Isometries on symmetric spaces associated with semi-finite von Neumann algebras

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Outline

Aim

Suppose $U : E \to F$ is an isometry. Describe its structure.

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Isometries of L^p-spaces

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ-finite measure spaces
$U : L_p(\Omega_1, \Sigma_1, \mu_1) \to L_p(\Omega_2, \Sigma_2, \mu_2)$, $(1 \leq p < \infty, p \neq 2)$ a linear isometry.
Then there exists a regular set isomorphism $\eta : \Sigma_1 \to \Sigma_2$ and a function $h : \Omega_2 \to \mathbb{F}$ such that

$$U(f) = h.T_\eta(f) \qquad \forall f \in L_p(\mu_1)$$

where T_η is the transformation induced by η (i.e. $T_\eta(\chi_A) = \chi_{\eta(A)}$)

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Key ingredient

If f and g belong to $L_p(\mu)$, then

$$\begin{aligned} \|f+g\|_p^p + \|f-g\|_p^p &\leq 2\|f\|_p^p + 2\|g\|_p^p \qquad 0<p\leq 2 \\ \|f+g\|_p^p + \|f-g\|_p^p &\geq 2\|f\|_p^p + 2\|g\|_p^p \qquad p\geq 2 \end{aligned}$$

For $p \neq 2$, equality holds if and only if $f.g = 0$.

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Isometries of Lorentz spaces

If U is a surjective isometry of the space $L_{w,1}[0,1]$, then there exist a ± 1-valued Borel measurable function h and a trace-preserving Borel measurable map $\sigma : [0,1] \to [0,1]$ such that

$$(Uf)(t) = h(t)(C_\sigma(f))(t) \qquad 0 \leq t \leq 1,$$

where C_σ is the composition operator induced by σ.

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Let w be a strictly decreasing weight function and let E be the Lorentz spaces $L_{w,1}(0,\infty)$. f is an extreme point of the unit ball B_E of E if and only if $|f| = \frac{1}{\psi(m(A))}\chi_A$ for some $A \subset (0,\infty)$ with $0 < m(A) < \infty$.

Isometries of rearrangement invariant spaces

Suppose $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are purely non-atomic σ-finite measure spaces and suppose E_1 and E_2 are rearrangement invariant spaces of functions on $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, respectively. Assume that the norm on E_1 is not proportional to the norm of the space $L_2(\mu_1)$. If U is a surjective isometry from E_1 onto E_2, then there exist a measurable function h and a regular set isomorphism η from Σ_1 onto Σ_2 such that

$$U(f) = h.T_\eta(f) \qquad \forall f \in E_1.$$

Key ingredient

Suppose (Ω, Σ, μ) is a purely non-atomic σ-finite measure space and let E be a rearrangement invariant space such that the norm on E is not proportional to the norm on $L_2(\mu)$. H is a bounded Hermitian operator on E, if and only if there exists an $h \in L_\infty(\mu)$ such that $H(f) = h.f = M_h(f)$ for all $f \in E$. In this case, $\|H\| = \|h\|_\infty$.

The non-commutative setting

Ingredients:

von Neumann algebra: $\mathscr{A} \subseteq B(H)$ (we will stick to semi-finite ones!)

Faithful normal semi-finite (fns) trace: $\tau : \mathscr{A}^+ \to [0, \infty]$

τ-measurable operators: $S(\tau)$

- closed, densely defined operators
- affiliated with the von Neumann algebra
- $\tau(e^{|x|}(\lambda, \infty)) < \infty$ for some $\lambda > 0$

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The generalized singular value

For $x \in S(\mathscr{A}, \tau)$, let

$$d(x)(s) := \tau(e^{|x|}(s, \infty)), \qquad s \geq 0$$

In the commutative setting we have that

$$d(f)(s) = \mu(\{t \in \Omega : |f(t)| > s\})$$

For $x \in S(\mathscr{A}, \tau)$, let

$$\mu_x(t) := \inf\{s \geq : d(x)(s) \leq t\}, \qquad t \geq 0$$

In the commutative setting

$$\mu_f(t) = f^*(t)$$

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Non-commutative L_p-space

Suppose $\mathscr{A}$ is a von Neumann algebra equipped with a fns trace.

$$L_p(\tau) := \{x \in S(\tau) : \tau(|x|^p) < \infty\}$$

$$\|x\|_p := \tau(|x|^p)^{1/p}$$

Generalizations of these spaces: Symmetric spaces

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Generalizations of these spaces: Symmetric spaces

Symmetric space

A linear subspace $E \subseteq S(\mathscr{A}, \tau)$, equipped with a norm $\left\|\cdot\right\|_E$, is called a symmetric space if

- $uxv \in E$ and $\left\|uxv\right\|_E \leq \left\|u\right\|_{\mathscr{A}} \left\|x\right\|_E \left\|v\right\|_{\mathscr{A}}$ whenever $u, v \in \mathscr{A}$ and $x \in E$
- $x \in S(\mathscr{A}, \tau), y \in E$ with $\mu_x \leq \mu_y$ implies that $x \in E$ and $\left\|x\right\| \leq \left\|y\right\|$
- E is complete

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Using commutative spaces to generate non-commutative ones

Suppose $\mathscr{A}$ is semi-finite von Neumann algebra equipped with a fns trace τ
$E \subseteq L^0(0,\infty)$ a Banach function space
Define $E(\tau) = \{x \in S(\mathscr{A},\tau) : \mu_x \in E\}$
Norm $\|x\|_{E(\tau)} := \|\mu_x\|_E$

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Isometries on non-commutative L_p-spaces

$(\mathscr{A}, \tau)$ and $(\mathscr{B}, \nu)$ semi-finite von Neumann algebras ($1 \leq r < \infty$, $r \neq 2$)

If $U : L_r(\tau) \to L_r(\nu)$ is an isometry, then there exist, uniquely,

- a partial isometry $w \in \mathscr{B}$,
- a positive operator b affiliated with $\mathscr{B}$, and
- a Jordan $*$-isomorphism Φ of $\mathscr{A}$ onto a weakly closed $*$-subalgebra of $\mathscr{B}$ such that

$$U(x) = wb\Phi(x) \qquad \text{for all } x \in L_r(\tau) \cap \mathscr{A}.$$

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Isometries on non-commutative L_p-spaces

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Key ingredient

Suppose $(\mathscr{A}, \tau)$ is a semi-finite von Neumann algebra and $1 \leq p < \infty$, $p \neq 2$. If $x, y \in L_p(\tau)$, then equality

$$\|x+y\|_p^p + \|x-y\|_p^p = 2\|x\|_p^p + 2\|y\|_p^p$$

holds in Clarkson's inequality

$$\begin{array}{rcll} \|x+y\|_p^p + \|x-y\|_p^p & \leq & 2\|x\|_p^p + 2\|y\|_p^p & (1 \leq p < 2) \\ \|x+y\|_p^p + \|x-y\|_p^p & \geq & 2\|x\|_p^p + 2\|y\|_p^p & (2 < p < \infty) \end{array}$$

if and only if $xy^* = 0 = x^*y$.

If $T(e)^*T(f) = 0 = T(e)T(f)^*$ whenever e and f are orthogonal projections, then we will call T disjointness-preserving.

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Isometries on Lorentz spaces

Let $(\mathscr{A},\tau)$ and $(\mathscr{B},\nu)$ be finite von Neumann algebras with $\tau(\mathbf{1})=1=\nu(\mathbf{1})$ and suppose $\psi:[0,1]\to[0,\infty)$ is a strictly concave continuous increasing function with $\psi(0)=0$. A continuous surjective linear mapping $U:\Lambda_\psi(\tau)\to\Lambda_\psi(\nu)$ is an isometry if and only if there exist uniquely a unitary operator $a\in\mathscr{B}$ and a Jordan $*$-isomorphism Φ of $\mathscr{A}$ onto $\mathscr{B}$ such that

$$\tau(x)=\nu(\Phi(x))\qquad\forall x\in\mathscr{A}$$

and

$$U(x)=a\Phi(x)\qquad\forall x\in\mathscr{A}.$$

Key ingredient

Let $(\mathscr{A}, \tau)$ be a finite von Neumann algebras with $\tau(\mathbf{1}) = 1$ and suppose $\psi : [0,1] \to [0,\infty)$ is a strictly concave continuous increasing function with $\psi(0) = 0$. An element $x \in \Lambda_\psi(\tau)$ is an extreme point of the unit ball of $\Lambda_\psi(\tau)$ if and only if $x = \frac{1}{\psi(\tau(|v|))} v$ for some partial isometry $v \in \mathscr{A}$.

Isometries on symmetric spaces

Suppose $(\mathscr{A}, \tau)$ is an AFD factor of type II_1 or II_∞ and suppose $E(0,\infty)$ is a separable symmetric space such that the norms on $E(\tau)$ and $L_2(\tau)$ are not proportional. Then a continuous linear mapping U of $E(\tau)$ onto itself is an isometry if and only if there exist a unitary operator $a \in \mathscr{A}$ and a Jordan $*$-automorphism Φ of $\mathscr{A}$ such that

$$U(x) = a\Phi(x) \qquad \forall x \in \mathscr{A} \cap E(\tau).$$

Key ingredient

Let H be a Hermitian operator on the separable symmetric space $E(\mathscr{A}, \tau)$. If $E(\mathscr{A}, \tau) \neq L_2(\mathscr{A}, \tau)$, then H^2 is Hermitian if and only if H can be represented as either a left multiplication or a right multiplication by a self-adjoint operator in $\mathscr{A}$.

Isometries on symmetric spaces

Let $(\mathscr{A},\tau)$ and $(\mathscr{B},\nu)$ be trace-finite von Neumann algebras and suppose $E \subseteq S(\mathscr{A},\tau)$ is a symmetric space and $F \subseteq S(\mathscr{B},\nu)$ is a fully symmetric space. If $U : E \to F$ is a positive linear isometry from E onto F, then there exist uniquely a positive operator $a \in S(Z(B),\nu)$ and a Jordan $*$-isomorphism Φ of $\mathscr{A}$ onto $\mathscr{B}$ such that $s(a) = 1$ and

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Extension procedures

Aim: Define a map on projections. Extend it to a Jordan $*$-homomorphism

Difficulties

- Linearity of extension to linear combination of projections
- Linearity of extension from $\mathscr{F}(\tau)$ to $\mathscr{A}$

Typically we will define $\Phi(p) = s(U(p))$

Why not use Dye's Theorem?

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If U is a positive linear isometry from E onto F such that $\nu(s(U(p))) < \infty$ whenever $p \in \mathscr{A}$ is a projection with finite trace, then there exist uniquely

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Two natural questions

Is this result useful?
What can we say if some of these conditions are not met?

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What are Orlicz spaces?

An Orlicz function is a convex function $\phi : [0\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$.
For a measurable function f define

$$I_\phi(f) = \int_\Omega \phi(|f(t)|)d\mu$$

The Orlicz space $L^\phi(\mu)$ is defined as

$$\{f \in L^0(\mu) : I_\phi(\lambda f) < \infty \text{ for some } \lambda > 0\}$$

This is a Banach space when equipped with the norm

$$||f|| := \inf\{\lambda > 0 : I_\phi(f/\lambda) \leq 1\}$$

Non-commutative Orlicz spaces can be generated using the process described earlier

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An Orlicz function is a convex function $\phi : [0\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$.
For a measurable function f define

$$I_\phi(f) = \int_\Omega \phi(|f(t)|)d\mu$$

The Orlicz space $L^\phi(\mu)$ is defined as

$$\{f \in L^0(\mu) : I_\phi(\lambda f) < \infty \text{ for some } \lambda > 0\}$$

This is a Banach space when equipped with the norm

$$||f|| := \inf\{\lambda > 0 : I_\phi(f/\lambda) \leq 1\}$$

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Isometries on Orlicz spaces

We will be interested in positive surjective modular isometries
A map $U : L^{\phi}(\tau) \to L^{\phi}(\nu)$ is called a modular isometry if

$$I_{\phi}(\mu_{U(x)}) = I_{\phi}(\mu_x) \qquad \forall x \in L^{\phi}(\tau)$$

It is easily checked that every modular isometry is an isometry
In the commutative setting, all surjective isometries of Orlicz spaces (over the complex numbers) are modular isometries

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We will consider three scenarios corresponding to three types of Orlicz functions

To distinguish these types, we will use the following quantities $a_\phi := \sup\{t \geq 0 : \phi(t) = 0\}$, $b_\phi := \sup\{t \geq 0 : \phi(t) < \infty\}$

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$\mathscr{A}$ and $\mathscr{B}$ semi-finite von Neumann algebras equipped with fns traces τ and ν (with $\tau(\mathbf{1}) = \infty = \nu(\mathbf{1})$).
$0 < a_\phi < \infty$, $b_\phi = \infty$, ϕ has at least one point of discontinuity
In this case $\mathscr{A} \subseteq L^\phi(\tau)$ and $\mathscr{B} \subseteq L^\phi(\nu)$.

Theorem

Suppose these conditions hold. If $U : L^\phi(\tau) \to L^\phi(\nu)$ is a positive surjective modular isometry, then the restriction of U to $\mathscr{A}$ is a Jordan $*$-isomorphism from $\mathscr{A}$ onto $\mathscr{B}$

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Sketch of the proof

Show that $\mu_{U(\mathbf{1})} = \chi_{[0,\infty)}$. This can be used to show that U is unital.
Show that $U(\mathscr{A}) \subseteq \mathscr{B}$
Show that U^{-1} is positive and that $U^{-1}(\mathscr{B}) \subseteq \mathscr{A}$
Remark: It is interesting to note that one can show directly that U is disjointness-preserving.

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If $\phi(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ \infty & \text{if } t > 1 \end{cases}$, then $L^{\phi}(\tau) = L^1 \cap L^{\infty}(\tau)$ with equality of norms.

Theorem

Suppose $(\mathscr{A}, \tau)$ and $(\mathscr{B}, \nu)$ are non-atomic semi-finite von Neumann algebras with $\tau(\mathbf{1}) = \infty = \nu(\mathbf{1})$. If U is a positive surjective isometry from $L^1 \cap L^{\infty}(\tau)$ onto $L^1 \cap L^{\infty}(\nu)$, then U is the restriction of a trace-preserving Jordan $*$-isomorphism from $\mathscr{A}$ onto $\mathscr{B}$.

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Show that the extreme points of the unit balls of these spaces can be characterized as the partial isometries with unit trace.
Show that U is disjointness-preserving and finiteness-preserving.
Let $\Phi(p) = U(p)$ for projections p with finite-trace
Φ can be extended to a map which is square-preserving and L_∞-isometric on self-adjoint elements of $\mathscr{F}(\tau)$.
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