BALANCE BETWEEN QUANTUM MARKOV SEMIGROUPS

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Abstract. The concept of balance between two state preserving quantum Markov semigroups is introduced and studied as an extension of conditions appearing in the theory of quantum detailed balance. This is partly motivated by the theory of joinings. Basic properties of balance are derived and the connection to correspondences in the sense of Connes is discussed. Some applications and possible applications, including to non-equilibrium statistical mechanics, are briefly explored.

Contents

1. Introduction

Motivated by quantum detailed balance, we define and study the notion of balance between pairs of quantum Markov semigroups on von Neumann algebras, where each semigroup preserves a faithful normal state. Ideas related to quantum detailed balance continue to play an important role in studying certain aspects of non-equilibrium statistical

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mechanics, in particular non-equilibrium steady states. See for example [2], [3] and [5]. A theory of balance as introduced here is therefore potentially applicable to non-equilibrium statistical mechanics. In this paper, however, we just lay the foundations by developing the basics of a theory of balance. Non-equilibrium is only touched on.

The papers on quantum detailed balance that most directly lead to the work presented in this paper are [27], [28], [29], and [25]. Of particular relevance are ideas connected to standard quantum detailed balance conditions mentioned in [20], and discussed and developed in [29] and [28]. However, a number of other papers develop ideas related to standard quantum detailed balance and dualities, of which [12], [13] and [49] contributed to our line of investigation.

The theory of balance can be viewed as being parallel to the theory joinings for W*-dynamical systems. The latter was developed in [22, 23, 24], and studied further in [10], for the case where the dynamics are given by $*$-automorphism groups. Some aspects of noncommutative joinings also appeared in [56] and [44] related to entropy, and in [32] related to certain ergodic theorems. In [46] results closely related to joinings were presented regarding a coupling method for quantum Markov chains and mixing times.

The theory of joinings is already a powerful tool in classical ergodic theory, which is what motivated its study in the noncommutative case (see the book [36] for an exposition). Analogously, we expect a theory of balance between quantum Markov semigroups to be of use in the study of such semigroups.

The definition of balance is given in Section 2, along with relevant mathematical background, in particular regarding the definition of a dual of certain positive maps. Couplings of states on two von Neumann algebras are also defined here.

In Section 3 we show how couplings lead to unital completely positive (u.c.p.) maps from one von Neumann algebra to another. These maps play a key role in developing the theory of balance. This is related to [10, Section 4], although in the latter, certain assumptions involving modular groups are built into the framework, while analogous assumptions do not form part of the theory developed here.

Section 4 gives a characterization of balance in terms of intertwinement with the u.c.p. maps defined in Section 3. The role of KMS-duals and the special case of KMS-symmetry are also briefly discussed in the context of symmetry of balance. Two simple applications are then given to illustrate the possible use of balance. One is to characterize a certain ergodicity condition in a way analogous to the theory of joinings. The other is related to the convergence of states to steady states in open quantum systems and non-equilibrium statistical mechanics.

The development of the theory of balance continues in Section 5, where balance is shown to be transitive. The connection to correspondences in the sense of Connes is also discussed. The connection to correspondences in the context of joinings was already pointed out in [10] and [44, Section 5].

Next, in Section 6, we discuss a quantum detailed balance condition (namely standard quantum detailed balance with respect to the reversing operation from [29] and [28]) in terms of balance. Based on this, we briefly speculate on non-equilibrium steady states in the context of balance.

We turn to a simple example to illustrate a number of the ideas from this paper in Section 7.

In the final section, possible further directions of study are mentioned.

2. The definition of balance

This section gives the definition of balance, but for convenience and completeness also collects some related known results that we need in the formulation of this definition as well as later on in the paper. Some of the notation used in the rest paper is also introduced.

In this paper we consider systems defined as follows:

Definition 2.1. A *system* $\mathbf{A} = (A, \alpha, \mu)$ consists of a faithful normal state μ on a (necessarily σ-finite) von Neumann algebra A, and a unital completely positive (u.c.p.) map $\alpha : A \to A$, such that $\mu \circ \alpha = \mu$.

Remark 2.2. Note that we only consider a single u.c.p. map, since throughout the paper we can develop the theory at a single point in time. This can then be applied to a semigroup of u.c.p. maps by applying the definitions and results to each element of the semigroup separately (also see Remarks 2.6, 2.11, 4.4 and 6.6, Proposition 4.8, and Section 7).

In the rest of the paper the symbols $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ will denote systems (A, α, μ), (B, β, ν) and (C, γ, ξ) respectively. The unit of a von Neumann algebra will be denoted by 1. When we want to emphasize it is the unit of, say, A, the notation 1_A will be used. We naturally also assume that $A \neq 0$ for all the von Neumann algebras A that we consider, i.e. $1_A \neq 0$.

Without loss of generality, in this paper we always assume that these von Neumann algebras are in the cyclic representations associated with the given states, i.e. the cyclic representation of (A, μ) is of the form $(G_\mu, \mathrm{id}_A, \Lambda_\mu)$, where G_μ is the Hilbert space, id_A denotes the identity map of A into $B(G_\mu)$, and Λ_μ is the cyclic and separating vector such that $\mu(a) = \langle \Lambda_\mu, a\Lambda_\mu \rangle$. Often one uses the notation H_μ instead of G_μ, and Ω_μ instead of Λ_μ, but we reserve these symbols for cyclic representations which will appear in the next section and onwards.

The dynamics α of a system $\mathbf{A}$ is necessarily a contraction, since it is positive and unital (see for example [11, Proposition II.6.9.4]. Furthermore, α is automatically normal. This is due to the following result:

Theorem 2.3. *Let M and N be von Neumann algebras on the Hilbert spaces H and K respectively, and consider states on them respectively given by $\mu(a) = \langle \Omega, a\Omega \rangle$ and $\nu(b) = \langle \Lambda, b\Lambda \rangle$, with $\Omega \in H$ and $\Lambda \in K$ cyclic vectors, i.e. $\overline{M\Omega} = H$ and $\overline{N\Lambda} = K$. Assume that ν is faithful and consider a positive linear (but not necessarily unital) $\eta : M \to N$ such that*

$$\nu(\eta(a)^*\eta(a)) \leq \mu(a^*a)$$

for all $a \in M$. Then it follows that η is normal, i.e. σ-weakly continuous.

Results of this type appear to be well known, so we omit the proof. This result applies to a system $\mathbf{A}$, since from the Stinespring dilation theorem [58] one obtains Kadison's inequality $\alpha(a)^*\alpha(a) \leq \alpha(a^*a)$ for all $a \in A$, i.e. α is a Schwarz mapping; see for example [11, Proposition II.6.9.14]

A central notion in our work is the dual of a system, defined as follows:

Definition 2.4. The *dual* of the system $\mathbf{A}$, is the system $\mathbf{A}' = (A', \alpha', \mu')$ where A' is the commutant of A (in $B(H_\mu)$), μ' is the state on A' given by $\mu'(a') = \langle \Lambda_\mu, a'\Lambda_\mu \rangle$ for all $a' \in A$, and $\alpha' : A' \to A'$ is the unique map such that

$$\langle \Lambda_\mu, a\alpha'(a')\Lambda_\mu \rangle = \langle \Lambda_\mu, \alpha(a)a'\Lambda_\mu \rangle$$

for all $a \in A$ and all $a' \in A'$.

Note that in this definition we have

$$\mu' = \mu \circ j_\mu$$

where

$$j_\mu := J_\mu(\cdot)^* J_\mu \tag{1}$$

with J_μ the modular conjugation associated to μ.

The dual of a system is well-defined because of the following known result:

Theorem 2.5. *Let H and K be Hilbert spaces, M a (not necessarily unital) $*$-subalgebra of $B(H)$, and N a (not necessarily unital) C*-subalgebra of $B(K)$. Let $\Omega \in H$ with $\|\Omega\| = 1$ be cyclic for M, i.e. $M\Omega$ is dense in H, and let $\Lambda \in K$ be any unit vector. Set*

$$\mu : M \to \mathbb{C} : a \mapsto \langle \Omega, a\Omega \rangle$$

and

$$\nu : N \to \mathbb{C} : b \mapsto \langle \Lambda, b\Lambda \rangle .$$

Consider any positive linear $\eta : M \to N$, i.e. for a positive operator $a \in M$, we have that $\eta(a)$ is a positive operator. Assume furthermore that

$$\nu \circ \eta = \mu$$

Then there exists a unique map, called the dual *of η,*

$$\eta' : N' \to M'$$

such that

$$\langle \Omega, a\eta'(b')\Omega \rangle = \langle \Lambda, \eta(a)b'\Lambda \rangle$$

for all $a \in M$ and $b' \in N'$. The map η' is necessarily linear, positive and unital, i.e. $\eta'(1) = 1$, and $\|\eta'\| = 1$. Furthermore the following two results hold under two different sets of additional assumptions:

(a) If η is n-positive, then η' is n-positive as well. In particular, if η is completely positive, then η' is as well.

(b) If M and N contain the identity operators on H and K respectively, and η is unital (i.e. $\eta(1) = 1$), then it follows that

$$\mu' \circ \eta' = \nu',$$

where $\mu'(a') := \langle \Omega, a'\Omega \rangle$ and $\nu'(b') := \langle \Lambda, b'\Lambda \rangle$ for all $a' \in M'$ and $b' \in N'$. If in addition Λ is separating for N', then η' is faithful in the sense that when $\eta'(b'^*b') = 0$, it follows that $b' = 0$.

Proof. This is proven using [21, Lemma 1 on p. 53]. See [1, Proposition 3.1] and [8, Theorem 2.1]. □

Strictly speaking, η' is the dual of η with respect to μ and ν, but the states will always be implicitly clear.

In particular, with $M = N = A$ and $\Omega = \Lambda = \Omega_\mu$, we see from this theorem that the dual of the system $\mathbf{A}$ is well-defined.

Remark 2.6. If instead of the single map α we have a semigroup of u.c.p. maps $(\alpha_t)_{t\geq 0}$ leaving μ invariant, then $\alpha'_t \equiv (\alpha_t)'$ also gives a semigroup of u.c.p. maps leaving μ' invariant. The continuity or measurability properties of this dual semigroup (as function of t) will depend on those of α_t. Consider for example the standard assumption made for (continuous time) quantum Markov semigroups, namely that $t \mapsto \alpha_t(a)$ is σ-weakly continuous for every $a \in A$. Then it can be shown that $t \mapsto \varphi(\alpha'_t(a'))$ is continuous for every $a' \in A'$ and every normal state φ on A', so $t \mapsto \alpha'_t(a')$ is σ-weakly continuous for every $a' \in A'$. I.e. $(\alpha'_t)_{t\geq 0}$ is also a quantum Markov semigroup (with the same type of continuity property). If we were to include these assumptions in our definition of a system, then the dual of such a system would therefore still be a system. Our example in Section 7 will indeed be for semigroups indexed by $t \geq 0$, with even stronger continuity properties. Also, see for example the dynamical flows considered in [8], where weaker assumptions are made.

It is helpful to keep the following fact about duals in mind:

Corollary 2.7. *If in addition to the assumptions in Theorem 2.5 (prior to parts (a) and (b)), we have that M and N are von Neumann algebras, and Λ is cyclic for N', then we have*

$$\eta'' = \eta.$$

Proof. This follows directly from the theorem itself, since $\eta'' : M \to N$ is then the unique map such that $\langle \Lambda, b'\eta''(a)\Lambda \rangle = \langle \Omega, \eta'(b')a\Omega \rangle$ for all $a \in M$ and $b' \in N'$, while we know (again from the theorem) that $\langle \Lambda, b'\eta(a)\Lambda \rangle = \langle \Omega, \eta'(b')a\Omega \rangle$ for all $a \in M$ and $b' \in N'$. □

We also record the following simple result:

Proposition 2.8. *If in Theorem 2.5 we assume in addition that μ and ν are faithfull normal states on von Neumann algebras M and N (so Ω and Λ are the corresponding cyclic and separating vectors), then*

$$(j_\nu \circ \eta \circ j_\mu)' = j_\mu \circ \eta' \circ j_\nu$$

for the map $j_\nu \circ \eta \circ j_\mu : M' \to N'$ obtained in terms of Eq. (1).

Proof. It is a straightforward calculation to show that

$$\langle \Omega, a' j_\mu \circ \eta' \circ j_\nu(b)\Omega \rangle = \langle \Lambda, j_\nu \circ \eta \circ j_\mu(a')b\Lambda \rangle$$

for all $a' \in M'$ and $b \in N$. □

This proposition is related to KMS-duals and KMS-symmetry which appear in Sections 4 and 6 via the following definition:

Definition 2.9. The map $\eta^\sigma := j_\mu \circ \eta' \circ j_\nu : N \to M$ in Proposition 2.8 will be refered to as the *KMS-dual* of the positive linear map $\eta : M \to N$.

Combining Corollary 2.7 and Proposition 2.8, we see that

$$(\eta^\sigma)^\sigma = \eta.$$

Further remarks and references on the origins of KMS-duals can be found in Section 4.

Let us now finally turn to our main concern in this paper:

Definition 2.10. Let μ and ν be faithful normal states on the von Neumann algebras A and B respectively. A *coupling* of (A, μ) and (B, ν), is a state ω on the algebraic tensor product $A \odot B'$ such that

$$\omega(a \otimes 1) = \mu(a) \quad \text{and} \quad \omega(1 \otimes b') = \nu'(b')$$

for all $a \in A$ and $b \in B'$. We also call such an ω a coupling of μ and ν. Let $\mathbf{A}$ and $\mathbf{B}$ be systems. We say that $\mathbf{A}$ and $\mathbf{B}$ (in this order) are in *balance* with respect to a coupling ω of μ and ν, expressed in symbols as

$$\mathbf{A}\omega\mathbf{B},$$

if

$$\omega(\alpha(a) \otimes b') = \omega(a \otimes \beta'(b'))$$

for all $a \in A$ and $b' \in B'$.

Notice that this definition is in terms of the dual $\mathbf{B}'$ rather than in terms of $\mathbf{B}$ itself. To define balance in terms of $\omega(\alpha(a)\otimes b) = \omega(a\otimes \beta(b))$, for $a \in A$ and $b \in B$, turns out to be a less natural convention, in particular with regards to transitivity (see Section 5). Also, strictly speaking, saying that $\mathbf{A}$ and $\mathbf{B}$ are in balance, implies a direction, say from $\mathbf{A}$ to $\mathbf{B}$. These points will become more apparent in subsequent sections. For example, symmetry of balance will be explored in Section 4 in terms of KMS-symmetry of the dynamics α and β.

Remark 2.11. For systems given by quantum Markov semigroups $(\alpha_t)_{t\geq 0}$ and $(\beta_t)_{t\geq 0}$, instead of a single map for each system, we note that balance is defined by requiring $\omega(\alpha_t(a)\otimes b') = \omega(a\otimes \beta_t'(b'))$ at every $t \geq 0$.

Remark 2.12. For comparison to the theory of joinings [22, 23, 24], note that a joining of systems $\mathbf{A}$ and $\mathbf{B}$, where α and β $*$-automorphisms, is a state ω on $A \odot B$ such that $\omega(a\otimes 1) = \mu(a)$, $\omega(1\otimes b) = \nu(b)$ and $\omega\circ(\alpha\odot\beta) = \omega$. In addition [10] also assumes that $\omega\circ(\sigma_t^{\mu}\odot\sigma_t^{\nu}) = \omega$, where σ_t^{μ} and σ_t^{ν} are the modular groups associated to μ and ν. In [10], however, it is formulated in terms of the opposite algebra of B, which is in that sense somewhat closer to the conventions used above for balance.

3. Couplings and u.c.p. maps

Here we define and study a map E_ω associated to a coupling ω. This map is of fundamental importance in the theory of balance, as will be seen the next two sections. We do not consider systems in this section, only couplings. At the end of Section 5 we discuss how E_ω appears in the theory of correspondences. Some aspects of this section and the next are closely related to [10, Section 4] regarding joinings (see Remark 2.12).

Let ω be a coupling of (A, μ) and (B, ν) as in Definition 2.10. To clarify certain points later on in this and subsequent sections, we consider multiple (but necessarily unitarily equivalent) cyclic representations of a given von Neumann algebra and state. This requires us to have corresponding notations. We assume without loss of generality that (B, ν) is in its cyclic representation, denoted here by $(G_\nu, \mathrm{id}_B, \Lambda_\nu)$, which means that $(G_\nu, \mathrm{id}_{B'}, \Lambda_\nu)$ is a cyclic representation of (B', ν'). Similarly, we assume that (A, μ) is in the cyclic representation $(G_\mu, \mathrm{id}_A, \Lambda_\mu)$.

Denoting the cyclic representation of $(A\odot B', \omega)$ by $(H_\omega, \pi_\omega, \Omega_\omega)$, we obtain a second cyclic representation $(H_\mu, \pi_\mu, \Omega_\mu)$ of (A, μ) by setting

$$H_\mu := \overline{\pi_\omega(A\otimes 1)\Omega_\omega},\ \pi_\mu(a) := \pi_\omega(a\otimes 1)|_{H_\mu} \text{ and } \Omega_\mu := \Omega_\omega \tag{2}$$

for all $a \in A$, since

$$\langle \Omega_\mu, \pi_\mu(a)\Omega_\mu\rangle = \langle \Omega_\omega, \pi_\omega(a\otimes 1)\Omega_\omega\rangle = \omega(a\otimes 1) = \mu(a).$$

Similarly

(3) $H_\nu := \overline{\pi_\omega(1\otimes B')\Omega_\omega}$, $\pi_{\nu'}(b') := \pi_\omega(1\otimes b')|_{H_\nu}$ and $\Omega_\nu := \Omega_\omega$,

gives a second cyclic representation $(H_\nu, \pi_{\nu'}, \Omega_\nu)$ of (B', ν'). In particular H_μ and H_ν are subspaces of H_ω.

We can define a unitary equivalence

$$u_\nu : G_\nu \to H_\nu \tag{4}$$

from $(G_\nu, \mathrm{id}_{B'}, \Lambda_\nu)$ to $(H_\nu, \pi_{\nu'}, \Omega_\nu)$ by

$$u_\nu b'\Lambda_\nu := \pi_{\nu'}(b')\Omega_\nu$$

for all $b' \in B'$. Then

$$\pi_{\nu'}(b') = u_\nu b' u_\nu^* \tag{5}$$

for all $b' \in B'$. By setting

$$\pi_\nu(b) := u_\nu b u_\nu^* \tag{6}$$

for all $b \in B$, we also obtain a second cyclic representation $(H_\nu, \pi_\nu, \Omega_\nu)$ of (B, ν), which has the property

$$\pi_\nu(B) = \pi_{\nu'}(B')'$$

as is easily verified.

Let

$$P_\nu \in B(H_\omega)$$

be the projection of H_ω onto H_ν.

Proposition 3.1. *In terms of the notation above, we have*

$$u_\nu^*\iota_{H_\nu}^*\pi_\omega(a\otimes 1)\iota_{H_\nu}u_\nu = u_\nu^* P_\nu \pi_\omega(a\otimes 1)u_\nu \in B$$

for all $a \in A$, where $\iota_{H_\nu} : H_\nu \to H_\omega$ is the inclusion map, and $\iota_{H_\nu}^* : H_\omega \to H_\nu$ its adjoint.

Proof. Note that $P_\nu = \iota_{H_\nu}^*$, so indeed $u_\nu^*\iota_{H_\nu}^*\pi_\omega(a\otimes 1)\iota_{H_\nu}u_\nu = u_\nu^* P_\nu \pi_\omega(a\otimes 1)u_\nu$. We now show that this is in B.

For any $b' \in B'$ we have $\pi_\omega(1\otimes b')H_\nu^\perp \subset H_\nu^\perp$, since $\pi_\omega(1\otimes b'^*)H_\nu \subset H_\nu$. It follows that $P_\nu\pi_\omega(1\otimes b') = \pi_\omega(1\otimes b')P_\nu$. Therefore

$$\begin{aligned} P_\nu\pi_\omega(a\otimes 1)|_{H_\nu}\pi_{\nu'}(b') &= P_\nu\pi_\omega(a\otimes 1)\pi_\omega(1\otimes b')|_{H_\nu} \\ &= P_\nu\pi_\omega(1\otimes b')\pi_\omega(a\otimes 1)|_{H_\nu} \\ &= \pi_\omega(1\otimes b')P_\nu\pi_\omega(a\otimes 1)|_{H_\nu} \\ &= \pi_{\nu'}(b')P_\nu\pi_\omega(a\otimes 1)|_{H_\nu} \end{aligned}$$

for all $a \in A$ and $b' \in B'$. So $P_\nu\pi_\omega(a\otimes 1)|_{H_\nu} \in \pi_{\nu'}(B')' = \pi_\nu(B)$. Hence $u_\nu^* P_\nu \pi_\omega(a\otimes 1)u_\nu \in B$ by Eq. (6). □

This proposition proves part of the following result, which defines the central object of this section, namely the map $E_\omega : A \to B$.

Theorem 3.2. *In terms of the notation above we have the following well-defined linear map*

$$E_\omega : A \to B : a \mapsto u_\nu^* \iota_{H_\nu}^* \pi_\omega(a \otimes 1)\iota_{H_\nu} u_\nu \tag{7}$$

which is normal and completely positive. It has the following properties:

$$E_\omega(1) = 1$$

$$\|E_\omega\| = 1$$

$$\nu \circ E_\omega = \mu \tag{8}$$

Proof. The map $a \mapsto \pi_\omega(a \otimes 1)$ is completely positive, since it is a $*$-homomorphism. Therefore E_ω is completely positive, as it is the composition of the completely positive maps $a \mapsto \pi_\omega(a \otimes 1)$, $\iota_{H_\nu}^*(\cdot)\iota_{H_\nu}$ and $u_\nu^*(\cdot)u_\nu$.

From Eq. (7) we have $E_\omega(1) = u_\nu^* \iota_{H_\nu}^* \iota_{H_\nu} u_\nu = 1$ as well as $\|E_\omega\| \le 1$, thus it follows that $\|E_\omega\| = 1$. Furthermore,

$$\nu \circ E_\omega(a) = \langle \Lambda_\nu, E_\omega(a)\Lambda_\nu \rangle = \langle \Omega_\omega, \pi_\omega(a \otimes 1)\Omega_\omega \rangle = \omega(a \otimes 1) = \mu(a)$$

for all $a \in A$.

Lastly, Kadison's inequality, $E_\omega(a)^* E_\omega(a) \le E_\omega(a^*a)$, holds, since E_ω is a completely positive contraction, so $\nu(E_\omega(a)^* E_\omega(a)) \le \nu(E_\omega(a^*a)) = \mu(a^*a)$, for all $a \in A$. Hence, E_ω is normal, due to Theorem 2.3. □

Remark 3.3. The map $a \mapsto \pi_\omega(a \otimes 1)$ itself can also be shown to be normal (see for example the proof of [10, Theorem 3.3]).

We proceed by discussing some further general properties of E_ω which will be useful for us later.

The map E_ω is closely related to the *diagonal coupling* of ν with itself, which we now define: Let

$$\varpi_B : B \odot B' \to B(G_\nu)$$

be the unital $*$-homomorphism defined by extending $\varpi_B(b \otimes b') = bb'$ via the universal property of tensor products. Here $B(G_\nu)$ is the von Neumann algebra of all bounded linear operators $G_\nu \to G_\nu$. Now set

$$\delta_\nu(d) = \langle \Lambda_\nu, \varpi_B(d)\Lambda_\nu \rangle \tag{9}$$

for all $d \in B \odot B'$. Then δ_ν is a coupling of ν with itself, which we call the *diagonal coupling for* ν. In terms of this coupling we have the following characterization of E_ω which will often be used:

Proposition 3.4. *The map E_ω is the unique function from A to B such that*

$$\omega(a \otimes b') = \delta_\nu(E_\omega(a) \otimes b')$$

for all $a \in A$ and $b' \in B'$.

Proof. We simply calculate:

$$\begin{aligned}\delta_\nu(E_\omega(a)\otimes b') &= \langle \Lambda_\nu, E_\omega(a)b'\Lambda_\nu\rangle = \langle \Lambda_\nu, u_\nu^* P_\nu \pi_\omega(a\otimes 1)u_\nu b'\Lambda_\nu\rangle \\ &= \langle P_\nu\Omega_\nu, \pi_\omega(a\otimes 1)\pi_{\nu'}(b')\Omega_\nu\rangle \\ &= \langle \Omega_\nu, \pi_\omega(a\otimes b')\Omega_\nu\rangle = \omega(a\otimes b')\end{aligned}$$

for all $a\in A$ and $b'\in B'$. Secondly, suppose that for some $b_1, b_2\in B$ we have $\delta_\nu(b_1\otimes b') = \delta_\nu(b_2\otimes b')$ for all $b'\in B'$. Then $\langle b_1^*\Lambda_\nu, b'\Lambda_\nu\rangle = \langle b_2^*\Lambda_\nu, b'\Lambda_\nu\rangle$ for all $b'\in B'$, so $b_1^*\Lambda_\nu = b_2^*\Lambda_\nu$, since $B'\Lambda_\nu$ is dense in G_ν. But Λ_ν is separating for B, hence $b_1 = b_2$. Therefore E_ω is indeed the unique function as stated. □

This has three simple corollaries:

Corollary 3.5. *If ω_1 and ω_2 are both couplings of μ and ν, then $\omega_1 = \omega_2$ if and only if $E_{\omega_1} = E_{\omega_2}$.*

Corollary 3.6. *The map E_ω is faithful in the sense that if $E_\omega(a^*a) = 0$, then $a = 0$.*

Proof. If $E_\omega(a^*a) = 0$, then $\mu(a^*a) = \omega((a^*a)\otimes 1) = \delta_\nu(E_\omega(a^*a)\otimes 1) = 0$, but μ is faithful, hence $a = 0$. □

The latter also follows from Theorem 2.5(b) and $E''_\omega = E_\omega$.

The next corollary is relevant when we consider cases of trivial balance, i.e. balance with respect to $\mu\odot\nu'$, and will be applied toward the end of the next section, in relation to ergodicity:

Corollary 3.7. *Let ω be a coupling of (A,μ) and (B,ν). If $\omega = \mu\odot\nu'$, then $E_\omega(a) = \mu(a)1_B$ for all $a\in A$. Conversely, if $E_\omega(A) = \mathbb{C}1_B$, then $\omega = \mu\odot\nu'$.*

Proof. If $\omega = \mu\odot\nu'$, then $E_\omega(a) = \mu(a)1_B$ follows from Proposition 3.4. Conversely, again using Proposition 3.4, if $E_\omega(A) = \mathbb{C}1_B$, then $\omega(a\otimes b')1_B = \delta_\nu(E_\omega(a)\otimes b')1_B = E_\omega(a)\delta_\nu(1\otimes b') = E_\omega(a)\nu'(b')$. In particular, setting $b' = 1$, $E_\omega(a) = \mu(a)1_B$, so $\omega = \mu\odot\nu'$. □

Next we point out that u.c.p. maps from A to B with specific additional properties can be used to define couplings:

Proposition 3.8. *Let μ and ν be faithful normal states on the von Neumann algebras A and B respectively. Consider a linear map $E : A\to B$ and define a linear functional $\omega_E : A\odot B'\to\mathbb{C}$ by*

$$\omega_E := \delta_\nu\circ(E\odot \mathrm{id}_{B'}),$$

i.e.

$$\omega_E(a\otimes b') = \delta_\nu(E(a)\otimes b')$$

for all $a\in A$ and $b\in B'$. Then ω_E is a coupling of μ and ν if and only if E is completely positive, unital and $\nu\circ E = \mu$. In this case $E = E_{\omega_E}$.

Proof. Consider a completely positive linear map $E : A \to B$. Then $E \odot \mathrm{id}_{B'}$ is positive, so ω_E is positive, since δ_ν is. If we furthermore assume that E is unital, then $\omega_E(1\otimes 1) = 1$, so ω_E is a state. Assuming in addition that $\nu \circ E = \mu$, we conclude that $\omega_E(a\otimes 1) = \nu(E(a)) = \mu(a)$ and $\omega_E(1\otimes b') = \nu'(b')$, so ω_E is indeed a coupling of μ and ν. Because of Proposition 3.4 we necessarily have $E = E_{\omega_E}$. The converse is covered by Theorem 3.2 and Proposition 3.4. □

So in effect we can define couplings as maps E of the form described in this proposition.

Lastly we study the dual E'_ω of E_ω, given by Theorem 2.5. Given a coupling ω of μ and ν, we define

$$\omega' := \delta_{\mu'} \circ (E'_\omega \odot \mathrm{id}_A) : B' \odot A \to \mathbb{C}$$

where $\delta_{\mu'}(d') := \langle \Lambda_\mu, \varpi_{A'}(d')\Lambda_\mu \rangle$ for all $d' \in A' \odot A$, i.e. $\delta_{\mu'}(a' \otimes a) = \langle \Lambda_\mu, a' a \Lambda_\mu \rangle$. Since E'_ω is a u.c.p. map, it then follows, using Theorem 2.5, Proposition 3.8 and Proposition 3.4, that ω' is a coupling of ν' and μ' such that

$$\omega'(b' \otimes a) = \omega(a \otimes b') \tag{10}$$

for all $a \in A$ and $b' \in B'$.

Proposition 3.9. *In terms of the above notation we have*

$$E'_\omega = E_{\omega'} : B' \to A'$$

and

$$E_{\omega'}(b') = u_\mu^* \iota_{H_\mu}^* \pi_\omega(1 \otimes b') \iota_{H_\mu} u_\mu$$

for all $b' \in B'$, where $u_\mu : G_\mu \to H_\mu$ is the unitary operator defined by

$$u_\mu a \Lambda_\mu := \pi_\mu(a)\Omega_\mu$$

for all $a \in A$, $\iota_{H_\mu} : H_\mu \to H_\omega$ is the inclusion map, and $\iota_{H_\mu}^* : H_\omega \to H_\mu$ its adjoint.

Proof. That $E'_\omega = E_{\omega'}$, follows from the definition of ω' and Proposition 3.4 applied to ω' and $\delta_{\mu'}$ instead of ω and δ_ν.

Note that u_μ is defined in perfect analogy to u_ν in Eq. (4): As the cyclic representation of $(B' \odot A, \omega')$ we can use $(H_\omega, \pi_{\omega'}, \Omega_\omega)$ with $\pi_{\omega'}$ defined via

$$\pi_{\omega'}(b' \otimes a) := \pi_\omega(a \otimes b')$$

(and the universal property of tensor products) for all $b' \in B'$ and $a \in A$. Then, referring to the form of Eq. (3), we see that in the place of $(H_\nu, \pi_{\nu'}, \Omega_\nu)$ we have $(H_\mu, \pi_\mu, \Omega_\mu)$, as we would expect, since $\overline{\pi_{\omega'}(1 \otimes A)\Omega_\omega} = \overline{\pi_\omega(A \otimes 1)\Omega_\omega} = H_\mu$, $\pi_{\omega'}(1 \otimes a)|_{H_\mu} = \pi_\omega(a \otimes 1)|_{H_\mu} = \pi_\mu(a)$ and $\Omega_\mu = \Omega_\omega$ for all $a \in A$.

So u_μ plays the same role for $E_{\omega'}$ as u_ν does for E_ω, i.e. by definition (see Theorem 3.2)

$$E_{\omega'}(b') = u_\mu^* \iota_{H_\mu}^* \pi_{\omega'}(b' \otimes 1) \iota_{H_\mu} u_\mu = u_\mu^* \iota_{H_\mu}^* \pi_\omega(1 \otimes b') \iota_{H_\mu} u_\mu$$

for all $b' \in B'$. □

We are now in a position to apply E_ω to balance in subsequent sections. Also see Section 8 for brief remarks on how E_ω may be related to ideas from quantum information.

4. A characterization of balance

In this section we derive a characterization of balance in terms of the map E_ω from the previous section and consider some of its consequences, including a condition for symmetry of balance in terms of KMS-symmetry. This gives insight into the meaning and possible applications of balance. We continue with the notation from Section 3.

The dynamics α of a system $\mathbf{A}$ can be represented by a contraction U on H_μ defined as the unique extension of

$$U\pi_\mu(a)\Omega_\mu := \pi_\mu(\alpha(a))\Omega_\mu \tag{11}$$

for $a \in A$. Note that U is indeed a contraction, since from Kadison's inequality mentioned in Section 2, we have $\mu(\alpha(a)^*\alpha(a)) \leq \mu(a^*a)$. (It is also simple to check from the definition of the dual system that U^* is the corresponding representation of α' on H_μ.) Similarly

$$V\pi_\nu(b)\Omega_\nu := \pi_\nu(\beta(b))\Omega_\nu$$

for all $b \in B$, to represent β on H_ν by the contraction V.

Also set

$$P_\omega := P_\nu|_{H_\mu} : H_\mu \to H_\nu, \tag{12}$$

where P_ν is again the projection of H_ω onto H_ν. Note that from Eqs. (7) and (6) it follows that

$$P_\omega\pi_\mu(a)\Omega_\nu = \pi_\nu(E_\omega(a))\Omega_\nu \tag{13}$$

for all $a \in A$, so P_ω is a Hilbert space representation of E_ω.

The characterization of balance in terms of E_ω is the following:

Theorem 4.1. *For systems* $\mathbf{A}$ *and* $\mathbf{B}$*, let* ω *be a coupling of* μ *and* ν*. Then* $\mathbf{A}\omega\mathbf{B}$*, i.e.* $\mathbf{A}$ *and* $\mathbf{B}$ *are in balance with respect to* ω*, if and only if*

$$E_\omega \circ \alpha = \beta \circ E_\omega$$

holds, or equivalently, if and only if $P_\omega U = V P_\omega$*.*

Proof. We prove it on Hilbert space level. Note that P_ω as defined in Eq. (12) is the unique function $H_\mu \to H_\nu$ such that $\langle P_\omega x, y\rangle = \langle x, y\rangle$ for all $x \in H_\mu$ and $y \in H_\nu$. (This is a Hilbert space version of Proposition 3.4, but it follows directly from the definition of P_ω.)

Assume that **A** and **B** are in balance with respect to ω. Then, for $x = \pi_\mu(a)\Omega_\omega \in H_\mu$ and $y = \pi_{\nu'}(b')\Omega_\omega \in H_\nu$, where $a \in A$ and $b' \in B'$,

$$\begin{aligned}\langle P_\omega Ux, y\rangle &= \langle Ux, y\rangle = \langle \pi_\omega(\alpha(a)\otimes 1)\Omega_\omega, \pi_\omega(1\otimes b')\Omega_\omega\rangle \\ &= \langle \Omega_\omega, \pi_\omega(\alpha(a^*)\otimes b')\Omega_\omega\rangle = \omega(\alpha(a^*)\otimes b') \\ &= \omega(a^*\otimes \beta'(b')) = \langle \pi_\omega(a\otimes 1)\Omega_\mu, \pi_\omega(1\otimes\beta'(b'))\Omega_\omega\rangle \\ &= \langle x, V^*y\rangle = \langle P_\omega x, V^*y\rangle = \langle VP_\omega x, y\rangle\end{aligned}$$

which implies that $P_\omega U = VP_\omega$. Therefore, using Eqs. (7), (2) and (6), and since $u_\nu\Lambda_\nu = \Omega_\omega$,

$$\begin{aligned}E_\omega\circ\alpha(a)\Lambda_\nu &= u_\nu^* P_\omega\pi_\mu(\alpha(a))\Omega_\omega = u_\nu^* P_\omega U\pi_\mu(a)\Omega_\omega \\ &= u_\nu^* VP_\omega\pi_\mu(a)\Omega_\omega = u_\nu^* Vu_\nu E_\omega(a)u_\nu^*\Omega_\omega \\ &= u_\nu^* V\pi_\nu(E_\omega(a))\Omega_\omega = u_\nu^*\pi_\nu(\beta\circ E_\omega(a))\Omega_\omega \\ &= \beta\circ E_\omega(a)\Lambda_\nu\end{aligned}$$

but since Λ_ν is separating for B, this means that $E_\omega\circ\alpha(a) = \beta\circ E_\omega(a)$.

Conversely, if $E_\omega\circ\alpha = \beta\circ E_\omega$, then by Eq. (13),

$$\begin{aligned}P_\omega U\pi_\mu(a)\Omega_\mu &= P_\omega\pi_\mu(\alpha(a))\Omega_\omega = \pi_\nu(E_\omega(\alpha(a)))\Omega_\nu \\ &= \pi_\nu(\beta\circ E_\omega(a))\Omega_\omega = V\pi_\nu(E_\omega(a))\Omega_\omega \\ &= VP_\omega\pi_\mu(a)\Omega_\mu\end{aligned}$$

so $P_\omega U = VP_\omega$. Therefore, similar to the beginning of this proof,

$$\omega(\alpha(a^*)\otimes b') = \langle P_\omega Ux, y\rangle = \langle VP_\omega x, y\rangle = \omega(a^*\otimes\beta'(b'))$$

for all $a \in A$ and $b' \in B'$, as required. □

Remark 4.2. This theorem can be compared to the case of joinings in [10, Theorems 4.1 and 4.3]. Keep in mind that in [10] the dynamics of systems are given by $*$-automorphisms, and secondly an additional assumption is made involving the modular groups (see Remark 2.12). The u.c.p. map obtained in [10] from a joining then also intertwines the modular groups, not just the dynamics.

A natural question is whether or not balance is symmetric. I.e., are **A** and **B** in balance with respect to ω if and only if **B** and **A** are in balance with respect to some coupling (related in some way to ω)? Below we derive balance conditions equivalent to $\mathbf{A}\omega\mathbf{B}$, but where (duals of) the systems **A** and **B** appear in the opposite order. This is then used to find conditions under which balance is symmetric.

As before, let

$$j_\mu : B(G_\mu) \to B(G_\mu) : a \mapsto J_\mu a^* J_\mu,$$

where as in the previous section we assume that (A,μ) is in the cyclic representation $(G_\mu, \mathrm{id}_A, \Lambda_\mu)$ and J_μ is the corresponding modular conjugation. Similarly for j_ν.

Given a coupling ω of μ and ν, this allows us to define

$$\omega^\sigma := \delta_\mu \circ (E \odot \mathrm{id}_{A'}) : B \odot A' \to \mathbb{C},$$

where

$$E := j_\mu \circ E'_\omega \circ j_\nu : B \to A$$

and $\delta_\mu(d) := \langle \Lambda_\mu, \varpi_A(d)\Lambda_\mu \rangle$ for all $d \in A \odot A'$, i.e. $\delta_\mu(a \otimes a') = \langle \Lambda_\mu, aa'\Lambda_\mu \rangle$. Since j_μ is a anti-$*$-automorphism, the conjugate linear map $j_\mu^* : B(G_\mu) \to B(G_\mu)$ obtained by composing j_μ with the involution, i.e.

$$j_\mu^*(a) := j_\mu(a^*)$$

for all $a \in B(G_\mu)$, is completely positive in the sense that if it is applied entry-wise to elements of the matrix algebra $M_n(A)$, then it maps positive elements to positive elements for every n, just like complete positivity of linear maps. It follows that $E = j_\mu^* \circ E'_\omega \circ j_\nu^*$ is a u.c.p. map, since E'_ω is. Consequently, since $\mu \circ E = \mu' \circ E'_\omega \circ j_\nu = \nu' \circ j_\nu = \nu$, it follows from Proposition 3.8 that ω^σ is a coupling of ν and μ.

It is then also clear that

$$E_{\omega^\sigma} = E = j_\mu \circ E'_\omega \circ j_\nu \tag{14}$$

by applying Proposition 3.4. Therefore E_{ω^σ} is the KMS-dual of E_ω; see Definition 2.9. The KMS-dual of α is given by

$$\alpha^\sigma = j_\mu \circ \alpha' \circ j_\mu \tag{15}$$

and similarly for β. This means that

$$\langle \Lambda_\mu, a_1 j_\mu(\alpha^\sigma(a_2))\Lambda_\mu \rangle = \langle \Lambda_\mu, \alpha(a_1) j_\mu(a_2)\Lambda_\mu \rangle$$

for all $a_1, a_2 \in A$, which corresponds to the definition of the KMS-dual given in [28, Section 2], in connection with quantum detailed balance. (In [28], however, the KMS-dual is indicated by a prime rather than the symbol σ.) Also see [52] and [50, Proposition 8.3]. In the latter the KMS-dual is defined in terms of the modular conjugation as well, as is done above, rather than just in terms of an analytic continuation of the modular group, as is often done in other sources (including [28]).

Proposition 4.3. *In terms of the notation above,*

$$\mathbf{A}^\sigma := (A, \alpha^\sigma, \mu)$$

is a system, called the KMS-dual *of* $\mathbf{A}$.

Proof. Simply note that α^σ is indeed a u.c.p. map (by the same argument as for E above) such that $\mu \circ \alpha^\sigma = \mu' \circ \alpha' \circ j_\mu = \mu' \circ j_\mu = \mu$. □

Remark 4.4. For a QMS $(\alpha_t)_{t\geq 0}$ with the σ-weak continuity property as in Remark 2.6, we again have that the same σ-weak continuity property holds for $(\alpha_t^\sigma)_{t\geq 0}$ as well, where $\alpha_t^\sigma := (\alpha_t)^\sigma$ for every t. This follows from the corresponding property of $(\alpha'_t)_{t\geq 0}$.

In terms of this notation, we have the following consequence of Theorem 4.1:

Corollary 4.5. *For systems* $\mathbf{A}$ *and* $\mathbf{B}$*, let* ω *be a coupling of* μ *and* ν*. Then*

$$\mathbf{A}\omega\mathbf{B} \Leftrightarrow \mathbf{B}'\omega'\mathbf{A}' \Leftrightarrow \mathbf{B}^{\sigma}\omega^{\sigma}\mathbf{A}^{\sigma}.$$

Proof. By the definition of the dual of a map in Theorem 2.5 (which tells us that $(E_\omega \circ \alpha)' = \alpha' \circ E'_\omega$, etc.), as well as Proposition 3.9 and Eqs. (14) and (15), we have

$$E_\omega \circ \alpha = \beta \circ E_\omega \Leftrightarrow E_{\omega'} \circ \beta' = \alpha' \circ E_{\omega'} \Leftrightarrow E_{\omega^\sigma} \circ \beta^\sigma = \alpha^\sigma \circ E_{\omega^\sigma}$$

which completes the proof by Theorem 4.1. □

This is not quite symmetry of balance. However, we say that the system $\mathbf{A}$ (and also α itself) is *KMS-symmetric* when

$$\alpha^\sigma = \alpha \tag{16}$$

holds. If both α and β are KMS-symmetric, we see that

$$\mathbf{A}\omega\mathbf{B} \Leftrightarrow \mathbf{B}\omega^{\sigma}\mathbf{A},$$

which expresses symmetry of balance in this special case.

KMS-symmetry was studied in [37], [38] and [17], and in [29] it was considered in the context of the structure of generators of norm-continuous quantum Markov semigroups on $B(h)$ and standard quantum detailed balance conditions.

We have however not excluded the possibility that there is some coupling other than ω^σ that could be used to show symmetry of balance more generally. This possibility seems unlikely, given how natural the foregoing arguments and constructions are.

We end this section by studying some simple applications of balance that follow from Theorem 4.1 and the facts derived in the previous section.

First we consider *ergodicity* of a system $\mathbf{B}$, which we define to mean

$$B^\beta := \{b \in B : \beta(b) = b\} = \mathbb{C}1_B \tag{17}$$

in analogy to the case for $*$-automorphisms instead of u.c.p. maps. This is certainly not the only notion of ergodicity available; see for example [8] for an alternative definition which implies Eq. (17), because of [8, Lemma 2.1]. The definition we give here is however convenient to illustrate how balance can be applied: this form of ergodicity can be characterized in terms of balance, similar to how it is done in the theory of joinings (see [22, Theorem 3.3], [23, Theorem 2.1] and [10, Theorem 6.2]), as we now explain.

Definition 4.6. A system $\mathbf{B}$ is said to be *disjoint* from a system $\mathbf{A}$ if the only coupling ω with respect to which $\mathbf{A}$ and $\mathbf{B}$ (in this order) are in balance, is the trivial coupling $\omega = \mu \odot \nu'$.

In the next result, an *identity system* is a system $\mathbf{A}$ with $\alpha = \mathrm{id}_A$.

Proposition 4.7. *A system is ergodic if and only if it is disjoint from all identity systems.*

Proof. Suppose $\mathbf{B}$ is ergodic and $\mathbf{A}$ an identity system. If $\mathbf{A}\omega\mathbf{B}$ for some coupling ω, then $\beta \circ E_\omega = E_\omega$ by Theorem 4.1. So $E_\omega(A) = \mathbb{C}1_B$, since $\mathbf{B}$ is ergodic. By Proposition 3.7 we conclude that $\omega = \mu \odot \nu'$.

Conversely, suppose that $\mathbf{B}$ is disjoint from all identity systems. Recall that $A := B^\beta$ is a von Neumann algebra (see for example [10, Lemma 6.4] for a proof). Therefore $\mathbf{A} := (A, \mathrm{id}_A, \mu)$ is an identity system, where $\mu := \nu|_A$. Define a coupling of μ and ν by $\omega := \delta_\nu|_{A\odot B'}$ (see Eq. (9)), then from Proposition 3.4 we have $E_\omega = \mathrm{id}_A$. So $E_\omega \circ \alpha = \mathrm{id}_A = \beta \circ E_\omega$, implying that $\mathbf{A}$ and $\mathbf{B}$ are in balance with respect to ω by Theorem 4.1. Hence, by our supposition and Corollary 3.7, $B^\beta = E_\omega(A) = \mathbb{C}1_B$, which means $\mathbf{B}$ is ergodic. □

It seems plausible that some other ergodic properties can be similarly characterized in terms of balance, but that will not be pursued further in this paper.

Our second application is connected to non-equilibrium statistical mechanics, in particular the convergence of states to steady states. See for example the early papers [57] and [34] on the topic, as well as more recent papers like [31] and [26]. To clarify the connection between these results (which are expressed in terms of continuous time $t \geq 0$) and the result below, we formulate the latter in terms of continuous time as well. Compare it in particular to results in [34, Section 3]. It is an example of how properties of one system can be partially carried over to other systems via balance.

Proposition 4.8. *Assume that* $\mathbf{A}$ *and* $\mathbf{B}$ *are in balance with respect to* ω*. Suppose that*

$$\lim_{t\to\infty} \varkappa(\alpha_t(a)) = \mu(a)$$

for all normal states $\varkappa$ *on* A*, and all* $a \in A$*. Then*

$$\lim_{t\to\infty} \lambda(\beta_t(b)) = \nu(b)$$

for all normal states λ *on* B*, and all* $b \in E_\omega(A)$*.*

Proof. Applying Theorem 4.1 and setting $\varkappa := \lambda \circ E_\omega$, we have

$$\lim_{t\to\infty} \lambda(\beta_t(E_\omega(a))) = \lim_{t\to\infty} \varkappa(\alpha_t(a)) = \mu(a) = \nu(E_\omega(a))$$

for all $a \in A$, by Theorem 3.2. □

We expect various results of this sort to be possible, namely where two systems are in balance, and properties of the one then necessarily hold in a weaker form for the other.

Conversely, one can in principle use balance as a way to impose less stringent alternative versions of a given property, by requiring a system to be in balance with another system having the property in question. We expect that such conditions need not be directly comparable (and strictly weaker) than the property in question. This idea will be discussed further in relation to detailed balance in Section 6.

5. Composition of couplings and transitivity of balance

Here we show transitivity of balance: if **A** and **B** are in balance w.r.t. ω, and **B** and **C** are in balance w.r.t. ψ, then **A** and **C** are in balance with respect to a certain coupling obtained from ω and ψ, and denoted by $\omega \circ \psi$. The coupling $\omega \circ \psi$ is the composition of ω and ψ, as defined and discussed in detail below. Furthermore, we discuss the connection between couplings and correspondences in the sense of Connes.

Let ω be a coupling of (A, μ) and (B, ν), and let ψ be a coupling of (B, ν) and (C, ξ). Note that $E_\psi \circ E_\omega : A \to C$ is a u.c.p. map such that $\xi \circ E_\psi \circ E_\omega = \mu$ by Theorem 3.2. Therefore, by Proposition 3.8, setting

$$\omega \circ \psi := \delta_\xi \circ ((E_\psi \circ E_\omega) \odot \mathrm{id}_{C'}), \tag{18}$$

i.e.

$$\omega \circ \psi(a \otimes c') = \delta_\xi(E_\psi(E_\omega(a)) \otimes c')$$

for all $a \in A$ and $c \in C'$, we obtain a coupling $\omega \circ \psi$ of μ and ξ such that

$$E_{\omega \circ \psi} = E_\psi \circ E_\omega. \tag{19}$$

This construction forms the foundation for the rest of this section.

We call the coupling $\omega \circ \psi$ the *composition* of the couplings ω and ψ. We can view it as an analogue of a construction appearing in the theory of joinings in classical ergodic theory; see for example [36, Definition 6.9].

We can immediately give the main result of this section, namely that we have transitivity of balance in the following sense:

Theorem 5.1. *If* **A** *and* **B** *are in balance w.r.t.* ω*, and* **B** *and* **C** *are in balance w.r.t.* ψ*, then* **A** *and* **C** *are in balance w.r.t.* $\omega \circ \psi$*.*

Proof. By Theorem 4.1 we have $E_\omega \circ \alpha = \beta \circ E_\omega$ and $E_\psi \circ \beta = \gamma \circ E_\psi$, so

$$E_{\omega \circ \psi} \circ \alpha = E_\psi \circ \beta \circ E_\omega = \gamma \circ E_{\omega \circ \psi},$$

which again by Theorem 4.1 means that **A** and **C** are in balance w.r.t. $\omega \circ \psi$. □

In order to gain a deeper understanding of the transitivity of balance, we now study properties of the composition of couplings.

Proposition 5.2. *For the diagonal coupling* δ_ν *in Eq. (9), we have* $E_{\delta_\nu} = \mathrm{id}_B$*. Consequently* δ_ν *is the identity for composition of couplings in the sense that* $\delta_\nu \circ \psi = \psi$ *and* $\omega \circ \delta_\nu = \omega$*.*

Proof. From Proposition 3.3 it follows that $E_{\delta_\nu} = \mathrm{id}_B$. Hence, from Eq. (19), we obtain $E_{\delta_\nu \circ \psi} = E_\psi \circ E_{\delta_\nu} = E_\psi$ and $E_{\omega \circ \delta_\nu} = E_{\delta_\nu} \circ E_\omega = E_\omega$, which concludes the proof by Corollary 3.5. □

In order to treat further properties of $\omega \circ \psi$ and the connection with the theory of correspondences, we need to set up the relevant notation:

Continuing with the notation in the previous two sections, also assuming (C, ξ) to be in its cyclic representation $(G_\xi, \mathrm{id}_C, \Lambda_\xi)$, and denoting the cyclic representation of $(B \odot C', \psi)$ by $(K_\psi, \varphi_\psi, \Psi_\psi)$, it follows that

$$K_\nu := \overline{\pi_\psi(B \otimes 1)\Psi_\psi},\ \varphi_\nu(b) := \varphi_\psi(b \otimes 1)|_{K_\nu} \text{ and } \Psi_\nu := \Psi_\psi$$

gives a third cyclic representation $(K_\nu, \varphi_\nu, \Lambda_\nu)$ of (B, ν), and that

$$K_\xi := \overline{\pi_\psi(1 \otimes C')\Psi_\psi},\ \varphi_{\xi'}(c') := \varphi_\psi(1 \otimes c')|_{K_\xi} \text{ and } \Psi_\xi := \Psi_\psi \tag{20}$$

gives a cyclic representation $(K_\xi, \varphi_{\xi'}, \Psi_\xi)$ of (C', ξ'). Note that to help keep track of where we are, we use the symbol K instead of H for the Hilbert spaces originating from ψ (as opposed to ω), and similarly we use φ instead of π, and Ψ instead of Ω.

We can define a unitary equivalence

$$v_\nu : G_\nu \to K_\nu \tag{21}$$

from $(G_\nu, \mathrm{id}_B, \Lambda_\nu)$ to $(K_\nu, \varphi_\nu, \Psi_\nu)$ by

$$v_\nu b \Lambda_\nu := \varphi_\nu(b)\Psi_\nu$$

for all $b \in B$. Then

$$\varphi_\nu(b) := v_\nu b v_\nu^*$$

for all $b \in B$.

By Theorem 3.2 we can then define the normal u.c.p. map $E_{\psi'} : C' \to B'$. By Proposition 3.9 this map is the dual E'_ψ of E_ψ, and we can write it as

$$E'_\psi : C' \to B' : c' \mapsto v_\nu^* \iota_{K_\nu}^* \varphi_\psi(1 \otimes c') \iota_{K_\nu} v_\nu = v_\nu^* Q_\nu \varphi_\psi(1 \otimes c') v_\nu \tag{22}$$

where Q_ν is the projection of K_ψ onto K_ν, and $Q_\nu = \iota_{K_\nu}^*$ with $\iota_{K_\nu} : K_\nu \to K_\psi$ the inclusion map.

The coupling $\omega \circ \psi$ can now be expressed in various ways:

Proposition 5.3. *The coupling $\omega \circ \psi$ is given by the following formulas:*

$$\omega \circ \psi = \delta_\nu \circ (E_\omega \odot E'_\psi) \tag{23}$$

and

$$\omega \circ \psi = \delta_\mu \circ (\mathrm{id}_A \odot (E'_\omega \circ E'_\psi))$$

in terms of Eq. (9), as well as

$$\omega \circ \psi(a \otimes c') = \psi(E_\omega(a) \otimes c') = \omega(a \otimes E'_\psi(c')) \tag{24}$$

and

$$\omega \circ \psi(a \otimes c') = \langle u_\nu^* P_\nu \pi_\mu(a^*)\Omega_\omega, v_\nu^* Q_\nu \varphi_{\xi'}(c')\Psi_\psi \rangle \tag{25}$$

(in the inner product of the Hilbert space G_ν) for all $a \in A$ and $c' \in C'$.

Proof. From Eqs. (18) and (9), and Theorem 2.5, we have

$$\begin{aligned} \omega \circ \psi(a \otimes c') &= \langle \Lambda_\xi, E_\psi(E_\omega(a))c'\Lambda_\xi\rangle \\ &= \left\langle \Lambda_\nu, E_\omega(a)E'_\psi(c')\Lambda_\nu \right\rangle \end{aligned} \tag{26}$$

from which Eq. (23) follows. Continuing with the last expression above, we respectively have by Theorem 2.5 that

$$\begin{aligned} \omega \circ \psi(a \otimes c') &= \left\langle \Lambda_\mu, aE'_\omega(E'_\psi(c'))\Lambda_\mu \right\rangle \\ &= \delta_\mu \circ (\mathrm{id}_A \odot (E'_\omega \circ E'_\psi))(a \otimes c'), \end{aligned}$$

by Proposition 3.4 that

$$\omega \circ \psi(a \otimes c') = \omega(a \otimes E'_\psi(c'))$$

and by Proposition 3.9 that

$$\begin{aligned} \omega \circ \psi(a \otimes c') &= \langle \Lambda_\nu, E_{\psi'}(c')E_\omega(a)\Lambda_\nu\rangle \\ &= \psi'(c' \otimes E_\omega(a)) \\ &= \psi(E_\omega(a) \otimes c'), \end{aligned}$$

where in the second line we again applied Proposition 3.4, while the last line follows from the definition of ψ', as in Eq. (10).

On Hilbert space level we again have from Eq. (26) that

$$\begin{aligned} \omega \circ \psi(a \otimes c') &= \left\langle E_\omega(a^*)\Lambda_\nu, E'_\psi(c')\Lambda_\nu \right\rangle \\ &= \langle u_\nu^* P_\nu \pi_\omega(a^* \otimes 1)u_\nu\Lambda_\nu, v_\nu^* Q_\nu \varphi_\psi(1 \otimes c')v_\nu\Lambda_\nu\rangle \\ &= \langle u_\nu^* P_\nu \pi_\mu(a^*)\Omega_\omega, v_\nu^* Q_\nu \varphi_{\xi'}(c')\Psi_\psi\rangle \end{aligned}$$

for all $a \in A$ and $c' \in C'$, using Theorem 3.2 (and Proposition 3.1) as well as Eqs. (22), (2) and (20). □

At the end of this section $\omega \circ \psi$ will also be expressed in terms of the theory of relative tensor products of bimodules; see Proposition 5.7.

Next we consider triviality of transitivity, namely when $\omega \circ \psi = \mu \odot \xi'$, in which case we also say that the couplings ω and ψ are *orthogonal*, in analogy to the case of classical joinings [36, Definition 6.9]. We first note the following:

Proposition 5.4. *If either* $\omega = \mu \odot \nu'$ *or* $\psi = \nu \odot \xi'$, *then* $\omega \circ \psi = \mu \odot \xi'$.

Proof. By Proposition 3.4, $E_{\mu\odot\nu'} = \mu(\cdot)1_B$ and $E_{\nu\odot\xi'} = \nu(\cdot)1_C$, so $(\mu\odot\nu')\circ\psi(a\otimes c') = \delta_\xi(\mu(a)1_C\otimes c') = \mu(a)\xi'(c')$ and $\omega\circ(\nu\odot\xi')(a\otimes c') = \delta_\xi(\nu(E_\omega(a))1_C \otimes c') = \mu(a)\xi'(c')$ according to Eq. (18) and Theorem 3.2. □

However, as will be seen by example in Section 7, in general it is possible that $\omega \circ \psi = \mu \odot \xi'$ even when $\omega \neq \mu \odot \nu'$ and $\psi \neq \nu \odot \xi'$. In order for $\omega \circ \psi \neq \mu \odot \xi'$ to hold, there has to be sufficient "overlap" between ω and ψ. The following makes this precise on Hilbert space level and also explains the use of the term "orthogonal" above:

Proposition 5.5. *We have $\omega \circ \psi = \mu \odot \xi'$ if and only if*

$$u_\nu^*[P_\nu H_\mu \ominus \mathbb{C}\Omega_\omega] \perp v_\nu^*[Q_\nu K_\xi \ominus \mathbb{C}\Psi_\psi]$$

in the Hilbert space G_ν (see Section 3), where P_ν and Q_ν are the projections of H_ω onto H_ν and K_ψ onto K_ν respectively, and u_ν and v_ν are the unitaries defined above (see Eqs. (4) and (21)).

Proof. In terms of the projections P_{Ω_ω} and Q_{Ψ_ψ} of H_ω and K_ψ onto $\mathbb{C}\Omega_\omega$ and $\mathbb{C}\Psi_\psi$ respectively, we have

$$\begin{aligned}\left\langle u_\nu^* P_{\Omega_\omega}\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q_{\Psi_\psi}\varphi_{\xi'}(c')\Psi_\psi\right\rangle &= \left\langle \left\langle \Omega_\omega, \pi_\mu(a^*)\Omega_\omega\right\rangle u_\nu^*\Omega_\omega, \left\langle \Psi_\psi, \varphi_{\xi'}(c')\Psi_\psi\right\rangle v_\nu^*\Psi_\psi\right\rangle \\ &= \mu(a)\xi'(c')\left\langle \Lambda_\nu, \Lambda_\nu\right\rangle \\ &= \mu \odot \xi'(a \otimes c')\end{aligned}$$

for all $a \in A$ and $c' \in C'$. In terms of $P := P_\nu - P_{\Omega_\omega}$ and $Q := Q_\nu - Q_{\Psi_\psi}$, it then follows from Eq. (25) that

$$\begin{aligned}&\omega \circ \psi(a \otimes c') - \mu \odot \xi'(a \otimes c') \\ &= \left\langle u_\nu^* P\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q\varphi_{\xi'}(c')\Psi_\psi\right\rangle \\ &+ \left\langle u_\nu^* P\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q_{\Psi_\psi}\varphi_{\xi'}(c')\Psi_\psi\right\rangle + \left\langle u_\nu^* P_{\Omega_\omega}\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q\varphi_{\xi'}(c')\Psi_\psi\right\rangle \\ &= \left\langle u_\nu^* P\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q\varphi_{\xi'}(c')\Psi_\psi\right\rangle.\end{aligned}$$

For the last line we used $u_\nu^* P H_\omega = G_\nu \ominus \mathbb{C}\Lambda_\nu$ and $v_\nu^* Q_{\Psi_\psi} K_\psi = \mathbb{C}\Lambda_\nu$ to obtain the one term as zero, while the other term is zero, since $v_\nu^* Q K_\psi = G_\nu \ominus \mathbb{C}\Lambda_\nu$ and $u_\nu^* P_{\Omega_\omega} H_\omega = \mathbb{C}\Lambda_\nu$. Therefore $\omega \circ \psi(a \otimes c') - \mu \odot \xi'(a \otimes c')$ is zero for all $a \in A$ and $c' \in C'$ if and only if $u_\nu^*[P_\nu H_\mu \ominus \mathbb{C}\Omega_\omega] \perp v_\nu^*[Q_\nu K_\xi \ominus \mathbb{C}\Psi_\psi]$. □

To conclude this section, we discuss bimodules and correspondences, in particular showing how $\omega \circ \psi$ can be expressed in terms of the relative tensor product of bimodules obtained from ω and ψ. The goal is to give an indication of the connection between couplings and correspondences. Also see [10] for a related discussion of correspondences in the context of joinings.

The theory of correspondences was originally developed by Connes, but never published in full, although it is discussed briefly in his book [18, Appendix V.B]. In short, a *correspondence* from one von Neumann algebra, M, to another, N, is a M-N-bimodule (where the direction *from M to N*, is the convention used in this paper).

See for example [60, Section IX.3] and [30] for details on the relative tensor product, but also [55] for some of the early work on this topic. We only outline some of the most pertinent aspects of relative tensor products, and the reader is refered to these sources, in particular [60, Section IX.3], for a more systematic exposition.

As before, let

$$j_\nu(b) := J_\nu b^* J_\nu$$

for all $b \in B(G_\nu)$, with $J_\nu : G_\nu \to G_\nu$ the modular conjugation associated with (B, Λ_ν). Similarly, with (C, ξ) in its cyclic representation $(G_\xi, \mathrm{id}_C, \Lambda_\xi)$, let

$$j_\xi(c) := J_\xi c^* J_\xi$$

for all $c \in B(G_\xi)$, with $J_\xi : G_\xi \to G_\xi$ the modular conjugation associated with (C, Λ_ξ).

Given a coupling ω of the systems **A** and **B**, we can view $H = H_\omega$ as an A-B-bimodule by setting

$$\pi_H(a) := \pi_\omega(a \otimes 1)$$

and

$$\pi'_H(b) := \pi_\omega(1 \otimes j_\nu(b)),$$

and writing

$$axb := \pi_H(a)\pi'_H(b)x$$

for all $a \in A$, $b \in B$, and $x \in H$. As already mentioned in Remark 3.3, π_H is normal, as required for it to give a left A-moduled, and similarly π'_H gives a normal right action of B on H; again see [10, Theorem 3.3]. When viewing H as the A-B-bimodule thus defined, we also denote it by ${}_AH_B$. This module is therefore an example of a correspondence from A to B.

From $K = K_\psi$ we analogously obtain the B-C-bimodule ${}_BK_C$ via π_K and π'_K given by

$$\pi_K(b) := \varphi_\psi(b \otimes 1)$$

and

$$\pi'_K(c) := \varphi_\psi(1 \otimes j_\xi(c))$$

which enables us to write

$$byc := \pi_K(b)\pi'_K(c)y$$

for all $b \in B$, $c \in C$, and $y \in K$.

Now we form the relative tensor product (see [60, Definition IX.3.16])

$${}_AX_C := H \otimes_\nu K$$

with respect to the faithful normal state ν. This is also a Hilbert space (its inner product will be discussed below) and, as the notation on the left suggests, the relative tensor product is itself a A-C-bimodule. This is a special case of [60, Corollary IX.3.18]. The reason it works is that since H is a A-B-bimodule, any element of $\pi_H(A)$ can be viewed as an element of $\mathcal{L}(H_B)$, the space of all bounded (in the usual sense of linear operators on Hilbert spaces) right B-module maps. Similarly for the right action of C. So ${}_AX_C$ is a correspondence from A to C, which can be viewed as the compostion of the correspondences ${}_AH_B$ and ${}_BK_C$.

As one may expect, the actions of A and C on $H \otimes_\nu K$ are given by

$$a(x \otimes_\nu y)c = (ax) \otimes_\nu (yc)$$

for all $a \in A$ and $c \in C$. However, in general this does not hold for all $x \in H$ and $y \in K$. In fact the elementary tensor $x \otimes_\nu y$ does not exist for all $x \in H$ and $y \in K$. However, it does work if we restrict either x or y to a certain dense subspace, say $x \in \mathfrak{D}(H,\nu) \subset H$ and $y \in K$. (See below for further details on the space $\mathfrak{D}(H,\nu)$.) We correspondingly use $x \in H$ and $y \in \mathfrak{D}'(K,\nu) \subset K$ if we rather want to restrict y to a dense subspace of K.

In particular we have $\Omega_\omega \in \mathfrak{D}(H,\nu)$ and $\Psi_\psi \in \mathfrak{D}'(K,\nu)$, so we set

$$\Omega := \Omega_\omega \otimes_\nu \Psi_\psi \in H \otimes_\nu K,$$

which we use to define a state, denoted by $\omega \diamond \psi$, on $A \odot C'$ as follows:

$$\omega \diamond \psi(d) := \langle \Omega, \pi_X(d)\Omega \rangle \tag{27}$$

for all $d \in A \odot C'$, where π_X is the representation of $A \odot C'$ on ${}_AX_C$ given in terms of its bimodule structure by

$$\pi_X(a \otimes c')x := axj_\xi(c')$$

for all $x \in {}_AX_C$. Below we show that $\omega \diamond \psi = \omega \circ \psi$, so have the composition of couplings expressed in terms of the relative tensor product of bimodules, i.e. in terms of the composition of correspondences.

We first review the inner product of the relative tensor product in more detail, in order to clarify its use below. Write

$$\eta'_\nu(b) := j_\nu(b)\Lambda_\nu = J_\nu b^* \Lambda_\nu \tag{28}$$

for all $b \in B$.

For every $x \in \mathfrak{D}(H,\nu)$, define the bounded linear operator $L_\nu(x) : G_\nu \to H$ by settting

$$L_\nu(x)\eta'_\nu(b) = xb \equiv \pi'_H(b)x$$

for all $b \in B$, and uniquely extending to G_ν. We note that the space $\mathfrak{D}(H,\nu)$ is defined to ensure that $L_\nu(x)$ is indeed bounded:

$$\mathfrak{D}(H,\nu) = \{x \in H : \|xb\| \leq k_x \|\eta'_\nu(b)\| \text{ for all } b \in B, \text{ for some } k_x \geq 0\}$$

It then follows that $L_\nu(x_1)^* L_\nu(x_2) \in B$ for all $x_1, x_2 \in \mathfrak{D}(H,\nu)$. The space $H \otimes_\nu K$ and its inner product is obtained from a quotient construction such that we have

$$\langle x_1 \otimes_\nu y_1, x_2 \otimes_\nu y_2 \rangle = \langle y_1, \pi_K(L_\nu(x_1)^* L_\nu(x_2))y_2 \rangle_K \tag{29}$$

for $x_1, x_2 \in \mathfrak{D}(H,\nu)$ and $y_1, y_2 \in K$, where for emphasis we have denoted the inner product of K by $\langle \cdot, \cdot \rangle_K$. This is the "left" version, but there is also a corresponding "right" version of this formula for the inner product (see [60, Section IX.3]). It can be shown from the definition of $\mathfrak{D}(H,\nu)$, that $\pi_H(a)\pi_\nu(b)\Omega_\omega \in D(H,\nu)$ for all $a \in A$ and $b \in B$, from which in turn it follows that $\mathfrak{D}(H,\nu)$ is dense in H, and that $\Omega_\omega \in \mathfrak{D}(H,\nu)$. Similarly $D'(K,\nu)$, which is defined analogously, is dense in K.

From this short review of the inner product, we can show that it has the following property:

Proposition 5.6. *In* $H \otimes_\nu K$*,*

$$\langle a_1\Omega c_1, a_2\Omega c_2\rangle = \psi(E_\omega(a_1^*a_2) \otimes j_\xi(c_2c_1^*)) \tag{30}$$

for $a_1, a_2 \in A$ *and* $c_1, c_2 \in C$*.*

Proof. Firstly, we obtain a formula for $L_\nu(x)$ for elements of the form $x = \pi_H(a)\pi_\nu(b)\Omega_\omega \in D(H,\nu)$, where $a \in A$ and b. For all $b_1 \in B$ we have

$$\begin{aligned} L_\nu(x)\eta'_\nu(b_1) &= \pi'_H(b_1)\pi_H(a)\pi_\nu(b)\Omega_\omega \\ &= \pi_H(a)\pi_\nu(b)\pi_{\nu'}(j_\nu(b_1))\Omega_\omega \\ &= \pi_H(a)\pi_\nu(b)u_\nu\eta'_\nu(b_1), \end{aligned}$$

by Eqs. (5) and (28), which means that

$$L_\nu(\pi_H(a)\pi_\nu(b)\Omega_\omega) = \pi_H(a)\pi_\nu(b)u_\nu. \tag{31}$$

Applying the special case $L_\nu(\pi_H(a)\Omega_\omega) = \pi_H(a)u_\nu$ of this formula, for $a_1, a_2 \in A$ we have

$$\begin{aligned} L_\nu(\pi_H(a_1)\Omega_\omega)^*L_\nu(\pi_H(a_2)\Omega_\omega) &= u_\nu^*P_\nu\pi_H(a_1^*a_2)u_\nu \\ &= E_\omega(a_1^*a_2). \end{aligned}$$

by Theorem 3.2 and Proposition 3.1. From Eq. (29) we therefore have

$$\begin{aligned} \langle a_1\Omega c_1, a_2\Omega c_2\rangle &= \langle \pi'_K(c_1)\Psi_\psi, \pi_K(E_\omega(a_1^*a_2))\pi'_K(c_2)\Psi_\psi\rangle_K \\ &= \langle \Psi_\psi, \pi_K(E_\omega(a_1^*a_2))\pi'_K(c_2c_1^*)\Psi_\psi\rangle_K \\ &= \langle \Psi_\psi, \varphi_\psi(E_\omega(a_1^*a_2) \otimes j_\xi(c_2c_1^*))\Psi_\psi\rangle_K \\ &= \psi(E_\omega(a_1^*a_2) \otimes j_\xi(c_2c_1^*)). \end{aligned}$$

□

Now we can confirm that Eq. (27) is indeed equivalent to the original definition Eq. (18):

Corollary 5.7. *We have*

$$\omega \diamond \psi = \omega \circ \psi$$

in terms of the definitions Eq. (27) and Eq. (18).

Proof. From Eq. (27)

$$\begin{aligned} \omega \diamond \psi(a \otimes c') &= \langle \Omega, \pi_X(a \otimes c')\Omega\rangle = \langle \Omega, a\Omega j_\xi(c')\rangle \\ &= \psi(E_\omega(a) \otimes c')) \end{aligned}$$

by Eq. (30), for all $a \in A$ and $c' \in C'$. By Eq. (24), $\omega \diamond \psi = \omega \circ \psi$. □

So we have $\omega \circ \psi$ expressed in terms of the vector $\Omega \in H \otimes_\nu K$. Note, however, that in general $H \otimes_\nu K$ is not the GNS Hilbert space for the state $\omega \circ \psi$, although the former contains the latter. Consider for example the simple case where $\omega = \mu \odot \nu'$ and $\psi = \nu \odot \xi'$. Then, by Proposition 5.4, $\omega \circ \psi = \mu \odot \xi'$, and the GNS Hilbert space obtained from this state is $G_\mu \otimes G_\xi$, whereas $H \otimes_\nu K = G_\mu \otimes G_\nu \otimes G_\xi$.

When $(A, \mu) = (B, \nu)$ and ω is the diagonal coupling δ_ν in Eq. 9), then by [60, Proposition IX.3.19], ${}_A X_C$ is isomorphic to ${}_B K_C$, so in this case the correspondence ${}_A H_B$ acts as an identity from the left. Similarly from the left when ψ is the diagonal coupling. This is the correspondence version of Proposition 5.2.

Lastly, by Eq. (31) we have $L_\nu(\Omega_\omega) = \iota_{H_\nu} u_\nu$, therefore $L_\nu(\Omega_\omega)^* = u_\nu^* P_\nu$, which by Theorem 3.2 means that

$$E_\omega(a) = L_\nu(\Omega_\omega)^* \pi_H(a) L_\nu(\Omega_\omega)$$

for all $a \in A$. This is the form in which E_ω has appeared in the theory of correspondences, as a special case of maps of the form $a \mapsto L_\nu(x)^* \pi_H(a) L_\nu(x)$ for arbitrary $x \in \mathfrak{D}(H, \nu)$; see for example [53, Section 1.2].

6. Balance, detailed balance and non-equilibrium

Our main goal in this section is to suggest how balance can be used to define conditions that generalize detailed balance. We then speculate on how this may be of value in studying non-equilibrium steady states. In order to motivate these generalized conditions, we present a specific instance of how detailed balance can be expressed in terms of balance. We focus on only one form of detailed balance, namely standard quantum detailed balance with respect to a reversing operation, as defined in [29, Definition 3 and Lemma 1] and [28, Definition 1]. This form of detailed balance has only appeared in the literature relatively recently. The origins of quantum detailed balance, on the other hand, can be found in the papers [6], [7], [15], [45] and [48].

The basic idea of this section should also apply to properties other than detailed balance conditions, as will be explained.

We begin by noting the following simple fact in terms of the diagonal coupling δ_μ (see Eq. (9)):

Proposition 6.1. *A system* $\mathbf{A}$ *is in balance with itself with respect to the diagonal coupling* δ_μ*, i.e.* $\delta_\mu(\alpha(a) \otimes a') = \delta_\mu(a \otimes \alpha'(a'))$ *for all* $a \in A$ *and* $a' \in A'$*. Conversely, if two systems* $\mathbf{A}$ *and* $\mathbf{B}$*, with* $(A, \mu) = (B, \nu)$*, are in balance with respect to the diagonal coupling* δ_μ*, then* $\mathbf{A} = \mathbf{B}$*, i.e.* $\alpha = \beta$*.*

Proof. The first part is simply the definition of the dual (see Definition 2.4 and Theorem 2.5). The second part follows from the uniqueness of the dual, given by Theorem 2.5; alternatively use Theorem and 4.1 Proposition 5.2. □

So, if **A** and **B** are in balance with respect to the diagonal coupling and one of the systems has some property, then the other system has it as well, since the systems are necessarily the same.

One avenue of investigation is therefore to define weaker versions of a given property by demanding only that a system is in balance with another system with the given property, with respect to a coupling that is not necessarily the diagonal coupling. In particular we then do not need to assume that the two systems have the same algebra and state.

We demonstrate this idea below for a specific property, namely standard quantum detailed balance with respect to a reversing operation. In order to do so, we discuss this form of detailed balance along with Θ-KMS-duals:

Definition 6.2. Consider a system **A**. A *reversing operation* for **A** (or for (A, μ)), is a $*$-antihomorphism $\Theta : A \to A$ (i.e. Θ is linear, $\Theta(a^*) = \Theta(a)^*$, and $\Theta(a_1a_2) = \Theta(a_2)\Theta(a_1)$) such that $\Theta^2 = \mathrm{id}_A$ and $\mu \circ \Theta = \mu$. Furthermore we define the Θ-*KMS-dual*

$$\alpha^\Theta := \Theta \circ \alpha^\sigma \circ \Theta$$

of α in terms of the KMS-dual $\alpha^\sigma = j_\mu \circ \alpha' \circ j_\mu$ in Eq. (15).

The Θ-KMS-dual was introduced in [13] in the context of systems on $B(H)$, with H a separable Hilbert space, in order to study deviation from standard quantum detailed balance with respect to a reversing operation.

Using the Θ-KMS-dual, we can define this form of detailed balance in general as follows:

Definition 6.3. A system **A** satisfies *standard quantum detailed balance with respect to the reversing operation* Θ for (A, μ), or Θ-*sqdb*,when $\alpha^\Theta = \alpha$.

Note that [28] defines Θ-sqdb by $\alpha^\sigma = \Theta \circ \alpha \circ \Theta$, which is equivalent to the above definition, simply because $\Theta^2 = \mathrm{id}_A$.

To complete the picture, we state some straightforward properties related to reversing operations Θ and the Θ-KMS-dual:

Proposition 6.4. *Given a reversing operation Θ as in Definition 6.2, we define an anti-unitary operator $\theta : G_\mu \to G_\mu$ by extending*

$$\theta a \Lambda_\mu := \Theta(a^*)\Lambda_\mu$$

which in particular gives $\theta^2 = 1$ and $\theta\Lambda_\mu = \Lambda_\mu$. Then

$$\Theta(a) = \theta a^* \theta$$

for all $a \in A$, and consequently Θ is normal. This allows us to define

$$\Theta' : A' \to A' : a' \mapsto \theta a'^* \theta$$

which is the dual of Θ *in the sense that*

$$\langle \Lambda_\mu, a\Theta'(a')\Lambda_\mu \rangle = \langle \Lambda_\mu, \Theta(a)a'\Lambda_\mu \rangle$$

for all $a \in A$ *and* $a' \in A'$. *We also have*

$$\theta J_\mu = J_\mu \theta$$

from which

$$\alpha^\Theta = (\Theta \circ \alpha \circ \Theta)^\sigma$$

and

$$(\alpha^\Theta)^\Theta = \alpha$$

follow.

Proof. The first sentence is simple. From the definition of θ and the properties of Θ, $\theta\Lambda_\mu = \Lambda_\mu$ it follows that

$$\theta a^* \theta b \Lambda_\mu = \Theta((a^*\Theta(b^*))^*)\Lambda_\mu = \Theta(a)b\Lambda_\mu$$

for all $a, b \in A$, so $\Theta(a) = \theta a^* \theta$. Normality (i.e. σ-weak continuity) follows from this and the definition of the σ-weak topology. For $a \in A$ and $a' \in A'$ we now have $a\theta a'\theta = \theta\Theta(a^*)a'\theta = \theta a'\Theta(a^*)\theta = \theta a'\theta a$, hence $\theta a'\theta \in A'$. So Θ' is well-defined, and that it is the dual of Θ follows easily.

Denoting the closure of the operator

$$A\Lambda_\mu \to A\Lambda_\mu : a\Lambda_\mu \mapsto a^*\Lambda_\mu$$

by $S_\mu = J_\mu \Delta_\mu^{1/2}$, as usual in Tomita-Takesaki theory, we obtain $S_\mu = \theta S_\mu \theta = \theta J_\mu \theta \theta \Delta_\mu^{1/2} \theta$, hence $\theta J_\mu \theta = J_\mu$ by the uniqueness of polar decomposition, proving $\theta J_\mu = J_\mu \theta$.

Then by definition

$$\begin{aligned}\alpha^\Theta &= \Theta \circ j_\mu \circ \alpha' \circ j_\mu \circ \Theta = j_\mu \circ \Theta' \circ \alpha' \circ \Theta' \circ j_\mu = j_\mu \circ (\Theta \circ \alpha \circ \Theta)' \circ j_\mu \\ &= (\Theta \circ \alpha \circ \Theta)^\sigma\end{aligned}$$

follows. So $(\alpha^\Theta)^\Theta = \Theta \circ \Theta \circ \alpha \circ \Theta \circ \Theta = \alpha$. □

Returning now to the main goal of this section, it will be convenient for us to express the Θ-KMS dual as a system:

Proposition 6.5. *For a reversing operation* Θ *as in Definition 6.2,*

$$\mathbf{A}^\Theta := (A, \alpha^\Theta, \mu)$$

is a system, called the Θ-KMS-dual *of* $\mathbf{A}$.

Proof. Recall from Proposition 4.3 that $\mathbf{A}^\sigma$ is a system. Since α^σ is u.c.p., it can be checked as in Proposition 4.3 from $\alpha^\Theta = \Theta^* \circ \alpha^\sigma \circ \Theta^*$, where $\Theta^*(a) := \Theta(a^*)$ for all $a \in A$, that α^Θ is u.c.p. as well. From $\mu \circ \Theta = \mu$, we obtain $\mu \circ \alpha^\Theta = \mu$. □

Remark 6.6. Similar to before, for a QMS $(\alpha_t)_{t\geq 0}$ with the σ-weak continuity property as in Remark 2.6, we have that this continuity property also holds for $(\alpha_t^\Theta)_{t\geq 0}$, where $\alpha_t^\Theta := (\alpha_t)^\Theta$ for every t. This follows from the continuity of $(\alpha_t^\sigma)_{t\geq 0}$ in Remark 4.4, and the fact that Θ is normal (Proposition 6.4).

As a simple corollary of Proposition 6.1 we have:

Corollary 6.7. *The following are equivalent for a system* $\mathbf{A}$*:*
(a) $\mathbf{A}$ *satisfies* Θ*-sqdb.*
(b) $\mathbf{A}$ *and* $\mathbf{A}^\Theta$ *are in balance with respect to* δ_μ*.*
(c) $\mathbf{A}^\Theta$ *and* $\mathbf{A}$ *are in balance with respect to* δ_μ*.*

When two systems are in balance, we expect the one system to partially inherit properties of the other. We saw an example of this in Proposition 4.8. As mentioned there, this suggests that for any given property that a system may have, we can in principle consider generalized forms of the property via balance. In particular for Θ-sqdb:

- We can consider systems $\mathbf{A}$ and $\mathbf{B}$ which are in balance with respect to a coupling ω (or a set of couplings) other than $\mu\odot\nu'$, but not necessarily with respect to δ_μ. Assuming that either $\mathbf{A}$ or $\mathbf{B}$ satisfies Θ-sqdb, for some reversing operation Θ for $\mathbf{A}$ or $\mathbf{B}$ respectively, the other system can then be viewed as satisfying a weaker version of Θ-sqdb.

A second possible way of obtaining conditions generalizing Θ-sqdb for a system $\mathbf{A}$, is simply to adapt Corollary 6.7 more directly:

- We can require $\mathbf{A}$ and $\mathbf{A}^\Theta$ to be in balance with respect to some coupling ω (or a set of couplings) other than $\mu \odot \mu'$, but not necessarily with respect to δ_μ. Or $\mathbf{A}^\Theta$ and $\mathbf{A}$ to be in balance with respect to some coupling ω other than $\mu \odot \mu'$, but not necessarily with respect to δ_μ.

Under KMS-symmetry (see Eq. (16)), the two options in the second condition, namely $\mathbf{A}$ and $\mathbf{A}^\Theta$ in balance, versus $\mathbf{A}^\Theta$ and $\mathbf{A}$ in balance, are equivalent:

Proposition 6.8. *If the system* $\mathbf{A}$ *is KMS-symmetric, then* $\mathbf{A}\omega\mathbf{A}^\Theta$ *if and only if* $\mathbf{A}^\Theta\omega_E\mathbf{A}$*, where* $E := \Theta \circ E_\omega \circ \Theta$*. (See Proposition 3.8 for* ω_E*.)*

Proof. By KMS-symmetry $\alpha^\Theta = \Theta \circ \alpha \circ \Theta$. Note that for any coupling ω we have that $E = \Theta^* \circ E_\omega \circ \Theta^*$ is u.c.p. like α^Θ in the proof of Proposition 6.5, and $\mu \circ E = \mu$ by Corollary 3.2 and $\mu \circ \Theta = \mu$. Then ω_E is a coupling by Proposition 3.8. From Theorem 4.1 we have

$$\mathbf{A}\omega\mathbf{A}^\Theta \Leftrightarrow E_\omega \circ \alpha = \Theta \circ \alpha \circ \Theta \circ E_\omega \Leftrightarrow E \circ \alpha^\Theta = \alpha \circ E \Leftrightarrow \mathbf{A}^\Theta\omega_E\mathbf{A}.$$

□

The two types of conditions suggested above will be illustrated by a simple example in the next section, where the conditions obtained will in fact be weaker than Θ-sqdb.

A basic question we now have is the following: can weaker conditions like these be applied to characterize certain non-equilibrium steady states μ which have enough structure that one can successfully analized them mathematically, while also having physical relevance? This seems plausible, given that these conditions are structurally so closely related to detailed balance itself. We briefly return to this in Section 8.

7. An example

In this section we use an very simple example based on the examples in [2, Section 6], [12], [27, Section 5] and [28, Subsection 7.1] to illustrate some of the ideas discussed in this paper. Our main reason for considering this example is that it is comparatively easy to manipulate mathematically. We leave a more in depth study of relevant examples for further work.

Let $\mathfrak{H}$ be a separable Hilbert space with total orthonormal set $e_1, e_2, e_3, ...$. We are going to consider systems on the von Neumann algebra $B(\mathfrak{H})$. These systems will all have the same faithful normal state ζ on $B(\mathfrak{H})$ given by the diagonal (in the mentioned basis) density matrix

$$\rho = \begin{bmatrix} \rho_1 & & \\ & \rho_2 & \\ & & \ddots \end{bmatrix}$$

where $\rho_1, \rho_2, \rho_3, ... > 0$ satisfy $\sum_{n=1}^{\infty} \rho_n = 1$. I.e.

$$\zeta(a) = \mathrm{Tr}(\rho a)$$

for all $a \in B(\mathfrak{H})$.

We now briefly explain what the cyclic representation and modular conjugation look like for the state ζ:

The (faithful) cyclic representation of $(B(\mathfrak{H}), \zeta)$ can be written as (H, π, Ω) where $H = \mathfrak{H} \otimes \mathfrak{H}$,

$$\pi(a) = a \otimes 1$$

for all $a \in B(\mathfrak{H})$, and

$$\Omega = \sum_{n=1}^{\infty} \sqrt{\rho_n} e_n \otimes e_n$$

is the cyclic vector. Our von Neumann algebra is therefore represented as $A = \pi(B(\mathfrak{H}))$, and the state ζ is represented by the state μ on A given by

$$\mu(\pi(a)) = \zeta(a)$$

for all $a \in A$. However, we also consider a second representation π' given by

$$\pi'(a) = 1 \otimes a$$

for all $a \in B(\mathfrak{H})$, so $A' = \pi'(B(\mathfrak{H}))$. The state μ' on A' is then given by

$$\mu'(\pi'(a)) = \langle \Omega, \pi'(a)\Omega \rangle = \zeta(a)$$

for all $a \in A$.

The modular conjugation J associated to μ (and to ζ) is then obtained as the conjugate linear operator $J : H \to H$ given by

$$J(e_p \otimes e_q) = e_q \otimes e_p$$

for all $p, q = 1, 2, 3, ...$ Furthermore,

$$j(\pi(a)) := J\pi(a)^* J = \pi'(a^T)$$

for all $a \in B(\mathfrak{H})$, where a^T denotes the transpose of a in the basis $e_1, e_2, e_3, ...$

This allows us to apply the general notions from the earlier sections explicitly to this specific case.

Regarding notation: Instead of the notation $|x\rangle \langle y|$ for $x, y \in \mathfrak{H}$, we use $x \bowtie y$, i.e.

$$(x \bowtie y)z := x \langle y, z \rangle$$

for all $z \in \mathfrak{H}$.

7.1. **The couplings.** We consider couplings of ζ with itself. A coupling of ζ with itself corresponds to a coupling of μ with itself in the cyclic representation, which is a state ω on $A \odot A' = \pi(B(\mathfrak{H})) \odot \pi'(B(\mathfrak{H})) \cong B(\mathfrak{H}) \odot B(\mathfrak{H})$ such that

$$\omega(\pi(a) \otimes 1) = \mu(\pi(a))$$

and

$$\omega(1 \otimes \pi'(a)) = \mu'(\pi'(a))$$

for all $a \in B(\mathfrak{H})$. However, in this concrete example it is clearly equivalent, and notationally simpler, to view ω directly as a state on $B(\mathfrak{H}) \odot B(\mathfrak{H})$ such that

$$\omega(a \otimes 1) = \zeta(a) \tag{32}$$

and

$$\omega(1 \otimes a) = \zeta(a) \tag{33}$$

for all $a \in B(\mathfrak{H})$, rather than to work via the cyclic representation.

Consider any disjoint subsets $Y_1, Y_2, Y_3, ...$ of $\mathbb{N}_+ := \{1, 2, 3, 4, ...\}$ such that $\cup_{n=1}^{\infty} Y_n = \mathbb{N}_+$. We construct a coupling ω which is given by a density matrix $\kappa \in B(\mathfrak{H} \otimes \mathfrak{H})$, i.e.

$$\omega(c) = \text{Tr}(\kappa c)$$

for all $c \in B(\mathfrak{H}) \odot B(\mathfrak{H})$. Therefore we may as well allow $c \in B(\mathfrak{H} \otimes \mathfrak{H})$, and define ω on the latter algebra, even though our theory only needs it to be defined on the algebraic tensor product $B(\mathfrak{H}) \odot B(\mathfrak{H})$.

We begin by obtaining a positive trace-class operator κ_n corresponding to the set Y_n for every n. Each κ_n will be one of three types, namely a (maximally) entangled type, a mixed type, or a product type, each of which we now discuss in turn for any n.

First, the *entangled type*: We set

$$\Omega_n = \sum_{q \in Y_n} \sqrt{\rho_q} e_q \otimes e_q$$

and

$$\begin{aligned} \kappa_n &= \Omega_n \bowtie \Omega_n \\ &= \sum_{p \in Y_n} \sum_{q \in Y_n} \sqrt{\rho_p \rho_q} (e_p \bowtie e_q) \otimes (e_p \bowtie e_q) \end{aligned}$$

for all n. It is straightforward to verify that

$$\mathrm{Tr}(\kappa_n) = \sum_{q \in Y_n} \rho_q \tag{34}$$

and

$$\omega_n(a \otimes 1) = \omega_n(1 \otimes a) = \sum_{q \in Y_n} \rho_q \langle e_q, a e_q \rangle \tag{35}$$

for all $a \in B(\mathfrak{H})$.

Secondly, the *mixed type*: Setting

$$\kappa_n = \sum_{q \in Y_n} \rho_q (e_q \bowtie e_q) \otimes (e_q \bowtie e_q)$$

we again obtain Eqs. (34) and (35).

Thirdly, the *product type*: Setting

$$\kappa_n = d_n \otimes d_n$$

where

$$d_n := \left(\sum_{p \in Y_n} \rho_p \right)^{-1/2} \sum_{q \in Y_n} \rho_q (e_q \bowtie e_q)$$

we yet again obtain Eqs. (34) and (35).

For each type we take

$$\kappa_n = 0$$

if Y_n is empty (this allows for a partition of $\mathbb{N}_+$ into a finite number of non-empty subsets).

For each n, let κ_n be any of the three types above. Then κ_n is indeed trace-class and positive, so setting

$$\omega_n(c) = \mathrm{Tr}(\kappa_n c) \tag{36}$$

for all $c \in B(\mathfrak{H} \otimes \mathfrak{H})$, we obtain a well-defined positive linear functional ω_n on $B(\mathfrak{H} \otimes \mathfrak{H})$. Then

$$\omega := \sum_{n=1}^{\infty} \omega_n$$

converges in the norm of $B(\mathfrak{H} \otimes \mathfrak{H})^*$, since $\|\omega_n\| = \omega_n(1) = \mathrm{Tr}(\kappa_n)$, so $\sum_{n=1}^{\infty} \|\omega_n\| = 1$. Correspondingly,

$$\kappa := \sum_{n=1}^{\infty} \kappa_n \tag{37}$$

converges in the trace-class norm $\|\cdot\|_1$, since $\sum_{n=1}^{\infty} \|\kappa_n\|_1 = \sum_{n=1}^{\infty} \mathrm{Tr}(\kappa_n) = 1$. Then it indeed follows that

$$\omega(c) = \sum_{n=1}^{\infty} \mathrm{Tr}(\kappa_n c) = \mathrm{Tr}(\kappa c),$$

since $|\sum_{n=1}^{m} \mathrm{Tr}(\kappa_n c) - \mathrm{Tr}(\kappa c)| \leq \|\sum_{n=1}^{m} \kappa_n - \kappa\|_1 \|c\|$.

Furthermore $\omega(1) = \sum_{n=1}^{\infty} \omega_n(1) = \sum_{n=1}^{\infty} \rho_n = 1$, and from Eq. (35) it follows that Eqs. (32) and (33) hold. So ω is a coupling of ζ with itself as required.

For $Y_1 = \mathbb{N}_+$, i.e. $\kappa = \kappa_1$, we can get two extremes, namely the diagonal coupling ω if κ_1 is of the entangled type, and the product state $\omega = \zeta \otimes \zeta$ on $B(\mathfrak{H} \otimes \mathfrak{H})$ when κ_1 is of the product type. But the construction above gives many cases in between these two extremes. Then balance with respect to ω is non-trivial, but does not nessecarily force two systems **A** and **B** on the same algebra A to have the same dynamics as in Proposition 6.1.

7.2. **The dynamics.** We now construct dynamics in order to obtain examples of systems on the von Neumann algebra $B(\mathfrak{H})$. Let $r_j \in \{3, 4, 5, ...\}$ and $0 < k_j < 1$ for $j = 1, 2, 3, ...$, and write $k = (k_1, k_2, k_3, ...)$. In terms of the $n \times n$ matrix

$$O_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix}$$

with the blank spaces all being zero, we then define $R_k \in B(\mathfrak{H})$ by the infinite matrix

$$R_k = \begin{bmatrix} k_1^{1/2} O_{r_1} & & \\ & k_2^{1/2} O_{r_2} & \\ & & \ddots \end{bmatrix}$$

in the basis $e_1, e_2, e_3, ...,$ where again the blank spaces are zero. In other words, $R_k e_1 = k_1^{1/2} e_2$ etc. So R_k consists of a infinite direct sum of finite cycles, each cycle including its own factor $k_n^{1/2}$. Replacing k by $1 - k := (1 - k_1, 1 - k_2, 1 - k_3, ...)$, we similarly obtain R_{1-k}. In the same basis we consider a hermitian operator $g \in B(\mathfrak{H})$ defined by the diagonal matrix

$$g = \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & \ddots \end{bmatrix},$$

with $g_1, g_2, g_3, ...$ a bounded sequence in $\mathbb{R}$. Note that $R_k^* R_k + R_{1-k} R_{1-k}^* = 1$. So we can define the generator $\mathcal{K}$ of a uniformly continuous semigroup $\mathcal{S} = (\mathcal{S}_t)_{t \geq 0}$ in $B(\mathfrak{H})$ by

$$\mathcal{K}(a) = R_k^* a R_k + R_{1-k} a R_{1-k}^* - a + i[g, a]$$

for all $a \in B(\mathfrak{H})$. See for example [51, Corollary 30.13]; the original papers on generators for uniformly continuous semigroups are [39] and [47].

In the same way and still using the same basis, for $l = (l_1, l_2, l_3, ...)$ with $0 < l_j < 1$ we define the generator $\mathcal{L}$ of a second uniformly continuous semigroup $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ in $\mathfrak{H}$ by

$$\mathcal{L}(b) = R_l^* b R_l + R_{1-l} b R_{1-l}^* - b + i[h, b]$$

for all $b \in B(\mathfrak{H})$, where the diagonal matrix

$$h = \begin{bmatrix} h_1 & & \\ & h_2 & \\ & & \ddots \end{bmatrix},$$

with $h_1, h_2, h_3, ...$ a bounded sequence in $\mathbb{R}$, defines a selfadjoint operator $h \in B(\mathfrak{H})$.

We furthermore assume:

$$\begin{aligned} \rho_1 &= ... = \rho_{r_1} \\ \rho_{r_1+1} &= ... = \rho_{r_1+r_2} \\ \rho_{r_1+r_2+1} &= ... = \rho_{r_1+r_2+r_3} \\ &\vdots \end{aligned}$$

Then the state is seen to be invariant under both $\mathcal{S}$ and $\mathcal{T}$ by checking that $\zeta \circ \mathcal{K} = 0$ and $\zeta \circ \mathcal{L} = 0$.

It is going to be simpler (but equivalent) to work directly in terms of $B(\mathfrak{H})$, rather than its cyclic representation. Nevertheless, since much of the theory of this paper is expressed in the cyclic representation, it is worth expressing the various objects in this representation as well. In particular we can then see how to obtain duals directly in terms of $B(\mathfrak{H})$.

Our two systems $\mathbf{A}$ and $\mathbf{B}$, viewed in the cyclic representation, are in terms of $A = B = \pi(B(\mathfrak{H}))$, with the dynamics given by

$$\alpha_t(\pi(a)) = \pi(\mathcal{S}_t(a))$$

and

$$\beta_t(\pi(b)) = \pi(\mathcal{T}_t(b))$$

and the states μ and ν both given by

$$\mu(\pi(a)) = \nu(\pi(a)) = \zeta(a) = \mathrm{Tr}(\rho a)$$

for all $a, b \in B(\mathfrak{H})$. The corresponding diagonal coupling

$$\delta_\mu : \pi(B(\mathfrak{H})) \odot \pi'(B(\mathfrak{H})) \to \mathbb{C}$$

is given by

$$\begin{aligned}\delta_\mu(\pi(a) \odot \pi'(b)) &= \langle \Omega, \pi(a)\pi'(b)\Omega\rangle = \langle \Omega, (a \otimes b)\Omega\rangle \\ &= \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \left\langle e_p, \rho^{1/2} a e_q \right\rangle \left\langle e_q, \rho^{1/2} b^T e_p \right\rangle \\ &= \mathrm{Tr}(\rho^{1/2} a \rho^{1/2} b^T)\end{aligned}$$

where $b^T \in B(\mathfrak{H})$ is obtained as the transpose of the matrix representation of b in terms of the basis $e_1, e_2, e_3, \ldots$.

The dual $\beta'_t : \pi'(B(\mathfrak{H})) \to \pi'(B(\mathfrak{H}))$ of β_t is given by

$$\langle \Omega, \pi(b)\beta'_t(\pi'(b'))\Omega_\zeta\rangle = \langle \Omega, \beta_t(\pi(b))\pi'(b')\Omega\rangle$$

for all $b, b' \in B(\mathfrak{H})$.

We therefore define the dual $\mathcal{L}'$ of $\mathcal{L}$ via the representations by requiring

$$\langle \Omega, \pi(b)\pi'(\mathcal{L}'(b'))\Omega\rangle = \langle \Omega, \pi(\mathcal{L}(b))\pi'(b')\Omega\rangle$$

for all $b, b' \in B(\mathfrak{H})$, i.e.

$$Tr(\rho^{1/2} a \rho^{1/2} (\mathcal{L}'(b))^T) = Tr(\rho^{1/2} \mathcal{L}(a) \rho^{1/2} b^T)$$

for all $a, b \in B(\mathfrak{H})$. It is then straightforward to verify that

$$\mathcal{L}'(b) = R^*_{1-l} b R_{1-l} + R_l b R^*_l - b + i[h, b] \tag{38}$$

for all $b \in B(\mathfrak{H})$. From this one can see that $\mathcal{L}'$ is also the generator of a uniformly continuous semigroup $\mathcal{T}' = (\mathcal{T}'_t)_{t\geq 0}$ in $\mathfrak{H}$, which in addition satisfies

$$\langle \Omega, \pi(b)\pi'(\mathcal{T}'_t(b'))\Omega\rangle = \langle \Omega, \pi(\mathcal{T}_t(b))\pi'(b')\Omega\rangle$$

and therefore

$$\pi'(\mathcal{T}'_t(b')) = \beta'_t(\pi'(b'))$$

for all $b, b' \in B(\mathfrak{H})$. So we can call $\mathcal{T}'$ the *dual* of $\mathcal{T}$ with respect to ζ.

We now have a complete description of the systems we are interested in our example, as well as their duals.

7.3. **Balance.** We now show examples of balance between $\mathbf{A} := (B(\mathfrak{H}), \mathcal{S}, \zeta)$ and $\mathbf{B} := (B(\mathfrak{H}), \mathcal{T}, \zeta)$ and illustrate a number of points made in this paper. Remember that since we now now have a continuous time parameter $t \geq 0$, the balance condition in Definition 2.10 is required to hold at every t. However, it then follows that $\mathbf{A}$ and $\mathbf{B}$ are in balance w.r.t. ω if and only if

$$\mathrm{Tr}(\kappa(\mathcal{K}(a) \otimes b)) = \mathrm{Tr}(\kappa(a \otimes \mathcal{L}'(b))$$

for all $a, b \in B(\mathfrak{H})$. From this one can easily check that $\mathbf{A}$ and $\mathbf{B}$ are in balance w.r.t. ω if and only if

$$\begin{aligned}&(R_k \otimes 1)\kappa(R_k \otimes 1)^* + (R_{1-k} \otimes 1)^*\kappa(R_{1-k} \otimes 1) - i[g \otimes 1, \kappa] \\ &= (1 \otimes R_{1-l})\kappa(1 \otimes R_{1-l})^* + (1 \otimes R_l)^*\kappa(1 \otimes R_l) - i[1 \otimes h, \kappa]\end{aligned}$$

holds. However, equating the real and imaginary parts respectively (keeping in mind that κ as given in Subsection 7.1 is a real infinite matrix in the basis $e_p \otimes e_q$), we see that this is equivalent to

$$\begin{aligned}&(R_k \otimes 1)\kappa(R_k \otimes 1)^* + (R_{1-k} \otimes 1)^*\kappa(R_{1-k} \otimes 1) \\ &= (1 \otimes R_{1-l})\kappa(1 \otimes R_{1-l})^* + (1 \otimes R_l)^*\kappa(1 \otimes R_l)\end{aligned} \tag{39}$$

and

$$[g \otimes 1, \kappa] = [1 \otimes h, \kappa] \tag{40}$$

both being true.

To proceed, we refine the construction of κ in Subsection 7.1, by only allowing

$$Y_n = \bigcup_{p \in I_n} Z_p$$

where $Z_1 = \{1, 2, ..., r_1\}$, $Z_2 = \{r_1 + 1, r_1 + 2, ..., r_1 + r_2\}$, etc, and where $I_1, I_2, I_3, ...$ is any sequence of disjoint subsets of $\mathbb{N}_+$ such that $\cup_{n \in \mathbb{N}_+} I_n = \mathbb{N}_+$. Note that an I_n is allowed to be empty (then Y_n is empty), and it is also allowed to be infinite.

It then follows that $\mathbf{A}$ and $\mathbf{B}$ are in balance w.r.t. ω if and only if

$$\begin{aligned}&(R_k \otimes 1)\kappa_n(R_k \otimes 1)^* + (R_{1-k} \otimes 1)^*\kappa_n(R_{1-k} \otimes 1) \\ &= (1 \otimes R_{1-l})\kappa_n(1 \otimes R_{1-l})^* + (1 \otimes R_l)^*\kappa_n(1 \otimes R_l)\end{aligned} \tag{41}$$

and

$$[g \otimes 1, \kappa_n] = [1 \otimes h, \kappa_n] \tag{42}$$

both hold for every n. To see that Eq. (41) and Eq. (42) follow from Eq. (39) and Eq. (40) respectively, place the latter into $\langle e_p \otimes e_q, (\cdot)e_{p'} \otimes e_{q'}\rangle$ for $p, q, p', q' \in Y_n$. The converse holds, since Eq. (37) is convergent in the trace-class norm,

To evaluate these conditions in detail is somewhat tedious, so we just describe it in outline below.

Note that, roughly speaking, in a term like $(R_k \otimes 1)\kappa_n(R_k \otimes 1)^*$, for κ_n is of the entangled or mixing type, the first slot in the tensor product structure of κ_n is advanced by one step in each cycle appearing in R_k. In a term like $(1 \otimes R_l)^* \kappa_n (1 \otimes R_l)$, on the other hand, the second slot is rolled back by one step in each cycle, which is equivalent to the first slot being advanced by one step. So, if κ_n is of the entangled or mixing type, and

$$k_p = l_p \tag{43}$$

for each $p \in I_n$, then Eq. (41) holds.

Conversely, note that since $r_p > 2$ for all p, the terms $(R_k \otimes 1)\kappa_n (R_k \otimes 1)^*$ and $(1 \otimes R_l)^* \kappa_n (1 \otimes R_l)$ have to be equal for Eq. (41) to hold; the terms $(R_{1-k} \otimes 1)^* \kappa_n (R_{1-k} \otimes 1)$ and $(1 \otimes R_{1-l})\kappa_n (1 \otimes R_{1-l})^*$ involve other basis elements and therefore can not ensure Eq. (41) when $(R_k \otimes 1)\kappa_n (R_k \otimes 1)^* \neq (1 \otimes R_l)^* \kappa_n (1 \otimes R_l)$.

For the product type κ_n Eq. (41) always holds, since κ_n then commutes with R_k.

When κ_n is of the entangled type, one can verify by direct calculation that Eq. (42) holds if and only if

$$g_p - g_q = h_p - h_q \tag{44}$$

for all $p, q \in Y_n$. For the other two types Eq. (42) always holds, since then κ_n, $g \otimes 1$ and $1 \otimes h$ are diagonal, so the commutators are zero.

We conclude that $\mathbf{A}$ and $\mathbf{B}$ are in balance w.r.t. ω if and only if Eq. (43) holds for all $p \in I_n$ for every n for which κ_n is either of the entangled or mixing type, and Eq. (44) holds for all $p \in I_n$ for every n for which κ_n is of the entangled type.

We now have an example showing that the transity in Theorem 5.1 can be trivial in the sense that we can have $\omega \circ \psi = \mu \odot \xi'$ despite having $\omega \neq \mu \odot \nu'$ and $\psi \neq \nu \odot \xi'$. To see this, let $\mathbf{C}$ be a system constructed in the same way as $\mathbf{A}$ and $\mathbf{B}$ above, so it has the same von Neumann algebra and state, but the generator giving its dynamics can use different choices in place of k, g and l, h. As above, construct two couplings ω and ψ (giving balance of $\mathbf{A}$ and $\mathbf{B}$ w.r.t. ω, and of $\mathbf{B}$ and $\mathbf{C}$ w.r.t. ψ), but with entangled and mixed types not in overlapping parts of the two couplings respectively (i.e. the respective Y_n sets of the two couplings should be disjoint), while the rest of each coupling is a κ_n of the product type. Then it can be verified using Proposition 5.5 that we indeed obtain $\omega \circ \psi = \mu \odot \xi'$, despite having $\omega \neq \mu \odot \nu'$ and $\psi \neq \nu \odot \xi'$. This illustrates that to have $\omega \circ \psi \neq \mu \odot \xi'$, we need sufficient "overlap" between ω and ψ, where this overlap condition has been made precise in Hilbert space terms (in the cyclic representations) by Proposition 5.5.

7.4. **A reversing operation.** Let us consider in particular Θ-sqdb in Definition 6.3 and Corollary 6.7, as well as the two weaker conditions suggested at the end of Section 6. Take Θ to be transposition in the basis $e_1, e_2, e_3, ...$, i.e.

$$\Theta(a) := a^T$$

for all $a \in B(\mathfrak{H})$. It is simple to check that Θ is then indeed a reversing operation for $(B(\mathfrak{H}), \zeta)$. In the cyclic representation, Θ would be given by $\pi(a) \mapsto \pi(a^T)$. It is readily confirmed that in this case the Θ-KMS dual of $\mathbf{B}$ is $\mathbf{B}^\Theta = (B(\mathfrak{H}), \mathcal{T}', \zeta)$, i.e. in the cyclic representation we would have $\alpha_t^\Theta = \alpha_t'$ for all t.

If ω is the diagonal coupling, i.e. κ_1 is of the entangled type with $Y_1 = \mathbb{N}_+$, then from Eqs. (43) and (38) we see that $\mathbf{B}$ and $\mathbf{B}^\Theta = (B(\mathfrak{H}), \mathcal{T}', \zeta)$ are in balance with respect to ω, if and only if $l_p = 1 - l_p$, i.e. $l_p = 1/2$, for all p.

Now consider the situation where $\mathbf{B}$ satisfies Θ-sqdb, and $\mathbf{A}$ and $\mathbf{B}$ are in balance w.r.t. ω. It then follows that $k_p = 1/2$ for all p in I_n for which κ_n is of the entangled or mixed type, but we need not have $k_p = 1/2$ for other values of p. This illustrates that this is indeed a strictly weaker condition on $\mathbf{A}$ than Θ-sqdb as long as not all the κ_n are of the entangled or mixed type.

Next consider the situation where $\mathbf{A}$ and $\mathbf{A}^\Theta$ are in balance with respect to ω, where again not all the κ_n are of the entangled or mixed type. Then in a similar way we again see that $k_p = 1/2$ for all p in I_n for which κ_n is of the entangled or mixed type, but we need not have $k_p = 1/2$ for other values of p. So again this is a strictly weaker condition than Θ-sqdb.

This illustrates the two conditions suggested at the end of Section 6, albeit in a very simple situation. Here the two conditions are essentially equivalent when applied to $\mathbf{A}$, but we expect this not to be the case in general.

8. Further work

Balance seems to indicate some common structure in the two systems. However, this is a subtle issue. We note that already in the classical case, in the context of joinings, it has been shown that (translating into our context) two systems can be nontrivially in balance (by which we mean the coupling is not the product state), while the two systems have no "factor" (roughly speaking a subsystem) in common. This was a difficult problem in classical ergodic theory posed by Furstenberg in [35] in 1967, and was only solved a decade later by Rudolph in [54]. Therefore we suspect that balance between two systems is more general than the existence of some form of common system inside the two systems. This issue has not been pursued in this paper, but appears worth investigating.

It also seems natural to study joinings directly for systems as defined in Definition 2.1. The idea would be to replace the balance conditions in Definition 2.10, by the joining conditions (possibly adapted slightly) described in Remark 2.12.

In principle we can view E_ω as a quantum channel. It could be of interest to see what the physical significance of this map is, considering the well-known correspondence between completely positive maps and bipartite states in finite dimensions (see [16], but also [19] and [41] for earlier related work) which is of some importance in quantum information theory. See for example [61], [9] and [42]. Some related work has appeared in infinite dimensions for $B(H)$ and $B(H_1, H_2)$ as well [13], [40]. Also see [10, Section 1] for further remarks.

A related question is what the physical implications or applications of transitivity are. Transitivity appears to be a basic ingredient of the theory of balance, but we have not explored its consequences in this paper.

In Section 6 we only considered standard quantum detailed balance with respect to a reversing operation. It certainly seems relevant to investigate if balance can be used to give generalized forms of other types of detailed balance.

Furthermore, if balance can indeed be used to formulate certain types of non-equilibrium steady states, as asked in Section 6, then it seems natural to connect this to entanglement and correlated states more generally. Can results on entangled states be applied to a coupling ω of μ and ν to study or classify certain classes of non-equilibrium steady states μ (or ν) of quantum systems? Note that the two extremes are the product state $\omega = \mu \odot \nu'$, which is the bipartite state with no correlations, and the diagonal coupling δ_μ of μ with itself, which can be viewed as is the bipartite state which is maximally entangled while having μ and μ' as its reduced states, at least when the observable algebra is (a cyclic representation of) $B(H)$.

We have only studied one example in this paper. To gain a better understanding of balance, it is important to explore further examples, especially physical examples, in particular in relation to non-equilibrium.

Lastly we mention the dynamical, weighted and generalized detailed balance conditions studied in [5], [2] and [3] respectively, along with a local KMS-condition, which was explored further in [4] and [33]. We suspect that it would be of interest to explore if there are any connections between these, and balance as studied in this paper.

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