Translation invariant Dirichlet forms in the context of locally compact quantum groups

based on joint work with A. Viselter

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General aim

Our aim will be a study of a certain class of (noncommutative) probabilistic evolutions, understood as Markov semigroups on von Neumann algebras and associated L^p-spaces associated to locally compact quantum groups and describe their generators via quantum Dirichlet forms with specific invariance properties.

Classical Markov semigroups and Dirichlet forms

(X, μ) – classical measure space

Definition

A Markov semigroup $(P_t)_{t\geq 0}$ on (X, μ) is a family of operators acting on the von Neumann algebra $\mathrm{L}^\infty(X, \mu)$ and satisfying the following conditions:

- $P_0 = I,\ P_{t+s} = P_t \circ P_s, \quad s, t \geq 0$;
- $w^* - \lim_{t\to 0^+} P_t(f) = f, \quad f \in \mathrm{L}^\infty(X, \mu)$;
- $\forall_{t\geq 0}\ \ P_t$ is a contractive positive operator and $\mu \circ P_t \leq \mu$.

Markov semigroups as above yield contractive C_0-semigroups on all spaces $\mathrm{L}^p(X, \mu)$, $p \in [1, \infty)$; we will call the semigroup symmetric if the corresponding operators on $\mathrm{L}^2(X, \mu)$ consists of self-adjoint operators.

Classical Markov semigroups – continued

Markov semigroups can be studied via their $\mathrm{L}^p(X,\mu)$-generators; naturally the easiest case is that of $\mathrm{L}^2(X,\mu)$. Assume that $(P_t)_{t\geq 0}$ is a symmetric Markov semigroup and consider

$$Q(f) = \lim_{t\to 0^+} \frac{1}{t}\int_X \bar{f}(f - P_t f)d\mu, \quad f \in \mathrm{Dom}(Q) \subset \mathcal{L}^2(X,\mu)$$

This is a Dirichlet form: i.e. a densely defined quadratic form, which is closed, real and if we denote by $P_\wedge$ the projection from $L^2(X,\mu)_\mathbb{R}$ onto $\{f \in L^2(X,\mu)_\mathbb{R} : 0 \leq f \leq 1\}$ then

$$Q(P_\wedge f) \leq Q(f), \quad f \in \mathrm{Dom}(Q)_\mathbb{R}.$$

Theorem (Beurling-Deny)

There is a 1-1 correspondence between symmetric Markov semigroups on $\mathrm{L}^\infty(X,\mu)$ and Dirichlet forms on $\mathrm{L}^2(X,\mu)$.

Convolution semigroups of measures and Lévy processes

G – locally compact group

A family $(\mu_t)_{t\geq 0_+}$ of probability measures on G is called a convolution semigroup of measures if

1. $\mu_{t+s} = \mu_t \star \mu_s :, \quad t, s \geq 0;$
2. $\mu_t(f) \stackrel{t\to 0^+}{\longrightarrow} \mu_0(f) := f(e), \quad f \in \mathrm{C}_0(G).$

These are precisely distributions of Lévy processes (processes with independent, identically distributed increments). Further we can define

$$(P_t(f))(g) = \int_G f(gh) d_\mu(h), \quad f \in \mathrm{L}^\infty(G).$$

to get the Markov semigroup of the process. Corresponding Dirichlet forms are characterised by the translation invariance.

Quantum Markov semigroups

M - von Neumann algebra, with a fixed normal semifinite faithful weight ϕ

Definition

An operator $T : \mathsf{M} \to \mathsf{M}$ will be called Markov if it is a positive contraction and in addition we also have the condition

$$\phi \circ T \leq \phi.$$

Quantum Markov semigroups – L^p-versions

Tracial case – L^p-spaces are certain completions of M, with the $\|x\|_p := (\tau(|x|)^p)^{\frac{1}{p}}$.

Non-tracial state case – L^p-spaces are either interpolation spaces (which requires embedding M into $L^1(M) = M_*$) (Araki, Kosaki, Izumi) or certain concrete spaces of operators (Hilsum, Haagerup).

Non-tracial weight case:

$L^p(M, \phi)$ – Haagerup L^p-space. We consider symmetric embeddings $\iota_p : M^p \to L^p(M)$: these are informally defined as

$$\iota_p(x) = D^{\frac{1}{2p}} x D^{\frac{1}{2p}}, \quad x \in M^{(p)}.$$

Here $M^{(p)} \subset M$ is 'the set of p-integrable elements' and D can be thought of as 'the density matrix of the weight' – and formally is the unbounded generator of the implementing unitary group of the modular automorphism group of ϕ in the core of M.

Quantum Markov semigroups – L^p-versions continued

Given a map $T : M \to M$ we will say it is KMS-symmetric if its *KMS implementation*, the map $T^{(2)} : \iota_2(M^{(2)}) \to L^2(M, \phi)$

$$T^{(2)}(\iota_2(x)) = \iota_2(Tx)$$

extends to a *bounded self-adjoint* operator.
Note that we require in particular that $T^{(2)}(\iota_2(M^{(2)})) \subset \iota_2(M^{(2)})$.

Definition

A (quantum) KMS-symmetric Markov semigroup is a weak*-continuous semigroup $(T_t)_{t \geq 0^+}$ of normal KMS-symmetric Markov maps on M.

In fact *KMS*-symmetry plus being a positive contraction itself yields the weight inequality. We say that the family as above is completely Markov, if the same properties hold for $P_t \otimes \mathrm{id}_{M_n}$ for all $n \in \mathbb{N}$.

Quantum Dirichlet forms

Definition (Cipriani, Goldstein+Lindsay)

A quantum Dirichlet form for (M, ϕ) is a densely defined closed quadratic form on $\mathsf{L}^2(\mathsf{M})$ which is real (in a suitable sense) and satisfies the condition

$$Q(P_\wedge x) \leq Q(x), \;\; x \in \mathrm{Dom}(Q_\mathbb{R})$$

where $P_\wedge$ is the orthogonal projection from $\mathsf{L}^2(\mathsf{M})_\mathbb{R}$ onto the 'interval' $[0, D^{\frac{1}{2}}]$.

Theorem (Cipriani, Goldstein+Lindsay, Viselter+AS)

There is a 1-1 correspondence between completely Markov *KMS*-symmetric semigroups and quantum completely Dirichlet forms.

LCQGs

$\mathbb{G}$ – locally compact quantum group à la Kustermans-Vaes

$\mathrm{L}^\infty(\mathbb{G})$ – the von Neumann algebra, together with the *coproduct* (carrying all the information about $\mathbb{G}$)

$$\Delta : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G})\overline{\otimes}\mathrm{L}^\infty(\mathbb{G})$$

and a canonical *right Haar weight* ϕ

$\mathrm{C}_0(\mathbb{G})$ – the corresponding (reduced) C^*-object

$\mathrm{C}_0^u(\mathbb{G})$ – the universal version of $\mathrm{C}_0(\mathbb{G})$,

$\mathrm{L}^2(\mathbb{G})$ – the GNS Hilbert space of the right invariant Haar weight ϕ on $\mathrm{L}^\infty(\mathbb{G})$

$\mathrm{L}^1(\mathbb{G})$ – predual of $\mathrm{L}^\infty(\mathbb{G})$, with a natural Banach algebra structure.

$$\mathrm{C}_0(\mathbb{G}) \subset \mathrm{L}^\infty(\mathbb{G})$$
$$\mathrm{L}^2(\mathbb{G}) \approx \mathrm{L}^2(\mathrm{L}^\infty(\mathbb{G}), \phi)$$

Dual groups

Each LCQG $\mathbb{G}$ admits the dual LCQG $\widehat{\mathbb{G}}$.

$\mathrm{L}^\infty(\widehat{\mathbb{G}})$, $\mathrm{C}_0(\widehat{\mathbb{G}})$ – subalgebras of $B(\mathrm{L}^2(\mathbb{G}))$

In particular for G – locally compact group

$$\mathrm{L}^\infty(\widehat{G}) = \mathrm{VN}(G), \quad \mathrm{C}_0(\widehat{G}) = C_r^*(G), \quad \mathrm{C}_0^u(\widehat{G}) = C^*(G)$$

We sometimes write

$$\mathrm{L}^\infty(\widehat{\mathbb{G}}) = \mathrm{VN}(\mathbb{G}), \quad \mathrm{C}_0(\widehat{\mathbb{G}}) = C_r^*(\mathbb{G}), \quad \mathrm{C}_0^u(\widehat{\mathbb{G}}) = C^*(\mathbb{G})$$

Simplifications in the compact case

Definition

$\mathbb{G}$ is said to be compact if $\mathsf{C}_0(\mathbb{G})$ is unital (so written as $\mathsf{C}(\mathbb{G})$), equivalently, the weight ϕ is a state.

Any compact quantum group can be described purely algebraically via the Hopf *-algebra $\mathrm{Pol}(\mathbb{G}) \subset \mathsf{C}(\mathbb{G})$, with the counit ϵ.

Convolution semigroups of states on compact quantum groups

A family $(\mu_t)_{t\geq 0_+}$ of states on $\mathrm{Pol}(\mathbb{G})$ is called a convolution semigroup of states if

1. $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$
2. $\mu_t(a) \stackrel{t\to 0^+}{\longrightarrow} \mu_0(a) := \epsilon(a), \quad a \in \mathrm{Pol}(\mathbb{G}).$

Such convolution semigroups admit generating functionals:

$$L(a) = \lim_{t\to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in \mathrm{Pol}(\mathbb{G}).$$

We associate to it a convolution semigroup of operators $(R_{\mu_t})_{t\geq 0_+}$ on $\mathrm{Pol}(\mathbb{G})$:

$$R_{\mu_t} := (\mathrm{id} \otimes \mu_t) \circ \Delta$$

These extend to operators on $\mathrm{L}^\infty(\mathbb{G})$ which form a Markov semigroup. The corresponding Dirichlet forms contain $\mathrm{Pol}(\mathbb{G})$ in the domain and can be characterised/studied in the purely algebraic manner (see Cipriani, Franz, Kula).

Convolution semigroups of states revisited

$\mathbb{G}$ – locally compact quantum group

A family $(\mu_t)_{t\geq 0_+}$ of states on $\mathrm{C}_0^u(\mathbb{G})$ is called a convolution semigroup of states if

1. $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$
2. $\mu_t(a) \stackrel{t\to 0^+}{\longrightarrow} \mu_0(a) := \epsilon(a), \quad a \in \mathrm{C}_0^u(\mathbb{G}).$

We no longer have the 'algebraic domain' such as $\mathrm{Pol}(\mathbb{G})$. Generating functionals are densely defined, but that is all we know a priori.

Convolution semigroups of operators

The following is essentially a consequence of known results of the last 10 or so years, due to Daws, Junge, Neufang, Ruan and others.

Theorem

There exist $1-1$ correspondences between:

1. convolution semigroups $(\mu_t)_{t\geq 0}$ of states of $\mathrm{C}_0^u(\mathbb{G})$;
2. C_0-semigroups $(T_t^{\mathrm{u}})_{t\geq 0}$ of completely positive maps of norm 1 on $\mathrm{C}_0^u(\mathbb{G})$ that commute with the *left translation operators*;
3. semigroups $(T_t)_{t\geq 0}$ of normal, unital, completely positive maps on $\mathrm{L}^\infty(\mathbb{G})$ that are point–ultraweakly continuous at 0^+, and that satisfy $\Delta \circ T_t = (T_t \otimes \mathrm{id}) \circ \Delta$ for every $t \geq 0$;
4. C_0-semigroups $(M_t)_{t\geq 0}$ of norm 1 left module maps on $\mathrm{L}^1(\mathbb{G})$ with completely positive adjoints.

Convolution operators – revisited once again

$C_0^u(\mathbb{G})$ admits a canonical involutive operator R^u, so called universal unitary antipode (playing the role of the inverse operation).

Theorem

Let $\mu \in S(C_0^u(\mathbb{G}))$. The operator $R_\mu : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is unital, completely positive, ϕ-preserving. The map R_μ is KMS-symmetric iff $\mu = \mu \circ R^u$. Its KMS implementation (acting on $L^2(\mathbb{G})$) is always bounded and belongs to $L^\infty(\hat{\mathbb{G}})$.

Main result

We can now add the Dirichlet form part.

Theorem

Let $\mathbb{G}$ be a locally compact quantum group. There exist $1-1$ correspondences between:

1. w^*-continuous convolution semigroups $(\mu_t)_{t\geq 0}$ of $\mathrm{R^u}$-invariant states of $\mathsf{C}_0^u(\mathbb{G})$;
2. C_0^*-semigroups $(T_t)_{t\geq 0}$ of normal, unital, completely positive maps on $\mathsf{L}^\infty(\mathbb{G})$ that are KMS-symmetric with respect to ϕ and satisfy $\Delta \circ T_t = (T_t \otimes \mathrm{id}) \circ \Delta$ for every $t \geq 0$;
3. completely Dirichlet forms Q on $\mathsf{L}^2(\mathbb{G})$ with respect to ϕ that are invariant under $\mathcal{U}(\mathsf{L}^\infty(\hat{\mathbb{G}})')$ (modulo multiplication of forms by a positive number).

Applications

Theorem

Let $\mathbb{G}$ be a second countable locally compact quantum group. Then $\hat{\mathbb{G}}$ has Property (T) of Kazhdan if and only if every convolution semigroup of R^u-invariant states on $\mathrm{C}_0^u(\mathbb{G})$ has a bounded generator.

Theorem

Let $\mathbb{G}$ be a second countable locally compact quantum group. Then $\hat{\mathbb{G}}$ has the Hagerup property if and only if there exists a convolution semigroup of R^u-invariant states on $\mathrm{C}_0^u(\mathbb{G})$ such that the L^2-implementations of the associated convolution operators, acting on $\mathrm{L}^2(\mathbb{G})$, in fact belong to $\mathrm{C}_0(\hat{\mathbb{G}})$.

Examples

- Commutative case (G-classical): convolution semigroups on $\mathrm{L}^\infty(G)$ correspond to Lévy processes on G, are described via the Lévy-Khintchine formula.
- Dual case (G-classical, $\mathrm{L}^\infty(\hat{G}) = \mathrm{VN}(G)$): convolution semigroups are of the form

$$P_t(\lambda_g) = e^{t\psi(g)}\lambda_g, \;\; g \in G$$

where $\psi : G \to \mathbb{R}$ is a conditionally positive-definite function. Corresponding Dirichlet form on $\mathrm{L}^2(\hat{G}) = \mathrm{L}^2(G)$ as expected is equal to

$$Q(f) = \int_G |f(g)|^2 \psi(g) dg, \quad f \in \mathrm{Dom}(Q).$$

Examples continued – cocycle twists

Let $\mathbb{G}$ – locally compact quantum group, $\Omega \in \mathsf{L}^\infty(\mathbb{G})\overline{\otimes}\mathsf{L}^\infty(\mathbb{G})$ be a unitary 2-cocycle on $\mathbb{G}$. Then (by a result of De Commer) we can define $\mathbb{G}_\Omega$ via

$$\mathsf{L}^\infty(\mathbb{G}_\Omega) := \mathsf{L}^\infty(\mathbb{G}),$$

$$\Delta_\Omega(m) = \Omega^*\Delta(m)\Omega, \quad m \in \mathsf{L}^\infty(\mathbb{G}_\Omega).$$

Theorem

If $(T_t)_{t\geq 0^+}$ is a convolution semigroup of operators on $\mathsf{L}^\infty(\mathbb{G})$ in the sense studied earlier, and $(T_t \otimes \mathrm{id})(\Omega) = \Omega$ then $(T_t)_{t\geq 0^+}$ is also a convolution semigroup of operators on $\mathsf{L}^\infty(\mathbb{G}_\Omega)$.

Example of application: start from $\mathbb{G} = \hat{G}$, with G containing an abelian subgroup H admitting a non-trivial 2-cocycle and use a conditionally positive-definite function on G which vanishes on H.

Specifically: we can build interesting convolution semigroups on quantized Heisenberg groups or on quantized $SL_2(\mathbb{C})$.

Perspectives

- Cipriani and Sauvageot showed that in the tracial case all quantum Dirichlet forms (subject to technical conditions) arise canonically from certain derivations; this takes a simpler form in the case of convolution semigroups on compact quantum groups, and is related to Lévy-Khintchine decomposition. Is there such a result in the state/weight case?
- classical results of Hunt for Lie groups and then Heyer for general lc groups show that for each convolution semigroup say on $\mathrm{C}_0(G)$ the domain of its generator contains a canonical subalgebra. Can we have a result of this form for locally compact quantum groups?

References

Noncommutative Dirichlet forms:

F. Cipriani, Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras, JFA, 1997

S.Goldstein and M.Lindsay, Markov semigroups KMS-symmetric for a weight, Math. Ann., 1999.

Dirichlet forms for convolution semigroups of states – compact case

F. Cipriani, U. Franz and A. Kula, Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory, JFA, 2014

This talk:

A. Viselter and A.S., Translation invariant Dirichlet forms and locally compact quantum groups, in preparation