Relative weak mixing in W*-dynamical systems

Rocco Duvenhage & Malcolm King

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Outline

Statement of Main Result

Theorem 1 (Characterization of Relative Weak Mixing).

Let $\mathbf{A} = (A, \mu, \alpha)$ *be a* W^**-dynamical system and* $\mathbf{F} = (F, \lambda, \varphi)$ *a modular subsystem of* $\mathbf{A}$*. Then* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$ *if and only if* $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$*.*

W*-Dynamical Systems I

Theorem 1 (Characterization of Relative Weak Mixing).
Let $\mathbf{A} = (A, \mu, \alpha)$ be a W^*-dynamical system and $\mathbf{F} = (F, \lambda, \varphi)$ a modular subsystem of $\mathbf{A}$. Then $\mathbf{A}$ is weakly mixing relative to $\mathbf{F}$ if and only if $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$.

W*-dynamical system (A, μ, α) with

- A von Neumann algebra (represented in its GNS form on GNS Hilbert space $H \equiv \overline{A\Omega}$ with cyclic separating & vector Ω)
- μ faithful normal state $(\mu(a) = \langle \Omega, a\Omega \rangle)$
- α $*$-automorphism on A. $(U : H \to H \; U(a\Omega) = \alpha(a)\Omega)$

W*-dynamical modular subsystem of $\mathbf{A}$ (F, λ, φ) with

- F von Neumann subalgebra of A
- $\lambda := \mu|_F$

W*-Dynamical Systems II

- $\varphi := \alpha|_F$
- F is globally invariant under modular group associated to μ (Gives the existence of unique conditional expectation $D : A \to F$ such that $\lambda \circ D = \mu$. Additionally,
  - $\forall n \in \mathbb{Z}\ \alpha^n D = D\alpha^n$.
  - $\forall a \in A\ D(a) = PaP$ where $P : H \to \overline{F\Omega}$ is the orthogonal projection.

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Additional Notation

- Denote the GNS Hilbert space of F by H_λ.
- Let J denote the modular conjugation operator associated to the state μ and

$$j : B(H) \to B(H) : a \to Ja^*J.$$

Relative Weak Mixing I

Theorem 1 (Characterization of Relative Weak Mixing).

Let $\mathbf{A} = (A, \mu, \alpha)$ *be a* W^**-dynamical system and* $\mathbf{F} = (F, \lambda, \varphi)$ *a modular subsystem of* $\mathbf{A}$*. Then* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$ *if and only if* $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$*.*

Definition 1.

Let D be the conditional expectation from A onto D. We call a system $\mathbf{A}$ *weakly mixing relative to* the modular subsystem $\mathbf{F}$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda\left(D\left(\alpha^n(a^*)b^*\right) D\left(b\alpha^n(a)\right)\right) = 0 \qquad (1)$$

for all $a, b \in A$ with $D(a) = D(b) = 0$.

Joinings

Definition 2.

A *joining* of **A** and **B** is a state ω on the algebraic tensor product $A \odot B$ such that $\omega\left(a \otimes 1_B\right) = \mu(a)$, $\omega\left(1_A \otimes b\right) = \nu(b)$ and $\omega \circ \left(\alpha \odot \beta\right) = \omega$ for all $a \in A$ and $b \in B$.

Relative Independent Joining I

- $\mathbf{A}' := (A', \mu', \alpha')$
  $\forall b \in A'$
  - $\mu'(b) := \mu \circ j(b)$
  - $\alpha'(b) := j \circ \alpha \circ j(b)$
  - $\alpha'(b) = UbU^*$
- modular subsystem $\tilde{\mathbf{F}} = \left(\tilde{F}, \tilde{\lambda}, \tilde{\varphi}\right)$ of $\mathbf{A}'$
  - $\tilde{F} := j(F) \subseteq A'$
  - $\tilde{\lambda} := \tilde{\mu}|_{\tilde{F}}$
  - $\tilde{\varphi} := \tilde{\alpha}|_{\tilde{F}}$
- relatively independent joining of **A** and $\mathbf{A}'$ over **F**
  - $\tilde{D} := j \circ D \circ j : A' \to \tilde{F}$
  - $\delta : F \odot \tilde{F} \to B(H)$ linear extension of
    $F \times \tilde{F} \to B(H) : (a, b) \mapsto ab$
  - Diagonal State $\Delta_\lambda : F \odot \tilde{F} \to \mathbb{C}$ for all $c \in F \odot \tilde{F}$
    $\Delta_\lambda(c) := \langle \Omega, \delta(c)\Omega \rangle$

Relative Independent Joining II

- $\mu \odot_\lambda \mu'$ on $A \odot A'$ by

$$\omega := \mu \odot_\lambda \mu' := \Delta_\lambda \circ (D \odot \tilde{D}). \qquad (2)$$

- Relative Product system (of **A** and **A**$'$ over **F**) $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}' := (A \odot A', \mu \odot_\lambda \mu', \alpha \odot \alpha')$
- We let W denote the unitary representation of $\alpha \odot \alpha'$ on the GNS Hilbert space H_ω
- We let H_ω^W denote the fixed point space of W.

Relative Independent Joining III

Important Consequence: We can "embed" H (and, thus H_λ) into the GNS Hilbert space H_ω for $(A \odot B, \omega)$:

- $\overline{\pi_\omega(A \otimes 1)\Omega_\omega} \equiv H.$

  ($\pi_\omega : A \odot B \to B(H_\omega)$ is the GNS representation for $(A \odot B, \omega)$).

- $H_{\tilde{\lambda}} = H_\lambda \equiv \overline{F\Omega} \subseteq H.$

Relative Ergodicity

Theorem 1 (Characterization of Relative Weak Mixing).

Let $\mathbf{A} = (A, \mu, \alpha)$ *be a* W^**-dynamical system and* $\mathbf{F} = (F, \lambda, \varphi)$ *a modular subsystem of* $\mathbf{A}$*. Then* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$ *if and only if* $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$*.*

Definition 3.

We say that $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* the modular subsystem $\mathbf{F}$ of $\mathbf{A}$, if $H_\omega^W \subset H_\lambda$.

Relative Ergodicity implies Relative Weak Mixing I

Proposition 4.

If $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$, *then* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$.

Relative Ergodicity implies Relative Weak Mixing II

Proof.

- Let Q be the projection of H_ω onto the fixed point space H_ω^W of W.
- mean ergodic theorem: For all $s, t \in A \odot A'$,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \omega(((\alpha \odot \alpha')^n(s))t) \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle W^n \pi_\omega(s^*)\Omega_\omega, \pi_\omega(t)\Omega_\omega\rangle \\
&= \langle Q\pi_\omega(s^*)\Omega_\omega, \pi_\omega(t)\Omega_\omega\rangle
\end{aligned}$$

Relative Ergodicity implies Relative Weak Mixing III

Proof (Cont.)

- In particular, $s = a \otimes (JaJ)$ and $t = b \otimes (JbJ)$, where $a, b \in A$ and $D(a) = 0$ or $D(b) = 0$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(((\alpha \odot \alpha')^n (a \otimes (JaJ)))b \otimes (JbJ))$$
$$= \langle Q\pi_\omega((a \otimes (JaJ))^*)\Omega_\omega, \pi_\omega(b \otimes (JbJ))\Omega_\omega\rangle$$

- Suppose $D(a) = 0$ (the case $D(b) = 0$ is similar). As $\pi_\omega((a \otimes (JaJ))^*)\Omega_\omega \perp H_\omega^W$ we have $Q\pi_\omega(a \otimes (JaJ)^*)\Omega_\omega = 0$. Thus, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(((\alpha \odot \alpha')^n (a \otimes (JaJ)))b \otimes (JbJ)) = 0$

Relative Ergodicity implies Relative Weak Mixing IV

Proof (Cont.)

- Using the definition of $\omega = \mu \odot \mu'$ we calculate

$$
\begin{aligned}
0 &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(((\alpha \odot \alpha')^n(a \otimes (JaJ)))b \otimes (JbJ)) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left\langle \Omega, D(\alpha^n(a)b)\tilde{D}(\alpha'^n(JaJ)JbJ)\Omega \right\rangle \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(D(\alpha^n(a)b)D(b^*\alpha^n(a^*))),
\end{aligned}
$$

as required.

Characterization of Relative Ergodicity of Rel. Ind. System

Proposition 5.

$\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$ if and only if for every $s, t \in A \odot A'$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(s\tau^n(t)) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(E(s)\tau^n(E(t))) \tag{3}$$

where $\tau = \alpha \odot \alpha'$ and $E = D \odot \tilde{D}$.

Relative Weak mixing implies Relative Ergodicity I

Proposition 6.

Assume that μ is a trace. Then $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(D\left(\alpha^n(a^*)b^*\right) D(b\alpha^n(a))) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(D(\alpha^n(a^*))D(b^*)D(b)D(\alpha^n(a))) \tag{4}$$

for all $a, b \in A$.

Relative Weak mixing implies Relative Ergodicity II

Proposition 7.

If **A** *is ergodic relative to* **F**, *then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(b\alpha^n(a)) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda(D(b)\alpha^n(D(a)))$$

and, in particular both these limits exist, for all $a, b \in A$.

Relative Weak mixing implies Relative Ergodicity III

Proposition 8.

The system $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda\left(\left|D\left(b\alpha^n(a)\right) - D(b)D(\alpha^n(a))\right|^2\right) = 0$$

for all $a, b \in A$.

Corollary 9.

If $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$*, then* $\mathbf{A}$ *is ergodic relative to* $\mathbf{F}$*.*

Relative Weak mixing implies Relative Ergodicity IV

Proposition 10.

Assume that μ is tracial and that $\mathbf{A}$ is weakly mixing relative to $\mathbf{F}$. Then $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$.

Relative Weak mixing implies Relative Ergodicity V

Proof.

1. Proposition 8 gives us

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda\left(\left|D\left(b\alpha^n(a)\right)-D(b)D(\alpha^n(a))\right|^2\right)=0$$

2. $\lambda\left(\left|D\left(b\alpha^n(a)\right)-D(b)D(\alpha^n(a))\right|^2\right)$

$$=\lambda(D(\alpha^n(a^*)b^*)D(b\alpha^n(a)) \tag{5}$$

$$-\lambda(D(\alpha^n(a^*)b^*)D(b)D(\alpha^n(a))) \tag{6}$$

$$-\lambda(D(\alpha^n(a^*))D(b^*)D(b\alpha^n(a))) \tag{7}$$

$$+\lambda(D(\alpha^n(a^*))D(b^*)D(b)D(\alpha^n(a))). \tag{8}$$

Relative Weak mixing implies Relative Ergodicity VI

Proof.

5. Eq. (7): $\lambda(D(\alpha^n(a^*))D(b^*)D(b\alpha^n(a)))$

$$= \lambda(D[D(b^*)b\alpha^n(a)D(\alpha^n(a^*))]) = \mu(D(b^*)b\alpha^n(aD(a^*))).$$

Now,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(D(b^*)b\alpha^n(aD(a^*)))$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(b^*)D(b)\alpha^n(D(a)D(a^*))) \quad (9)$$

(Weak mixing assumption implies that **A** ergodic relative to **F** (Corollary 9). Proposition 7 gives the equality of limits).

Relative Weak mixing implies Relative Ergodicity VII

Proof.

7. We wish to show that the limit of the ergodic average of Eq. (5), and (9) are equal. (This will give us the required result using Proposition 6). We do this by showing that the limits of the ergodic averages of Eqs (6) and (8) are equal.

Relative Weak mixing implies Relative Ergodicity VIII

Proof.

8. Eq (6):

$$\lambda(D(\alpha^n(a^*)b^*)D(b)D(\alpha^n(a))) = \lambda(D[\alpha^n(a^*)b^*D(b)D(\alpha^n(a))])$$
$$= \mu(\alpha^n(a^*)b^*D(b)D(\alpha^n(a))) = \mu(b^*D(b)\alpha^n(D(a)a^*)).$$

Corollary 9 gives

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(\alpha^n(a^*)b^*)D(b)D(\alpha^n(a)))$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(b^*)D(b)\alpha^n(D(a)D(a^*))). \tag{10}$$

Relative Weak mixing implies Relative Ergodicity IX

Proof.

9. Eq (8):

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(\alpha^n(a^*))D(b^*)D(b)D(\alpha^n(a)))$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D[D(b^*)D(b)\alpha^n(aD(a^*))])$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(D(b^*)D(b)\alpha^n(aD(a^*)))$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(b^*)D(b)\alpha^n(D(a)D(a^*))$$

Last limit is equal to (10).

Bibliography I

- T. Austin, T. Eisner, T. Tao, *Nonconventional ergodic averages and multiple recurrence for von Neumann dynamical systems*, Pacific J. Math. 250 (2011), 1–60. MR2780387
- J. P. Bannon, J. Cameron and K. Mukherjee, *On noncommutative joinings*, Int. Math. Res. Not., to appear.
- R. Duvenhage, *Joinings of W*-dynamical systems*, J. Math. Anal. Appl. 343 (2008) 175–181. arXiv:0802.0827v1 [math.OA]. MR2409467 (2009b:46135)
- R. Duvenhage, *Ergodicity and mixing of W*-dynamical systems in terms of joinings*, Illinois J. Math. 54 (2010) 543–566.
- R. Duvenhage, *Relatively independent joinings and subsystems of W*-dynamical systems*, Studia Math. 209 (2012), 21–41.

Bibliography II

- H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49. MR 0213508
- H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, M. B. Porter Lectures. Princeton University Press, Princeton, N.J., 1981
- E. Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs 101, American Mathematical Society, Providence, RI, 2003.
- S. Popa, *Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups*, Invent. Math. 170 (2007), 243–295.

Bibliography III

M. Takesaki, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003.