Quantum stochastic semigroups

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Outline

Research collaboration

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Quantum dynamical semigroups (QDS)

Setup 1/3

- $\mathfrak{h}$ a fixed Hilbert space.

Definition: Quantum dynamical semigroup on $B(\mathfrak{h})$

A QDS is a pointwise ultraweakly continuous semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of normal, completely positive, contractions on $B(\mathfrak{h})$; it is called *conservative* if it is unital, i.e. identity preserving.

Theorem (Lindblad, 1976; Gorini–Kossakowski–Sudarshan, 1976)

The norm-continuous QDS's are all semigroups $(e^{t\mathcal{L}})_{t\geq 0}$ *where* $\mathcal{L}$ *has the form*

$$x \mapsto x\,K + K^*\,x + L^*(x \otimes I_{\mathsf{k}})L, \text{ in which}$$

$$K \in B(\mathfrak{h}), L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k}) \text{ and } K + K^* + L^*L \leq 0,$$

for some Hilbert space k.

Minimal QDS's

Setup 2/3

- $\mathfrak{h}$ and k, two fixed Hilbert spaces
- $\mathfrak{X}_2(\mathfrak{h}, \mathsf{k})$ denotes the collection of pairs (K, L) such that K is the generator of a contractive C_0-semigroup on $\mathfrak{h}$; L is an operator from $\mathfrak{h}$ to $\mathfrak{h} \otimes \mathsf{k}$ with $\operatorname{Dom} L \supset \operatorname{Dom} K$; $\|Lv\|^2 + 2\operatorname{Re}\langle v, Kv\rangle \leq 0$ for all $v \in \operatorname{Dom} K$.
- *Associated quadratic forms*: for $x \in B(\mathfrak{h})$,

  $\mathcal{L}_{K,L}(x)[v] := \langle v, x\,Kv\rangle + \langle Kv, xv\rangle + \langle Lv, x \otimes I_{\mathsf{k}}\, Lv\rangle, \quad v \in \operatorname{Dom} K.$

Definition (A QDS $\mathcal{T}$ is a *minimal QDS for* $(K, L) \in \mathfrak{X}_2(\mathfrak{h}, \mathsf{k})$ if:)

- For all $x \in B(\mathfrak{h})$, $v \in \operatorname{Dom} K$ and $t \in \mathbb{R}_+$,

$$\langle v, \mathcal{T}_t(x)v\rangle = \langle v, xv\rangle + \int_0^t ds\, \mathcal{L}_{K,L}(\mathcal{T}_s(x))[v]. \qquad (\star)$$

- For any other QDS $\mathcal{T}'$ satisfying $(\star)$,

$$\mathcal{T}_t(x) \leq \mathcal{T}'_t(x), \qquad \text{for all } t \in \mathbb{R}_+, x \in B(\mathfrak{h})_+.$$

Existence of minimal QDS's

Theorem (Davies, 1977; after Kato, 1954 and Feller, 1940)

Let $(K, L) \in \mathfrak{X}_2(\mathfrak{h}, \mathsf{k})$*.*
Then there is a unique minimal QDS associated to (K, L)*.*

Notation: $\mathcal{T}^{K,L}$.

Remark

If $\mathcal{T}^{K,L}$ is conservative then $\mathcal{L}_{K,L}(1) = 0$, in other words

$$\|Lv\|^2 + 2\,\mathrm{Re}\langle v, Kv\rangle = 0, \quad v \in \mathrm{Dom}\, K.$$

Setup

Setup 3/3

- $\mathfrak{h}$ and k, two fixed Hilbert spaces
- $\mathcal{F} := \Gamma(L^2(\mathbb{R}_+; \mathsf{k}))$, where $\Gamma(\mathsf{H}) := \mathbb{C} \oplus \mathsf{H} \oplus \mathsf{H}^{\vee 2} \oplus \cdots$
- $\varpi(f) := \exp(-\|f\|^2/2)\varepsilon(f)$, $f \in L^2(\mathbb{R}_+; \mathsf{k})$, where $\varepsilon(f) := (1, f, (2!)^{-1/2} f^{\otimes 2}, \cdots)$
- $\Delta := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \in B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}))$, $\binom{z}{c} \mapsto \binom{0}{c}$

$\mathcal{F} = \mathcal{F}_{[0,r[} \otimes \mathcal{F}_{[r,t[} \otimes \mathcal{F}_{[t,\infty[}$, where $\mathcal{F}_{[r,t[} := \Gamma(L^2([r,t[; \mathsf{k}))$

Setup 3/3 contd.

On $B(\mathfrak{h} \otimes \mathcal{F}) = B(\mathfrak{h}) \overline{\otimes} B(\mathcal{F})$, the (ampliated) *CCR flow* $(\sigma_t)_{t \geq 0}$ and *conditional expectations* $(\mathbb{E}_t)_{t \geq 0}$, satisfy

$$\operatorname{Ran} \sigma_t = B(\mathfrak{h}) \overline{\otimes} (I_{[0,t[} \otimes B(\mathcal{F}_{[t,\infty[})), \text{ and}$$

$$\operatorname{Ran} \mathbb{E}_t = B(\mathfrak{h}) \overline{\otimes} B(\mathcal{F}_{[0,t[}) \otimes I_{[t,\infty[}.$$

Quantum stochastic (QS) contraction cocycles on $\mathfrak{h}$

Definition (Notation: $\mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$.)

$V = (V_t)_{t \geq 0}$ contractions in $B(\mathfrak{h} \otimes \mathcal{F})$ satisfying

- $V_{s+t} = V_s \sigma_s(V_t)$ and $V_0 = I$
- $V_t \in B(\mathfrak{h} \otimes \mathcal{F}_{[0,t[}) \otimes I_{[t,\infty[}$
- $t \mapsto V_t$ is strongly continuous

$\sigma_s(V_t) \in B(\mathfrak{h}) \otimes I_{[0,s[} \overline{\otimes} B(\mathcal{F}_{[s,s+t[}) \otimes I_{[s+t,\infty[}$

Expectation semigroup of V

$$P^{0,0} := (\mathbb{E}[V_t])_{t \geq 0} \quad \text{where } \mathbb{E} := \mathrm{id}_{B(\mathfrak{h})} \overline{\otimes} \omega_{\varepsilon(0)}.$$

$\mathbb{E} = \mathbb{E} \circ \mathbb{E}_s$ and $\mathbb{E}_s \circ \sigma_s = \iota_{B(\mathcal{F})} \circ \mathbb{E}$.

Examples

Example 0: $V := (P_t \otimes I_{\mathcal{F}})_{t\geq 0}$

where $P = (P_t)_{t\geq 0}$ is a contractive C_0-semigroup on $\mathfrak{h}$.

Example 1: $V := (e^{iH\otimes M_{B_t}})_{t\geq 0}$ [Case: $\dim \mathsf{k} = 1$]

where H is a selfadjoint operator on $\mathfrak{h}$,
$B = (B_t)_{t\geq 0}$ is a Brownian motion, and
$L^2(\mathcal{W}) \cong \mathcal{F}$ (Wiener-Segal-Itô isomorphism).

Example 2: *Weyl cocycles*, $W^c := \big(I_{\mathfrak{h}} \otimes W(c_{[0,t[})\big)_{t\geq 0}$ $(c \in \mathsf{k})$

where $W(f)$ is the (unitary) *Fock-Weyl operator* determined by

$$W(f)\varpi(g) = e^{-i\,\mathrm{Im}\langle f,g\rangle}\varpi(f+g), \quad g \in L^2(\mathbb{R}_+;\mathsf{k}).$$

$W(c_{[0,r+t[}) = W(c_{[0,r[})W(c_{[r,r+t[})$ & $W(c_{[r,r+t[}) = \sigma_r(W(c_{[0,t[}))$.

Constructions: Associated cocycles and dual cocycles

Let $V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$,

Definition (Associated cocycles)

$V^{c,d} \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$, where

$$V^{c,d} := \left((W_t^c)^* V_t W_t^d\right)_{t \geq 0}, \quad c, d \in \mathsf{k}.$$

$\sigma_r(V_t) \smile I_{\mathfrak{h}} \otimes W(e_{[0,r[}) \text{ in } B(\mathfrak{h}) \otimes \mathcal{F}_{[0,r[} \otimes \mathcal{F}_{[r,\infty[}.$

Definition (Dual cocycle)

$\widetilde{V} \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$, where

$$\widetilde{V} := \left((I_{\mathfrak{h}} \otimes R_t) V_t^* (I_{\mathfrak{h}} \otimes R_t)\right)_{t \geq 0}$$

Here R_t is the (unitary) *time-reversal operator* determined by

$$R_t \varepsilon(f) := \varepsilon(r_t f), \quad f \in L^2(\mathbb{R}_+; \mathsf{k})$$

with $(r_t f)(s) := f(t-s)$ for $s \in [0, t[$ and $:= f(s)$ for $s \in [t, \infty[$.

Differential quotients and associated operators

Isometric embeddings: $\mathfrak{h} \otimes \mathsf{k} \to \mathfrak{h} \otimes \mathsf{k} \otimes L^2(\mathbb{R}_+) \subset \mathfrak{h} \otimes \mathcal{F}$

$$E_t : \xi \mapsto \xi \otimes t^{-1/2} 1_{[0,t[} \quad (t > 0).$$

Differential quotients. For $V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$, define

- $K^V(t) := t^{-1}\big(I_{\mathfrak{h}} \otimes \langle\Omega|\big)\big(V_t - I_{\mathfrak{h}\otimes\mathcal{F}}\big)\big(I_{\mathfrak{h}} \otimes |\Omega\rangle\big) \in B(\mathfrak{h})$
- $L^V(t) := t^{-1/2}(E_t)^*(V_t - I_{\mathfrak{h}\otimes\mathsf{k}})(I_{\mathfrak{h}} \otimes |\Omega\rangle) \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$
- $C^V(t) := I_{\mathfrak{h}\otimes\mathsf{k}} + (E_t)^*(V_t - I_{\mathfrak{h}\otimes\mathcal{F}})E_t \in B(\mathfrak{h} \otimes \mathsf{k})$.

Properties (JML–Wills, 2007). Set $K^V := \text{st-lim}_{t\to 0^+} K^V(t)$

- $\big\{v \in \mathfrak{h} : L^V v := \text{w-lim}_{t\to 0^+} L^V(t)v \text{ exists}\big\} \supset \operatorname{Dom} K^V$, and $\|L^V v\|^2 + 2\operatorname{Re}\langle v, K^V v\rangle \leq 0$. Thus $(K^V, L^V) \in \mathfrak{X}_2(\mathfrak{h}, \mathsf{k})$.
- $\big(C^V(t) = (E_t)^* V_t E_t\big)_{t>0}$ is a family of contractions in $B(\mathfrak{h} \otimes \mathsf{k})$.

Nonsingularity and Associated quadruple of operators

Definition

$V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$ is *nonsingular* if

$$(C^V(t))_{t>0} \text{ converges as } t \to 0^+ \text{ (W.O.T.);}$$

equiv., Dom $L^{V^{c,d}}$ is independent of d. Write C^V for the limit.

Remark

If $V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$ satisfies the QSDE

$$dV_t = V_t d\Lambda_F(t) = V_t(Kdt + LdA_t^* + MdA_t + (C - I)dN_t)$$

for an operator $F = \begin{bmatrix} K & M \\ L & C-I \end{bmatrix}$ (with dense domain of the form $\mathcal{D} \oplus (\mathcal{D} \underline{\otimes} \mathrm{D})$), then V is nonsingular and $C^V = C$.

Associated quadruple

Let $V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$ be nonsingular. Then
$\widetilde{V}$ is nonsingular, $C^{\widetilde{V}} = (C^V)^*$ and, setting $\widetilde{L}^V := L^{\widetilde{V}}$,
we have an *associated quadruple* $\mathbb{F}^V := (K^V, L^V, \widetilde{L}^V, C^V - I)$.

Elementary QS coycles

Definition

$V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$ is *elementary* if its expectation semigroup is norm continuous. Write $\mathbb{QS}_c\mathbb{C}_{\mathrm{Elem}}(\mathfrak{h}, \mathsf{k})$ for this class.

Theorem (H–P, 1984). Let $F = \begin{bmatrix} K & M \\ L & N \end{bmatrix} \in B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}))$.

Then the QSDE $dV_t = V_t d\Lambda_F(t)$, $V_0 = I$ has a unique (strong) solution. Notation: V^F.

Bounded QS generators

$$C_0(\mathfrak{h}, \mathsf{k}) := \{F \in B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k})) : r(F) \leq 0\},$$

where $r(F) := F^* + F + F^* \Delta F$. Fact: $r(F) \leq 0$ iff $r(F^*) \leq 0$.

Theorem (JML–Wills, 2000)

The map $F \mapsto V^F$ restricts to a bijection

$$C_0(\mathfrak{h}, \mathsf{k}) \to \mathbb{QS}_c\mathbb{C}_{\mathrm{Elem}}(\mathfrak{h}, \mathsf{k}).$$

The QS cocycle on $B(\mathfrak{h})$ induced by $V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$.

Definition

The *induced QS cocycle on* $B(\mathfrak{h})$, the *induced semigroup on* $B(\mathfrak{h} \otimes \mathcal{F})$; its *associated semigroups* are defined respectively by

$$\Big(k_t^V : x \mapsto \widetilde{V}_t(x \otimes I_{\mathcal{F}})\widetilde{V}_t^*\Big)_{t \geq 0}; \quad \Big(K_t^V := \widehat{\mathrm{k}}_t^V \circ \sigma_t\Big)_{t \geq 0};$$

$$\Big(\mathcal{P}_t^{c,d} : x \mapsto \mathbb{E}\big[(V_t^{e,c})^*(x \otimes I_{\mathcal{F}})V_t^{e,d}\big]\Big)_{t \geq 0}.$$

Remarks

$$k_t^V(I_{\mathfrak{h}}) = R_t V_t^* V_t R_t \quad \text{and} \quad \mathbb{E}\big[k_t^V(x)\big] = \mathbb{E}\big[V_t^*(x \otimes I_{\mathcal{F}})V_t\big]$$

Theorem

Let T *be a total subset of* k *containing* 0. *Then TFAE:*

(i) k^V *is unital, equivalently* V *is isometric (equiv.,* K^V *is unital);*

(ii) $\mathcal{P}^{c,c}$ *is conservative for all* $c \in \mathrm{T}$.

Theorem (.... ; Ouhabaz, 1992)

On $\mathfrak{h}$, there is a trijective correspondence between

(i) *semisectorial, m-accretive operators $-G$;*

(ii) *closed, densely defined, semisectorial, accretive quadratic forms q;*

(iii) *holomorphic contraction semigroups P;*

such that P is the semigroup generated by G, q is the form-generator of P, and $-G$ is the closed operator associated with q:

$$P_t v = \lim_{n\to\infty} (I - n^{-1} tG)^{-n} v \quad (v \in \mathfrak{h}),$$

$$\mathrm{Dom}\, q = \left\{ v \in \mathfrak{h} : \sup_{t>0} t^{-1} \,\mathrm{Re}\langle v, (I - P_t)v\rangle < \infty \right\}$$

$$q[v] = \lim_{t\to 0^+} t^{-1} \langle v, (I - P_t)v\rangle$$

$$\mathrm{Dom}\, G = \{ v \in \mathrm{Dom}\, q : \exists_{v'\in\mathfrak{h}} \forall_{u\in \mathrm{Dom}\, q}\ \langle u, v'\rangle = -q(u,v)\},\ \ Gv = v'.$$

Some definitions. The class $\mathfrak{X}_2^{\mathrm{Hol}}(\mathfrak{h}, \mathrm{k})$

For a quadratic form q on $\mathfrak{h}$ with domain $\mathcal{Q}$,

- q is *accretive* if

$$\operatorname{Re} q[v] \geq 0, \quad v \in \mathcal{Q}.$$

For an accretive quadratic form q with domain $\mathcal{Q}$,

- An inner-product norm on $\mathcal{Q}$ is given by

$$\|v\|_q := (\operatorname{Re} q[v] + \|v\|^2)^{1/2};$$

- q is *closed* if $\mathcal{Q}$ is complete in the norm $\|\cdot\|_q$;
- q is *semisectorial* if there is $C \geq 0$ such that

$$|\operatorname{Im} q[v]| \leq C\|v\|_q, \quad v \in \mathcal{Q}.$$

Set $\mathfrak{X}_2^{\mathrm{Hol}}(\mathfrak{h}, \mathrm{k})$ equal to

$$\{(K, L) \in \mathfrak{X}_2(\mathfrak{h}, \mathrm{k}) : -K \text{ is semisectorial and } \operatorname{Dom} L = \operatorname{Dom} q\}$$

where q is the quadratic form associated with $-K$.

Holomorphic QS contraction cocycles: definition

Definition

We call $V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h}, \mathsf{k})$ *holomorphic* if its expectation semigroup is holomorphic.
Write $\mathbb{QS}_c\mathbb{C}_{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$ for the collection of these.

Thus

$$\mathbb{QS}_c\mathbb{C}_{\mathrm{Elem}}(\mathfrak{h}, \mathsf{k}) \subset \mathbb{QS}_c\mathbb{C}_{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k}),$$

and

$\widetilde{V}$ is homolorphic if and only if V is.

Holomorphic QS contraction cocycles

Theorem

Let $V \in \mathbb{QS}_c\mathbb{C}_{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$. *Then* V *and let* γ^V *be the form-generator of its expectation semigroup. Then*

- V *is nonsingular,*
- $\mathrm{Dom}\, L^V, \mathrm{Dom}\, \widetilde{L}^V \supset \mathrm{Dom}\, \gamma^V$.

Therefore V has an associated quadruple

$$\mathbb{F}^V = (K^V, L^V, \widetilde{L}^V, C^V - I_{\mathfrak{h}\otimes\mathsf{k}}).$$

Theorem

Let $V \in \mathbb{QS}_c\mathbb{C}_{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$. *Then each of its associated cocycles* $V^{c,d}$ *is holomorphic.*

Structure relations

Definition

Set $\mathfrak{X}_4^{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$ equal to

the set of quadruples $\mathbb{F} = (K, L, \widetilde{L}, C - I)$ such that

- K is the generator of a holomorphic contraction semigroup on $\mathfrak{h}$ (let γ be the form-generator),
- L, $\widetilde{L}$ are operators from $\mathfrak{h}$ to $\mathfrak{h} \otimes \mathsf{k}$ with domain $\operatorname{Dom}\gamma$,
- C is a contraction in $B(\mathfrak{h} \otimes \mathsf{k})$,
- $\|\Delta F \zeta\|^2 \leq 2 \operatorname{Re} \Gamma[\zeta]$,

where ΔF and Γ are the operator and quadratic form given by:

$$\operatorname{Dom}\Gamma = \operatorname{Dom}\Delta F = \operatorname{Dom}\gamma \oplus (\mathfrak{h} \otimes \mathsf{k}),$$

$$\Delta F = \begin{bmatrix} 0 & 0 \\ L & C - I \end{bmatrix},$$

$$\Gamma\left[\begin{pmatrix} v \\ \xi \end{pmatrix}\right] = \gamma[v] - \left\{ \langle \xi, Lv \rangle + \langle \widetilde{L}v, \xi \rangle + \langle \xi, (C - I)\xi \rangle \right\}$$

Remarks on the structure relations

Remarks

- We have the inclusion

$$\mathfrak{X}_4^{\mathrm{Hol}}(\mathfrak{h},\mathsf{k}) \supset \left\{(K,L,M^*,C-I) : \left[\begin{smallmatrix} K & M \\ L & C-I \end{smallmatrix}\right] \in C_0(\mathfrak{h},\mathsf{k})\right\}$$

- $(K,L,\widetilde{L},C-I) \in \mathfrak{X}_4^{\mathrm{Hol}}(\mathfrak{h},\mathsf{k}) \implies (K,L) \in \mathfrak{X}_2^{\mathrm{Hol}}(\mathfrak{h},\mathsf{k})$.
- In the converse direction,
  if $(K,L) \in \mathfrak{X}_2^{\mathrm{Hol}}(\mathfrak{h},\mathsf{k})$ then, for any contraction $C \in B(\mathfrak{h} \otimes \mathsf{k})$, we have

$$(K,L,-C^*L,C-I) \in \mathfrak{X}_4^{\mathrm{Hol}}(\mathfrak{h},\mathsf{k}).$$

In particular, $(K,L,-L,0)$, $(K,L,0,-I) \in \mathfrak{X}_4^{\mathrm{Hol}}(\mathfrak{h},\mathsf{k})$.

The stochastic generator of a homomorphic QS cocycle

Theorem

The prescription

$$V \mapsto \mathbb{F}^V = (K^V, L^V, \widetilde{L}^V, C^V - I_{\mathfrak{h}\otimes\mathsf{k}})$$

defines a bijection

$$\mathbb{QS}_c\mathbb{C}_{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k}) \to \mathfrak{X}_4^{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k}),$$

extending (the inverse of) the earlier (L-W) bijection

$$C_0(\mathfrak{h}, \mathsf{k}) \to \mathbb{QS}_c\mathbb{C}_{\mathrm{Elem}}(\mathfrak{h}, \mathsf{k}), \quad F = \begin{bmatrix} K & M \\ L & C - I \end{bmatrix} \mapsto V^F.$$

This justifies the following.

Definition

For $V \in \mathbb{QS}_c\mathbb{C}_{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$,
we refer to $\mathbb{F}^V$ as the *stochastic generator* of V.

Holomorphic QS cocycles induce 'dilations' of minimal QDS's

Theorem

Let $V \in \mathbb{QS}_c\mathbb{C}_{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$. *Then*

$$\mathbb{E}\left[V_t^*(x \otimes I_{\mathcal{F}})V_t\right] = \mathcal{T}_t^{K,L}(x), \quad x \in B(\mathfrak{h}), t \geq 0. \qquad (\dagger)$$

where $(K, L) \in \mathfrak{X}_2^{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$ *is the truncation of the stochastic generator of* V *to its first two components*
[*i.e.* $\mathbb{F}^V$ *has the form* $(K, L, *, *)$].

Corollary

Let $(K, L) \in \mathfrak{X}_2^{\mathrm{Hol}}(\mathfrak{h}, \mathsf{k})$. *Then, letting* $V = V^{\mathbb{F}}$,
where $\mathbb{F} = (K, L, -C^*L, C - I)$ *for a contraction* $C \in B(\mathfrak{h} \otimes \mathsf{k})$,
e.g. $\mathbb{F} = (K, L, -L, 0)$,
$(\dagger)$ *holds.*