Isometries on symmetric spaces associated with semi-finite von Neumann algebras

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- A brief history: the non-commutative setting

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Aim

Suppose $U: E \rightarrow F$ is an isometry. Describe its structure.

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lsometries of L^p-spaces

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ -finite measure spaces $U: L_p(\Omega_1, \Sigma_1, \mu_1) \rightarrow L_p(\Omega_2, \Sigma_2, \mu_2)$, $(1 \le p < \infty, p \ne 2)$ a linear isometry.

Then there exists a regular set isomorphism $\eta: \Sigma_1 \to \Sigma_2$ and a function $h: \Omega_2 \to \mathbb{F}$ such that

$$U(f) = h.T_{\eta}(f) \qquad \forall f \in L_{\rho}(\mu_1)$$

where ${\cal T}_\eta$ is the transformation induced by η (i.e. ${\cal T}_\eta(\chi_A)=\chi_{\eta(A)})$

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Key ingredient

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If f and g belong to $L_p(\mu)$, then

$$\|f + g\|_{p}^{p} + \|f - g\|_{p}^{p} \leq 2\|f\|_{p}^{p} + 2\|g\|_{p}^{p} \qquad 0
$$\|f + g\|_{p}^{p} + \|f - g\|_{p}^{p} \geq 2\|f\|_{p}^{p} + 2\|g\|_{p}^{p} \qquad p \geq 2$$$$

For $p \neq 2$, equality holds if and only if $f \cdot g = 0$.

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Isometries of Lorentz spaces

If U is a surjective isometry of the space $L_{w,1}[0,1]$, then there exist a ± 1 -valued Borel measurable function h and a trace-preserving Borel measurable map $\sigma : [0,1] \rightarrow [0,1]$ such that

$(Uf)(t) = h(t)(C_{\sigma}(f))(t) \qquad 0 \le t \le 1,$

where C_{σ} is the composition operator induced by σ .

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Key ingredient

Let w be a strictly decreasing weight function and let E be the Lorentz spaces $L_{w,1}(0,\infty)$. f is an extreme point of the unit ball B_E of E if and only if $|f| = \frac{1}{\psi(m(A))}\chi_A$ for some $A \subset (0,\infty)$ with $0 < m(A) < \infty$.

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lsometries of rearrangement invariant spaces

Suppose $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are purely non-atomic σ -finite measure spaces and suppose E_1 and E_2 are rearrangement invariant spaces of functions on $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, respectively. Assume that the norm on E_1 is not proportional to the norm of the space $L_2(\mu_1)$. If U is a surjective isometry from E_1 onto E_2 , then there exist a measurable function h and a regular set isomorphism η from Σ_1 onto Σ_2 such that

$$U(f) = h.T_{\eta}(f) \qquad \forall f \in E_1.$$

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Key ingredient

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Suppose (Ω, Σ, μ) is a purely non-atomic σ -finite measure space and let E be a rearrangement invariant space such that the norm on E is not proportional to the norm on $L_2(\mu)$. H is a bounded Hermitian operator on E, if and only if there exists an $h \in L_{\infty}(\mu)$ such that $H(f) = h.f = M_h(f)$ for all $f \in E$. In this case, $\|H\| = \|h\|_{\infty}$.

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The non-commutative setting

Ingredients:

von Neumann algebra: $\mathscr{A} \subseteq B(H)$ (we will stick to semi-finite ones!)

Faithful normal semi-finite (fns) trace: $\tau : \mathscr{A}^+ \to [0,\infty]$ τ -measurable operators: $S(\tau)$

- closed, densely defined operators
- affiliated with the von Neumann algebra

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$$au(e^{|x|}(\lambda,\infty))<\infty$$
 for some $\lambda>0$

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The generalized singular value

For $x \in S(\mathscr{A}, \tau)$, let

$$d(x)(s) := \tau(e^{|x|}(s,\infty)), \qquad s \ge 0$$

In the commutative setting we have that

$$d(f)(s) = \mu(\{t \in \Omega : |f(t)| > s\})$$

For $x \in S(\mathscr{A}, \tau)$, let

$$\mu_x(t) := \inf\{s \ge d(x)(s) \le t\}, \qquad t \ge 0$$

In the commutative setting

$$\mu_f(t) = f^*(t)$$

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Non-commutative L_p -space

Suppose \mathscr{A} is a von Neumann algebra equipped with a fns trace.

$$L_p(\tau) := \{ x \in S(\tau) : \tau(|x|^p) < \infty \}$$

$$||x||_p := \tau(|x|^p)^{1/p}$$

Generalizations of these spaces: Symmetric spaces

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Generalizations of these spaces: Symmetric spaces

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Symmetric space

A linear subspace $E \subseteq S(\mathscr{A}, \tau)$, equipped with a norm $\|\cdot\|_E$, is called a symmetric space if

- $uxv \in E$ and $||uxv||_E \le ||u||_{\mathscr{A}} ||x||_E ||v||_{\mathscr{A}}$ whenever $u, v \in \mathscr{A}$ and $x \in E$
- $x \in S(\mathscr{A}, \tau), y \in E$ with $\mu_x \leq \mu_y$ implies that $x \in E$ and $||x|| \leq ||y||$
- *E* is complete

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- $uxv \in E$ and $||uxv||_E \le ||u||_{\mathscr{A}} ||x||_E ||v||_{\mathscr{A}}$ whenever $u, v \in \mathscr{A}$ and $x \in E$
- $x \in S(\mathscr{A}, \tau), y \in E$ with $\mu_x \leq \mu_y$ implies that $x \in E$ and $||x|| \leq ||y||$
- *E* is complete

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Using commutative spaces to generate non-commutative ones

Suppose \mathscr{A} is semi-finite von Neumann algebra equipped with a fns trace τ $E \subseteq L^0(0,\infty)$ a Banach function space Define $E(\tau) = \{x \in S(\mathscr{A}, \tau) : \mu_x \in E\}$ Norm $\|x\|_{E(\tau)} := \|\mu_x\|_E$

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lsometries on non-commutative L_p -spaces

 (\mathscr{A},τ) and (\mathscr{B},v) semi-finite von Neumann algebras ($1\leq r<\infty,$ r
eq 2)

If $U:L_r(au) ightarrow L_r(au)$ is an isometry, then there exist, uniquely,

- a partial isometry $w \in \mathscr{B}$,
- a positive operator b affiliated with *B*, and
- a Jordan *-isomorphism Φ of A onto a weakly closed *-subalgebra of B such that

 $U(x) = wb\Phi(x)$ for all $x \in L_r(\tau) \cap \mathscr{A}$.

(\mathscr{A}, τ) and (\mathscr{B}, ν) semi-finite von Neumann algebras $(1 \leq r < \infty, r \neq 2)$

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Key ingredient

Suppose (\mathscr{A}, τ) is a semi-finite von Neumann algebra and $1 \leq p < \infty$, $p \neq 2$. If $x, y \in L_p(\tau)$, then equality

$$||x+y||_{p}^{p} + ||x-y||_{p}^{p} = 2||x||_{p}^{p} + 2||y||_{p}^{p}$$

holds in Clarkson's inequality

$$\begin{aligned} & \|x+y\|_{p}^{p} + \|x-y\|_{p}^{p} &\leq 2\|x\|_{p}^{p} + 2\|y\|_{p}^{p} \qquad (1 \leq p < 2) \\ & \|x+y\|_{p}^{p} + \|x-y\|_{p}^{p} &\geq 2\|x\|_{p}^{p} + 2\|y\|_{p}^{p} \qquad (2 < p < \infty) \end{aligned}$$

if and only if $xy^* = 0 = x^*y$.

If $T(e)^*T(f) = 0 = T(e)T(f)^*$ whenever *e* and *f* are orthogonal projections, then we will call *T* disjointness-preserving.

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Isometries on Lorentz spaces

Let (\mathscr{A}, τ) and (\mathscr{B}, ν) be finite von Neumann algebras with $\tau(1) = 1 = \nu(1)$ and suppose $\psi : [0,1] \to [0,\infty)$ is a strictly concave continuous increasing function with $\psi(0) = 0$. A continuous surjective linear mapping $U : \Lambda_{\psi}(\tau) \to \Lambda_{\psi}(\nu)$ is an isometry if and only if there exist uniquely a unitary operator $a \in \mathscr{B}$ and a Jordan *-isomorphism Φ of \mathscr{A} onto \mathscr{B} such that

$$au(x) = v(\Phi(x)) \qquad \forall x \in \mathscr{A}$$

and

$$U(x) = a\Phi(x) \qquad \forall x \in \mathscr{A}.$$

Key ingredient

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Let (\mathscr{A}, τ) be a finite von Neumann algebras with $\tau(1) = 1$ and suppose $\psi : [0,1] \to [0,\infty)$ is a strictly concave continuous increasing function with $\psi(0) = 0$. An element $x \in \Lambda_{\psi}(\tau)$ is an extreme point of the unit ball of $\Lambda_{\psi}(\tau)$ if and only if $x = \frac{1}{\psi(\tau(|v|))}v$ for some partial isometry $v \in \mathscr{A}$.

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sometries on symmetric spaces

Suppose (\mathscr{A}, τ) is an AFD factor of type II_1 or II_{∞} and suppose $E(0,\infty)$ is a separable symmetric space such that the norms on $E(\tau)$ and $L_2(\tau)$ are not proportional. Then a continuous linear mapping U of $E(\tau)$ onto itself is an isometry if and only if there exist a unitary operator $a \in \mathscr{A}$ and a Jordan *-automorphism Φ of \mathscr{A} such that

$$U(x) = a\Phi(x)$$
 $\forall x \in \mathscr{A} \cap E(\tau).$

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Key ingredient

Let *H* be a Hermitian operator on the separable symmetric space $E(\mathscr{A}, \tau)$. If $E(\mathscr{A}, \tau) \neq L_2(\mathscr{A}, \tau)$, then H^2 is Hermitian if and only if *H* can be represented as either a left multiplication or a right multiplication by a self-adjoint operator in \mathscr{A} .

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sometries on symmetric spaces

Let (\mathscr{A}, τ) and (\mathscr{B}, v) be trace-finite von Neumann algebras and suppose $E \subseteq S(\mathscr{A}, \tau)$ is a symmetric space and $F \subseteq S(\mathscr{B}, v)$ is a fully symmetric space. If $U: E \to F$ is a positive linear isometry from E onto F, then there exist uniquely a positive operator $a \in S(Z(B), v)$ and a Jordan *-isomorphism Φ of \mathscr{A} onto \mathscr{B} such that s(a) = 1 and

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Extension procedures

Aim: Define a map on projections. Extend it to a Jordan *-homomorphism Difficulties

- Linearity of extension to linear combination of projections
- Linearity of extension from $\mathscr{F}(au)$ to \mathscr{A}

Typically we will define $\Phi(p) = s(U(p))$ Why not use Dye's Theorem?

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Positive surjective isometries (2015)

If U is a positive linear isometry from E onto F such that $v(s(U(p))) < \infty$ whenever $p \in \mathscr{A}$ is a projection with finite trace, then there exist uniquely

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Positive surjective isometries Surjective isometries on Symmetric spaces Positive surjective isometries on Orlicz spaces

Surjective isometries (2016)

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then there exist uniquely

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Surjective isometries (2016)

If U is a linear isometry from E onto F such that

- v(s(U(p))) < ∞ whenever p ∈ 𝔄 is a projection with finite trace
- U(p)*U(q) = 0 = U(p)U(q)*, whenever p and q are orthogonal projections with finite trace

then there exist uniquely

- a unitary operator $v \in \mathscr{B}$
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Positive surjective isometries Surjective isometries on Symmetric spaces Positive surjective isometries on Orlicz spaces

Two natural questions

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P. de Jager, J. Conradie, R. Martin Isometries on symmetric spaces associated with semi-finite

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Positive surjective isometries on Orlicz spaces

What are Orlicz spaces?

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What are Orlicz spaces?

An Orlicz function is a convex function $\phi : [0\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. For a measurable function f define

$$l_{\phi}(f) = \int_{\Omega} \phi(|f(t)|) d\mu$$

The Orlicz space $L^{\phi}(\mu)$ is defined as

 $\{f\in L^0(\mu): l_\phi(\lambda f)<\infty$ for some $\lambda>0\}$

This is a Banach space when equipped with the norm

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Non-commutative Orlicz spaces can be generated using the process described earlier

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Isometries on Orlicz spaces

We will be interested in positive surjective modular isometries A map $U: L^{\phi}(\tau) \rightarrow L^{\phi}(\nu)$ is called a modular isometry if

$$l_{\phi}(\mu_{U(x)}) = l_{\phi}(\mu_x) \qquad \forall x \in L^{\phi}(\tau)$$

It is easily checked that every modular isometry is an isometry In the commutative setting, all surjective isometries of Orlicz spaces (over the complex numbers) are modular isometries

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We will consider three scenarios corresponding to three types of Orlicz functions

To distinguish these types, we will use the following quantities $a_{\phi} := \sup\{t \ge 0 : \phi(t) = 0\}, \ b_{\phi} := \sup\{t \ge 0 : \phi(t) < \infty\}$

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Scenario 1

 \mathscr{A} and \mathscr{B} semi-finite von Neumann algebras equipped with fns traces τ and ν (with $\tau(1) = \infty = \nu(1)$). $0 < a_{\phi} < \infty$, $b_{\phi} = \infty$, ϕ has at least one point of discontinuity In this case $\mathscr{A} \subseteq L^{\phi}(\tau)$ and $\mathscr{B} \subseteq L^{\phi}(\nu)$.

Theorem

Suppose these conditions hold. If $U: L^{\phi}(\tau) \to L^{\phi}(\nu)$ is a positive surjective modular isometry, then the restriction of U to \mathscr{A} is a Jordan *-isomorphism from \mathscr{A} onto \mathscr{B}

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Sketch of the proof

Show that $\mu_{U(1)} = \chi_{[0,\infty)}$. This can be used to show that U is unital. Show that $U(\mathscr{A}) \subseteq \mathscr{B}$ Show that U^{-1} is positive and that $U^{-1}(\mathscr{B}) \subseteq \mathscr{A}$ Remark: It is interesting to note that one can show directly that U is disjointness-preserving.

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Show that $\mu_{U(1)} = \chi_{[0,\infty)}$. This can be used to show that U is unital. Show that $U(\mathscr{A}) \subseteq \mathscr{B}$ Show that U^{-1} is positive and that $U^{-1}(\mathscr{B}) \subseteq \mathscr{A}$ Remark: It is interesting to note that one can show directly that is disjoint as

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Scenario 2

 \mathscr{A} and \mathscr{B} semi-finite von Neumann algebras equipped with fns traces τ and v (with $\tau(1) = v(1) < \infty$). ϕ discontinuous Orlicz function with $\tau(1)\phi(b_{\phi}) < 1$. In this case $\mathscr{A} \subseteq L^{\phi}(\tau)$ and $\mathscr{B} \subseteq L^{\phi}(v)$.

Theorem

If U is a positive surjective modular isometry from $L^{\phi}(\tau)$ onto $L^{\phi}(v)$, then the restriction of U to \mathscr{A} is a Jordan *-isomorphism from \mathscr{A} onto \mathscr{B} .

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Show that there exists a Jordan *-isomorphism from \mathscr{A} onto \mathscr{B} and a positive operator *a* such that $U(x) = a\Phi(x)$ for all $x \in \mathscr{A}$ Note that a = U(1)Show that *U* is unital

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If
$$\phi(t) = \begin{cases} t & \text{if } 0 \le t \le 1\\ \infty & \text{if } t > 1 \end{cases}$$
, then $L^{\phi}(\tau) = L^1 \cap L^{\infty}(\tau)$ with equality of norms.

Theorem

Suppose (\mathscr{A}, τ) and (\mathscr{B}, v) are non-atomic semi-finite von Neumann algebras with $\tau(1) = \infty = v(1)$. If U is a positive surjective isometry from $L^1 \cap L^{\infty}(\tau)$ onto $L^1 \cap L^{\infty}(v)$, then U is the restriction of a trace-preserving Jordan *-isomorphism from \mathscr{A} onto \mathscr{B} .

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Sketch of the proof

Show that the extreme points of the unit balls of these spaces can be characterized as the partial isometries with unit trace. Show that U is disjointness-preserving and finiteness-preserving. Let $\Phi(p) = U(p)$ for projections p with finite-trace Φ can be extended to a map which is square-preserving and L_{∞} -isometric on self-adjoint elements of $\mathscr{F}(\tau)$. Show that U and Φ agree on $\mathscr{F}(\tau)$.

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- My supervisors, Dr. Jurie Conradie and Dr. Robert Martin for their help, insight and patience.

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