a Banach space with respect to the norm $\|\eta\|_E := \|\langle\eta, \eta\rangle_E\|_{\mathfrak{A}}^{\frac{1}{2}}$, and E self-dual:

$$T \in \mathcal{B}^a(E, \mathfrak{B}) \quad \Longleftrightarrow \quad T\eta = \langle\eta, \eta_T\rangle \text{ for some } \eta_T \in E.$$

Correspondence-representation pair and the Fock space

A correspondence-representation pair (E, σ) consists of a W^*-correspondence E w.r.t. $(\mathfrak{A}, \mathfrak{A})$ and an faithful $*$-representation $\sigma : \mathfrak{A} \to \mathcal{B}(\mathcal{V})$.

$$\mathcal{F}^2(E) = \bigoplus_{n=0}^{\infty} E^{\otimes n} \quad \text{and} \quad \mathcal{F}^2(E, \sigma) = \mathcal{F}^2(E) \otimes_\sigma \mathcal{V}.$$

N.B. $\mathcal{F}^2(E)$ and $\mathcal{F}^2(E, \sigma)$ are W^*-corresp. w.r.t. $(\mathfrak{A}, \mathfrak{A})$, resp. $(\mathfrak{A}, \mathbb{C})$.

W^*-correspondence approach to H^∞ (Muhly-Solel)

Definition of $\mathcal{F}^\infty(E)$

Let the left $\mathfrak{A}$-action on $\mathcal{F}^2(E)$ be given by a normal $*$-rep. $\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{F}^2(E))$ and for $\eta \in E$ define the creation operator

$$C_\eta \in \mathcal{B}(\mathcal{F}^2(E)), \quad C_\eta\left(\bigoplus_{n=0}^{\infty} \xi^{(n)}\right) = 0 \oplus \bigoplus_{n=0}^{\infty} (\eta \otimes \xi^{(n)}) \quad (\xi^{(n)} \in E^{\otimes n}).$$

Then $\mathcal{F}^\infty(E)$ is the ultra-weak closure of the algebra generated by $\varphi(a)$, $a \in \mathfrak{A}$, and C_η, $\eta \in E$. Also $\mathcal{T}_+(E)$ is the norm closure (NC disc algebra).

Point-evaluation maps

A linear, completely contractive bimodule map $T : E \to \mathcal{B}(\mathcal{V})$ generates a completely contractive representation $\rho = \rho_T$ of $\mathcal{T}_+(E)$ on $\mathcal{B}(\mathcal{V})$ via

$$\rho(\varphi(a)) = \sigma(a), \quad \rho(C_\eta) = T(\eta) \qquad \text{(and all are obtained in this way)}$$

which may or may not extend to $\mathcal{F}^\infty(E)$.
Also, T induces a contractive bi-module maps from $\zeta_T : E \otimes \mathcal{V} \to \mathcal{V}$ via

$$\zeta_T(\eta \otimes v) = T(\eta)v \qquad \text{(and all are obtained in this way),}$$

and $\rho = \rho_T$ extends to $\mathcal{F}^\infty(E)$ at least whenever $\|\zeta_T\| < 1$.

Intertwining characterization

Dual correspondence-representation pair

Let $\iota : \sigma(\mathfrak{A})' \to \mathcal{B}(\mathcal{V})$ the embedding $*$-representation and

$$E^\sigma = \{\mu : \mathcal{V} \to E \otimes \mathcal{V} \colon \mu \text{ a bi-module map w.r.t. } \mathfrak{A}\}.$$

Then E^σ is a W^*-correspondence w.r.t. $(\sigma(\mathfrak{A})', \sigma(\mathfrak{A})')$:

$$\langle \mu, \mu' \rangle = \mu'^* \mu, \quad b \cdot \mu \cdot b' = (I_E \otimes b)\mu b'$$

and (E^σ, ι) is a CR pair which is dual to the CR pair (E, σ).

Intertwining characterization

For $\mu \in E^\sigma$, define a dual creation operator $\widehat{C}_\mu$ on $\mathcal{F}^2(E, \sigma)$ by

$$\widehat{C}_\mu(\oplus_{n=0}^\infty \xi^{(n)}) = 0 \oplus \bigoplus_{n=0}^{\infty} (\widehat{C}_\mu^{(n)} \xi^{(n)}), \quad \widehat{C}_\mu^{(n)}(\eta_n \otimes \cdots \otimes \eta_1 \otimes v) = \eta_n \otimes \cdots \otimes \eta_1 \otimes \mu v.$$

Then $R \in \mathcal{B}(\mathcal{F}^2(E, \sigma))$ is in $\mathcal{F}^\infty(E) \otimes I_{\mathcal{V}}$ if and only if it commutes with

$$I_{\mathcal{F}^2(E)} \otimes b, \; b \in \sigma(\mathfrak{A})', \quad \widehat{C}_\mu, \; \mu \in E^\sigma.$$

Toeplitz structure

Toeplitz structure characterization

Let $R = [R_{i,j}]_{i,j=0}^\infty \in \mathcal{B}(\mathcal{F}^2(E, \sigma))$. Then $R \in \mathcal{F}^\infty(E) \otimes I_{\mathcal{V}}$ iff

- R lower triangular: $R_{i,j} = 0$ if $i < j$;
- Local intertwining structure: For all $b \in \sigma(\mathfrak{A})'$, $\mu \in E^\sigma$:

$$R_{i,j}(I_{E^j} \otimes b) = (I_{E^i} \otimes b) R_{i,j}, \quad R_{i+1,j+1}(I_{E^j} \otimes \mu) = (I_{E^i} \otimes \mu) R_{i,j}.$$

- Toeplitz structure: There exist $\xi^{(n)} \in E^{\otimes n}$, $n = 0, 1, \ldots$, s.t. for $i \geq j$:

$$R_{i,j} : E^{\otimes j} \otimes \mathcal{V} \to E^{\otimes i} \otimes \mathcal{V}, \quad R_{i,j} : \xi \mapsto \xi^{(i-j)} \otimes \xi.$$

- Conversely, if $\xi^{(n)} \in E^{\otimes n}$, $n = 0, 1, \ldots$, are such that R defined as above is bounded on $\mathcal{F}^2(E, \sigma)$, then $R \in \mathcal{F}^\infty(E) \otimes I_{\mathcal{V}}$.

Point evaluation revisited

Set $\mathbb{D}((E^\sigma)^*) = \{\zeta \colon \zeta^* \in E^\sigma, \; \|\zeta\| < 1\}$. For $\zeta \in \mathbb{D}((E^\sigma)^*)$ define

$$\zeta^n = \zeta(I_E \otimes \zeta) \cdots (I_{E^{\otimes n-1}} \otimes \zeta) : E^{\otimes n} \otimes \mathcal{V} \to \mathcal{V}.$$

For $R \in \mathcal{F}^\infty(E)$ we define $\widehat{R} : \mathbb{D}((E^\sigma)^*) \to \mathcal{B}(\mathcal{V})$ via

$$\widehat{R}(\zeta)v = \rho_\zeta(T)v = \sum_{n=0}^{\infty} \eta^n(\zeta^{(n)} \otimes v); \quad \text{and set } H^\infty(E, \sigma) = \{\widehat{R} \colon R \in \mathcal{F}^\infty(E)\}.$$

Reproducing kernel correspondences I

Reproducing kernel correspondences

A reproducing kernel W^*-correspondences (RKW*C) on a set Ω w.r.t. $(\mathfrak{A}, \mathfrak{B})$ is a W^*-correspondence G w.r.t. $(\mathfrak{A}, \mathfrak{B})$ s.t. $f \in G$ is a functions $f : \Omega \times \mathfrak{A} \to \mathfrak{B}$ and there exist $k_w \in G$, $w \in \Omega$ s.t.

$$\langle a \cdot f, k_w \rangle_E = f(w, a).$$

The reproducing kernel is the map $K : \Omega \times \Omega \to \mathcal{B}(\mathfrak{A}, \mathfrak{B})$ given by

$$K(w, w')[a] = k_{w'}(w, a).$$

For $K : \Omega \times \Omega \to \mathcal{B}(\mathfrak{A}, \mathfrak{B})$ TFAE ([BBLS04] (2) $\Leftrightarrow$ (3); [BBFtH09]; [Marx17]):

(1) K is the reproducing kernel for a RKW*C on Ω w.r.t. $(\mathfrak{A}, \mathfrak{B})$.

(2) K is a completely positive kernel: For all $w_i \in \Omega$, $a_i \in \mathfrak{A}$, $b_i \in \mathfrak{B}$:

$$\sum_{i,j=0}^{n} b_i^* K(w_i, w_j)[a_i^* a_j] b_j \geq 0$$

and $K(w, w') \in \mathcal{B}(\mathfrak{A}, \mathfrak{B})$ is weak-$*$ continuous for all $w, w' \in \Omega$.

(3) K has a Kolmogorov decomposition: $\exists$ W^*-corresp. G w.r.t. $(\mathfrak{A}, \mathfrak{B})$ and $k_w \in G$, $w \in \Omega$, s.t. $K(w', w)[a] = \langle a k_w, k_{w'} \rangle$.

Reproducing kernel correspondences II: $\mathfrak{B} = \mathcal{B}(\mathcal{E})$

Theorem

For $K : \Omega \times \Omega \to \mathcal{B}(\mathfrak{A}, \mathcal{B}(\mathcal{E}))$ TFAE:

(1) K is the reproducing kernel for a (RKW*C) on Ω w.r.t. $(\mathfrak{A}, \mathfrak{B})$.

(2) K is a completely positive kernel: For all $w_i \in \Omega$, $a_i \in \mathfrak{A}$, $e_i \in \mathcal{E}$:

$$\sum_{i,j=0}^{n} \langle K(w_i, w_j)[a_i^* a_j] e_j, e_i \rangle \geq 0$$

(3) K has a Kolmogorov decomposition: $\exists$ W^*-corresp. $\mathcal{H}$ w.r.t. $(\mathfrak{A}, \mathbb{C})$ and $H : \Omega \to \mathcal{B}(\mathcal{H}, \mathcal{E})$ s.t.

$$K(w', w)[a] = H(w) a H(w')^*$$

The Hardy space $H^2(E, \sigma)$

For $f = (\xi_n)_{n=0}^\infty \in \mathcal{F}^2(E, \sigma)$ we define

$$\widehat{f} : \mathbb{D}((E^\sigma)^*) \times \sigma(\mathfrak{A})' \to \mathcal{V}, \quad \widehat{f}(\zeta, b) = \sum_{n=0}^{\infty} \zeta^n (I_{E^{\otimes n}} \otimes b) \xi_n$$

Then $H^2(E, \sigma) = \{\widehat{f} \colon f \in \mathcal{F}^2(E, \sigma)\}$ is a RKW*C with rep. kernel

$$K_{E,\sigma} : \mathbb{D}((E^\sigma)^*) \times \mathbb{D}((E^\sigma)^*) \to \mathcal{B}(\sigma(\mathfrak{A})', \mathcal{B}(\mathcal{V})), \quad K_{E,\sigma}(\zeta, \zeta')[b] = \sum_{n=0}^{\infty} \zeta^n (I_{E^{\otimes n}} \otimes b) \zeta'^{*n}$$

Main theorem

Theorem ([Muhly-Solel '08, Ball-Biswas-Fang-tH '09])
Let (E,σ) be a CR pair. For a function $F:\mathbb{D}((E^\sigma)^*)\to\mathcal{B}(\mathcal{V})$ TFAE

(0) $F=\widehat{R}$ for an $R\in\mathcal{F}^\infty(E)$ with $\|R\|\leq 1$;

(2) F defines a contractive multiplication operator on $H^2(E,\sigma)$ via
$$(M_F g)(\zeta,b)=F(\zeta)g(\zeta,b)\quad(h\in H^2(E,\sigma));$$

(3) $F=\widehat{R}$ for an $R\in\mathcal{F}^\infty(E)$ and for any injective $*$-representation $\sigma':\mathfrak{A}\to\mathcal{B}(\mathcal{V}')$ and $\zeta'\in\mathbb{D}((E^{\sigma'})^*)$ we have $\|\widehat{R}(\zeta')\|\leq 1$;

(4) The function $K:\mathbb{D}((E^\sigma)^*)\times\mathbb{D}((E^\sigma)^*)\to\mathcal{B}(\sigma(\mathcal{A})',\mathcal{B}(\mathcal{V}))$
$$K(\zeta,\zeta')[b]=K_{E,\sigma}(\zeta,\zeta')[b]-F(\zeta)K_{E,\sigma}(\zeta,\zeta')[b]F(\zeta')^*$$
is a completely positive kernel;

(5) $\exists$ W^*-corresp. $\mathcal{H}$ w.r.t. $(\sigma(\mathfrak{A})',\mathbb{C})$ and a co-isometric $\sigma(\mathfrak{A})'$-module map
$$\begin{bmatrix} A & B\\ C & D\end{bmatrix}:\begin{bmatrix}\mathcal{H}\\ \mathcal{V}\end{bmatrix}\to\begin{bmatrix}E^\sigma\otimes\mathcal{H}\\ \mathcal{V}\end{bmatrix}$$
so that, with $L_\zeta:E^\sigma\otimes\mathcal{H}\to\mathcal{H}$, $L_\zeta:\mu\otimes h\to\langle\mu,\zeta^*\rangle h$, we have
$$F(\zeta)=D+C(I-L_\zeta A)^{-1}L_\zeta B.$$

Comments

- Free semigroup case: $\mathfrak{A}=\mathbb{C}$, $E=\mathbb{C}^d$, $\Sigma:\lambda\mapsto\lambda I_{\mathcal{H}}$.
  Then $\sigma(\mathfrak{A})'=\mathcal{B}(\mathcal{H})$, $\mathbb{D}((E^\sigma)^*)=$ NC strict row contractions.
- Other examples: Semigroupoid (graph) algebras (Kribs-Power '04), analytic crossed products (Muhly-Solel '98).
- Completely positive kernel: In many examples positive kernel; Choi's theorem.
- Not currently covered: Schur-Agler class over the polydisk $\mathbb{D}^d$.
  No short cut, have to do (2) $\Rightarrow$ (3) $\Rightarrow$ (4) via variations on SzNF dilation, GNS construction and HB separation.
- Current theme: NC function theory (Vinnikov-Kaliuzhnyi-Verbovetskyi '14, et al)– Also Muhly-Solel ('12)

THANK YOU FOR YOUR ATTENTION