Interpolation of Markov maps on quantum Orlicz spaces

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Outline

Orlicz functions

Orlicz function: A convex function $\psi : [0,\infty) \to [0,\infty]$ satisfying

- $\psi(0) = 0$ and $\lim_{u\to\infty} \psi(u) = \infty$,
- neither identically zero nor infinite valued on all of $(0,\infty)$,
- left continuous at $b_\psi = \sup\{u > 0 : \psi(u) < \infty\}$.

Complementary Orlicz function: $\psi^*(u) = \sup_{v>0}(uv - \psi(v))$.

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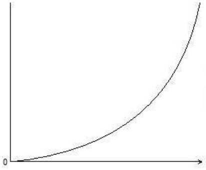

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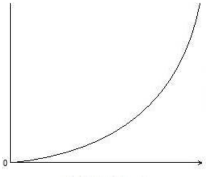

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Defining Orlicz spaces

L^0 the space of all measurable functions on σ-finite (Ω, Σ, m).

Definition (Orlicz space corresponding to ψ)

$f \in L^0$ belongs to $L^\psi \Leftrightarrow \psi(\lambda|f|)$ is integrable for some $\lambda = \lambda(f) > 0$.

Luxemburg-Nakano norm: $\|f\|_\psi = \inf\{\lambda > 0 : \|\psi(|f|/\lambda)\|_1 \leq 1\}$.

Orlicz norm: $\|f\|_\psi^O = \sup\{|\int_\Omega fg\, dm| : g \in L^{\psi^*}, \|g\|_{\psi^*} \leq 1\}$.

Notational convention: L^ψ (Luxemburg norm); L_ψ (Orlicz norm).

Köthe duality: A measurable function f belongs to $L_{\psi^*}(X, \Sigma, \nu)$ if and only if $fg \in L^1$ for every $g \in L^\psi$.

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New norms on $M_n(\mathbb{C})$

Example: Replace L^∞ by $M_n(\mathbb{C})$, and $\int \cdot d\nu$ by Tr, and see what happens:

- $L^p(M_n(\mathbb{C}), \mathrm{Tr})$ is just $M_n(\mathbb{C})$ equipped with the norm $Tr(|a|^p)^{1/p}$.
- Similarly $L^\Psi(M_n(\mathbb{C}), \mathrm{Tr})$ is $M_n(\mathbb{C})$ equipped with the norm $\|a\|_\Psi = \inf\{\lambda > 0 : \mathrm{Tr}(\psi(|a|/\lambda)) \leq 1\}$.

J von Neumann, Some matrix inequalities and metrization of matrix space, *Tomsk Univ Rev* **1**(1937), 286-300

Upping the ante: If we play essentially the same game but using a semifinite von Neumann algebra M and an associated fns trace τ instead of $(M_n(\mathbb{C}), \mathrm{Tr})$, the theory still works.

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Orlicz and L^p-spaces for semifinite algebras

- M a von Neumann algebra, equipped with a faithful normal semifinite trace $\tau_{\mathcal{M}} = \tau : M^+ \to [0, \infty]$.
- $\widetilde{M}$ the algebra of τ_M-measurable operators: operators affiliated to M, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(e) \leq \varepsilon$, and $a(\mathbb{1} - e) \in M$.

$\widetilde{M}$ plays the role of the completion of L^∞ under the topology of convergence in measure.

$f \in \widetilde{M}$ belongs to $L^p(M, \tau) \Leftrightarrow \tau(|f|^p) < \infty$ with $\|f\|_p = \tau(|f|^p)^{1/p}$.

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Let f be a fixed element (state) in
$M_\nu = \{f \in L^1 : f > 0, \int f\, d\nu = 1\}$ $(\nu(X) = 1)$.

Definition (Pistone-Sempi, 1995)

A measurable function u is said to be a regular observable (with respect to f) if the function $\hat{u}(t) = \int e^{tu} f.d\nu$ exists in a neighbourhood of 0, and $\int u\, f d\nu = 0$.

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Proposition (L, Majewski; 2014)

Let M be a semifinite algebra and $f \in L^1 \cap L\log(L+1)(M,\tau)$ with $f \geq 0$. Then $\tau(f\log(f+\epsilon))$ is well defined for any $\epsilon > 0$. Moreover

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is bounded above, and if in addition $f \in L^{1/2}$, it is also bounded from below.

Here $L\log(L+1)(M,\tau)$ is the Orlicz space corresponding to the function $\Psi(t) = t\log(t+1)$.

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Inspired by the controversial work of Boltzmann on the dynamics of rarefied gases [1872], von Neumann expressed entropy as $\mathrm{Tr}(\rho\log(\rho))$ in the context of $B(H)$ (here ρ is a norm 1 element of $\mathscr{S}^1(H)^+$ representing the state of the system).

Problem: For the specific case of $B(H)$ one gets a respectable theory for the action of this quantity on $\mathscr{S}^1(H)^+$. For more general tracial von Neumann algebras M, the quantity $\tau(\rho\log(\rho))$ ($\rho \in L^1(M,\tau)^+$) can be extremely badly behaved with respect to the L^1-topology. So $B(H)$ is somewhat exceptional!!

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So why does the space $L\log(L+1)(M,\tau) \cap L^1(M,\tau)$ not feature in the context of the pair $\langle \mathscr{S}^1(H), B(H)\rangle$?

In the case of $M = B(H)$, $\tau = \mathrm{Tr}$, we have that

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Implications

Achieved results: $L^{\cosh -1}(M, \tau)$ is a home for regular quantum observables, and $L\log(L+1)(M, \tau) \cap L^1(M, \tau)$ a home for states with good entropy.

Deeper truths: The space $L^{\cosh -1}(M, \tau)$ is actually an isomorphic copy of the Banach space dual of $L\log(L+1)(M, \tau)$. So up to isomorphism, $\langle L\log(L+1)(M, \tau), L^{\cosh -1}(M, \tau)\rangle$ is a dual pair.

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The strange ways of type III L^p spaces

Commutative	Quantum
$A = L^\infty(X, \Sigma, \nu) \otimes L^\infty(\mathbb{R})$	"enlarge" M by passing to $A = M \rtimes_\nu \mathbb{R}$
$\theta_s(f \otimes g)(x, t) = f(x)g(t - s)$	a dual action of $\mathbb{R}$ on A in the form of a group of *-automorphisms $\{\theta_s\}$ $(s \in \mathbb{R})$
$\int \cdot\, d\nu \otimes \int_{\mathbb{R}} \cdot\, e^{-t} dt$	a canonical trace τ_A on A characterised by $\tau_A \circ \theta_s = e^{-s}\tau_A$.

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(Haagerup, 1979): For any measurable function f on X (finite ν-almost everywhere) we have that

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(Haagerup, 1979): By analogy with the classical setting, we may define

$$L^p(M) = \{a \in \widetilde{A} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R}\}.$$

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- Replace $\mathcal{H}$ with $L^2(\mathbb{R}, \mathcal{H})$.
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Constructing $A = M \rtimes_\nu \mathbb{R}$

- Replace $\mathcal{H}$ with $L^2(\mathbb{R}, \mathcal{H})$.
- The map $a \to \pi(a)$ defines and embedding of M into $B(L^2(\mathbb{R}, \mathcal{H}))$, where $(\pi(a)(\eta))(t) = \sigma^\nu{}_{-t}(a)(\eta(t))$ for all $a \in M$ and all $\eta \in L^2(\mathbb{R}, \mathcal{H})$.
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Type III Orlicz spaces

Haagerup's construction of L^p-spaces for type III von Neumann algebras can be extended to also allow for the construction of Orlicz spaces. (L, 2014)

The classical roots of the construction: Let $M = L^\infty(X, \Sigma, \nu)$, and let $A = L^\infty(X, \Sigma, \nu) \otimes L^\infty(\mathbb{R})$ be as before.

Given an Orlicz function Ψ, define $\varphi_\Psi : [0, \infty) \to [0, \infty)$ by

$$\varphi_\Psi(t) = \frac{1}{\Psi^{-1}(1/t)}.$$

For any measurable function f on X, we then have that

$$f \otimes \varphi_\Psi(e^{(\cdot)}) \in \widetilde{A} \quad \Leftrightarrow \quad f \in L^\Psi(X, \Sigma, \nu).$$

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Given a von Neumann algebra M with fns weight ν,

- let $\widehat{\nu}$ be the dual weight on the crossed product $A = M \rtimes_\nu \mathbb{R}$,
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Then a τ_A-measurable operator $a \in \widetilde{A}$ belongs to $L^\Psi(M)$ $\Leftrightarrow$ for every $s \in \mathbb{R}$ we have that $\theta_s(a) = e^{-s} d_s^{1/2} a d_s^{1/2}$ where d_s is the operator $d_s = \varphi^*(e^{-s}h)^{-1}\varphi^*(h)$.

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Emergent challenge

Challenge: Given a Markov map T with a canonical action on M and $L^1(M)$, can we show that it has a nice action on a large enough class of Orlicz spaces? First pause to see what is known.

Theorem (Yeadon 1977; HJX 2010)

Let $T : M \to M$ be a positive map for which there exists some $C_1 > 0$ such that $\nu(T(x)) \leq C_1 \nu(x)$ for all $x \in M^+$. Then for each $1 \leq p < \infty$, T canonically extends to a positive bounded map $T_p : L^p(M) \to L^p(M)$ such that $\|T_p\| \leq C_\infty^{1-(1/p)}.C_1^{1/p}$ where $C_\infty = \|T(\mathbb{1})\|_\infty$.

Problem: The proof uses complex interpolation. To date complex interpolation does not work for quantum Orlicz spaces.

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An alternative strategy

Challenge 2: Can we overcome the lack of access to complex interpolation, by passing to a smaller class of Markov maps, namely the CP Markov map? If so how?

Idea:

- Show that under acceptable assumptions, $T : M \to M$ extends to a map $\widetilde{T}$ on $A = M \rtimes_{\nu} \mathbb{R}$,
- and from there to a map on $(L^{\infty} + L^{1})(A, \tau_A)$.
- Then see if any of the Orlicz spaces $L^{\Psi}(M)$ live inside $(L^{\infty} + L^{1})(A, \tau_A)$, and try to extract the action from that.

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HJX extension result

Theorem

Let $T : M \to M$ be a completely bounded normal map such that $T \circ \sigma_t^\nu = \sigma_t^\nu \circ T, \quad t \in \mathbb{R}$. Then the prescription $\widetilde{T}(\lambda(s)\pi(x)) = \lambda(s)\pi(T(x))$ $(x \in M, s \in \mathbb{R}$ generates a unique bounded normal extension $\widetilde{T}$ of T to $A = M \rtimes_{\sigma^\nu} \mathbb{R}$ with $\|T\| = \|\widetilde{T}\|$. Moreover:

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2. $\widetilde{T} \circ \sigma_t^{\widehat{\nu}} = \sigma_t^{\widehat{\nu}} \circ \widetilde{T}$ *for all $t \in \mathbb{R}$ ($\widehat{\nu}$ is the dual weight of ν).*
3. $T \geq 0 \Rightarrow \widetilde{T} \geq 0$.
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HJX extension result

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Real interpolation to the rescue

Corollary

Let T and $\widetilde{T}$ be as before. If each of (1)-(4) holds, then $\tau_A \circ \widetilde{T} \leq \tau_A$ where τ_A is the canonical trace on $A = M \rtimes_{\sigma^\nu} \mathbb{R}$. The map $\widetilde{T}$ then also canonically induces a map on the space $(L^\infty + L^1)(A, \tau_A)$.

Proof The first claim follows by applying some ideas from Pedersen and Takesaki's seminal paper. For the second claim apply Yeadon's ergodic result to see that $\widetilde{T}$ induces a bounded map on $L^1(A, \tau_A)$, and then apply real interpolation to get the conclusion.

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Markov dynamics

Definition

Let φ_ψ be the fundamental function of the space $L^\Psi(0,\infty)$, and let $M_\psi(t) = \sup_{s>0} \frac{\varphi_\Psi(st)}{\varphi_\Psi(s)}$. We call the quantity

$$\overline{\beta}_{L^\Psi} = \inf_{1<t} \frac{\log M_\psi(s)}{\log s}$$

the upper fundamental index of $L^\Psi(M)$.

Proposition

If $\overline{\beta}_{L^\Psi} < 1$, then $L^\psi(M) \subset (L^\infty + L^1)(A, \tau_A)$ (isomorphically). Moreover $L^\psi(M)$ is an invariant subspace of the extension $\widetilde{T}$ of T to $(L^\infty + L^1)(A, \tau_A)$. This class includes $L^{\cosh -1}(M)$!! (The space of regular observables.)

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The emergent picture

Based on the preceding analysis we may propose the following framework as an axiomatic foundation for Quantum Statistical Mechanics:

- Corresponding to each quantum system there is a pair (M, ν) (where M is a von Neumann algebra and ν an associated faithful normal semifinite weight) describing the system.
- The pair of spaces $\langle L\log(L+1)(M), L^{\cosh -1}(M)\rangle$ are respectively homes for good states and good observables of this system.
- There is a Dirichlet form $\mathscr{E}$ (representing an energy potential) describing Markov dynamics on the space $L^{\cosh -1}(M)$ of regular observables.

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