On Some Ergodic Properties for C^*-dynamical systems arising from Yang-Baxter-Hecke Quantisation

Vito Crismale
Dipartimento di Matematica, Università di Bari
(joint work with F. Fidaleo and Y.G. Lu)

Workshop Operator Algebras and Quantum Dynamics
Pretoria, 12.07.2017

Outline

- Preliminaries about some ergodic properties for C^* dynamical systems.
- Connections with Yang-Baxter-Hecke quantisation
- Classification of stationary states in the monotone case
- Classification of stationary and symmetric states in the Boolean case

Motivations

- Dykema-Fidaleo (HJM 2010)
  The shift on the C^*-algebras generated by Fock representation of q-commutation relations, for $|q| < 1$ is uniquely mixing

Motivations

- Dykema-Fidaleo (HJM 2010)
  The shift on the C^*-algebras generated by Fock representation of q-commutation relations, for $|q| < 1$ is uniquely mixing
- ...what happens for Fermi, Bose, Boolean, Monotone? They are examples of the so-called Yang-Baxter-Hecke deformations firstly studied by Bożejko (2012)

1. Symmetric and stationary states

Let $\mathfrak{A}$ be a C^*–algebra and G a group which acts as a group of automorphisms of $\mathfrak{A}$:

$$\alpha : g \in G \mapsto \alpha_g \in Aut(\mathfrak{A}).$$

1. Symmetric and stationary states

Let $\mathfrak{A}$ be a C^*–algebra and G a group which acts as a group of automorphisms of $\mathfrak{A}$:

$$\alpha : g \in G \mapsto \alpha_g \in Aut(\mathfrak{A}).$$

- The fixed point subalgebra

$$\mathfrak{A}^G := \{A \in \mathfrak{A} \mid \alpha_g(A) = A, g \in G\}$$

1. Symmetric and stationary states

Let $\mathfrak{A}$ be a C^*–algebra and G a group which acts as a group of automorphisms of $\mathfrak{A}$:

$$\alpha : g \in G \mapsto \alpha_g \in Aut(\mathfrak{A}).$$

- The fixed point subalgebra

$$\mathfrak{A}^G := \{A \in \mathfrak{A} \mid \alpha_g(A) = A, g \in G\}$$

- $\mathcal{S}_G(\mathfrak{A})$ is the set of the G–invariant states

$$\varphi = \varphi \circ \alpha_g \,, \quad \varphi \in \mathcal{S}(\mathfrak{A}) \,, \quad g \in G$$

  is $*$–weakly compact in $\mathcal{S}(\mathfrak{A})$.
  Its **extremal points** are called **ergodic** states.

1. Symmetric and stationary states

Let $\mathfrak{A}$ be a C^*–algebra and G a group which acts as a group of automorphisms of $\mathfrak{A}$:

$$\alpha : g \in G \mapsto \alpha_g \in Aut(\mathfrak{A}).$$

- The fixed point subalgebra

$$\mathfrak{A}^G := \{A \in \mathfrak{A} \mid \alpha_g(A) = A, g \in G\}$$

- $\mathcal{S}_G(\mathfrak{A})$ is the set of the G–invariant states

$$\varphi = \varphi \circ \alpha_g\,, \quad \varphi \in \mathcal{S}(\mathfrak{A})\,, \quad g \in G$$

  is $*$–weakly compact in $\mathcal{S}(\mathfrak{A})$.
  Its **extremal points** are called **ergodic** states.

$(\mathfrak{A}, \alpha_g)$ is a C^*-dynamical system.

Examples

- $G = \mathbb{P}_J$
  Let J be any set. The group of the permutations $\mathbb{P}_J$ of J is

$$\mathbb{P}_J := \bigcup \{\mathbb{P}_I \mid I \subset J \text{ finite}\}.$$

Examples

- $G = \mathbb{P}_J$

  Let J be any set. The group of the permutations $\mathbb{P}_J$ of J is

  $$\mathbb{P}_J := \bigcup \{\mathbb{P}_I \mid I \subset J \, \text{finite}\}.$$

  A state φ on $\mathfrak{A}$ is called *symmetric* if it is $\mathbb{P}_J$–invariant,i.e.

  $$\varphi \circ \alpha_g = \varphi \,, \quad \forall g \in \mathbb{P}_J \,.$$

Examples

- $G=\mathbb{P}_J$
  Let J be any set. The group of the permutations $\mathbb{P}_J$ of J is
  $$\mathbb{P}_J := \bigcup\{\mathbb{P}_I \mid I \subset J \,\text{finite}\,\}.$$
  A state φ on $\mathfrak{A}$ is called *symmetric* if it is $\mathbb{P}_J$–invariant,i.e.
  $$\varphi \circ \alpha_g = \varphi\,, \quad \forall g \in \mathbb{P}_J\,.$$

- $G=\mathbb{Z}$. We have a (discrete) C^*-dynamical system made by $(\mathfrak{A}, \alpha)$ based on a single automorphism α of $\mathfrak{A}$, which automatically generates the action of $\mathbb{Z}$.

Examples

- $G = \mathbb{P}_J$
  Let J be any set. The group of the permutations $\mathbb{P}_J$ of J is
  $$\mathbb{P}_J := \bigcup \{\mathbb{P}_I \mid I \subset J \,\text{finite}\,\}.$$
  A state φ on $\mathfrak{A}$ is called *symmetric* if it is $\mathbb{P}_J$–invariant,i.e.
  $$\varphi \circ \alpha_g = \varphi\,, \ \ \forall g \in \mathbb{P}_J\,.$$
- $G = \mathbb{Z}$. We have a (discrete) C^*-dynamical system made by $(\mathfrak{A}, \alpha)$ based on a single automorphism α of $\mathfrak{A}$, which automatically generates the action of $\mathbb{Z}$.
  Invariant states are said *stationary*.

1. Symmetric and stationary states

Focus on $G = \mathbb{Z}$. Suppose that $\mathcal{S}^{\mathbb{Z}}(\mathfrak{A}) = \{\omega\}$ is a singleton. $(\mathfrak{A}, \alpha)$ is said to be *uniquely ergodic*. One can see that unique ergodicity is equivalent to

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\alpha^k(a)) = f(\mathbf{1})\omega(a)\,, \quad a \in \mathfrak{A}\,, f \in \mathfrak{A}^*\,,$$

or to

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(a) = \omega(a)\mathbf{1}, \quad a \in \mathfrak{A}\,,$$

in norm.

1. Symmetric and stationary states

Some natural generalisations are the *unique weak mixing*:

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}|f(\alpha^k(a))-f(\mathbf{1})\omega(a)|=0\,,\quad a\in\mathfrak{A}\,,f\in\mathfrak{A}^*\,,$$

Some natural generalisations are the *unique weak mixing*:

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}|f(\alpha^k(a))-f(\mathbb{1})\omega(a)|=0\,,\quad a\in\mathfrak{A}\,,f\in\mathfrak{A}^*\,,$$

or the *unique mixing*

$$\lim_{n\to+\infty}f(\alpha^n(a))=f(\mathbb{1})\omega(a)\,,\quad a\in\mathfrak{A}\,,f\in\mathfrak{A}^*\,,$$

for some state $\omega\in\mathcal{S}(\mathfrak{A})$ which is necessarily invariant.

1. Symmetric and stationary states

Some natural generalisations are the *unique weak mixing*:

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}|f(\alpha^k(a))-f(\mathbb{1})\omega(a)|=0\,,\quad a\in\mathfrak{A}\,,f\in\mathfrak{A}^*\,,$$

or the *unique mixing*

$$\lim_{n\to+\infty}f(\alpha^n(a))=f(\mathbb{1})\omega(a)\,,\quad a\in\mathfrak{A}\,,f\in\mathfrak{A}^*\,,$$

for some state $\omega\in\mathcal{S}(\mathfrak{A})$ which is necessarily invariant.

Remark

Unique mixing $\Rightarrow$ unique weak mixing $\Rightarrow$ uniquely ergodic

1. Symmetric and stationary states

For all these cases, $\mathfrak{A}^{\mathbb{Z}} = \mathbb{C}\mathbf{1}$, and the (unique) invariant conditional expectation onto the fixed-point subalgebra is $E(a) = \omega(a)\mathbf{1}$.

1. Symmetric and stationary states

For all these cases, $\mathfrak{A}^{\mathbb{Z}} = \mathbb{C}\mathbf{1}$, and the (unique) invariant conditional expectation onto the fixed-point subalgebra is $E(a) = \omega(a)\mathbf{1}$.
But in general $\mathbb{C}\mathbf{1} \subset \mathfrak{A}^{\mathbb{Z}}$... Now suppose there exists a conditional expectation $E^{\mathbb{Z}} : \mathfrak{A} \to \mathfrak{A}^{\mathbb{Z}}$. Is it uniquely invariant?
So one has that, for $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$,

For all these cases, $\mathfrak{A}^{\mathbb{Z}} = \mathbb{C}\mathbf{1}$, and the (unique) invariant conditional expectation onto the fixed-point subalgebra is $E(a) = \omega(a)\mathbf{1}$.
But in general $\mathbb{C}\mathbf{1} \subset \mathfrak{A}^{\mathbb{Z}}$... Now suppose there exists a conditional expectation $E^{\mathbb{Z}} : \mathfrak{A} \to \mathfrak{A}^{\mathbb{Z}}$. Is it uniquely invariant?
So one has that, for $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\alpha^k(a)) = f(E^{\mathbb{Z}}(a)), \ (E^{\mathbb{Z}}\text{-ergodicity})$$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} |f(\alpha^k(a)) - f(E^{\mathbb{Z}}(a))| = 0, \ (E^{\mathbb{Z}}\text{-weak mixing})$$

$$\lim_{n\to+\infty} f(\alpha^n(a)) = f(E^{\mathbb{Z}}(a)), \ (E^{\mathbb{Z}}\text{-mixing})$$

2. Yang-Baxter Deformation

Let $\mathcal{H}$ be a Hilbert space. $T : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that $T = T^*$, $T \geq -I$ and

$$T_1 T_2 T_1 = T_2 T_1 T_2 \ ,$$

where $T_1 := T \otimes I$ and $T_2 := I \otimes T$ on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, is called a Yang-Baxter operator.

2. Yang-Baxter Deformation

Let $\mathcal{H}$ be a Hilbert space. $T : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that $T = T^*$, $T \geq -I$ and

$$T_1 T_2 T_1 = T_2 T_1 T_2 \ ,$$

where $T_1 := T \otimes I$ and $T_2 := I \otimes T$ on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, is called a Yang-Baxter operator. If

$$T_k := \underbrace{I \otimes \cdots \otimes I}_{k-1 \text{ times}} \otimes T \otimes \underbrace{I \otimes \cdots \otimes I}_{n-k-1 \text{ times}} \ \text{ on } \mathcal{H}^{\otimes n}.$$

then

$$T_i T_j = T_j T_i \quad \text{for } |i - j| \geq 2$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \, .$$

2. Yang-Baxter Deformation

For each n, the T-symmetrizator is defined as

$$P_T^{(n)} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n} ,$$

where $P_T^{(1)} := I$, $P_T^{(2)} := I + T_1$ and, for $n \geq 2$,

$$P_T^{(n+1)} := (I \otimes P_T^{(n)}) R^{(n+1)} = (R^{(n+1)})^* (I \otimes P_T^{(n)}) \; ,$$

2. Yang-Baxter Deformation

For each n, the T-symmetrizator is defined as

$$P_T^{(n)} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}\,,$$

where $P_T^{(1)} := I$, $P_T^{(2)} := I + T_1$ and, for $n \geq 2$,

$$P_T^{(n+1)} := (I \otimes P_T^{(n)}) R^{(n+1)} = (R^{(n+1)})^* (I \otimes P_T^{(n)})\ ,$$

$R^{(n)} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ is

$$R^{(n)} := I + T_1 + T_1 T_2 + \ldots + T_1 T_2 \cdots T_{n-1}\,.$$

2. Yang-Baxter Deformation

An operator $V : \mathcal{H} \to \mathcal{H}$ is called a Hecke operator if there exists $q \geq -1$ such that

$$V^2 = (q-1)V + qI \ .$$

2. Yang-Baxter Deformation

An operator $V : \mathcal{H} \to \mathcal{H}$ is called a Hecke operator if there exists $q \geq -1$ such that

$$V^2 = (q-1)V + qI \ .$$

We get T a selfadjoint Yang-Baxter-Hecke operator.

2. Yang-Baxter Deformation

An operator $V : \mathcal{H} \to \mathcal{H}$ is called a Hecke operator if there exists $q \geq -1$ such that

$$V^2 = (q-1)V + qI \ .$$

We get T a selfadjoint Yang-Baxter-Hecke operator.

Bożejko proved that for each n

$$(P_T^{(n)})^2 = \underline{n}! P_T^{(n)} = \underline{n}!(P_T^{(n)})^* \geq 0 \,,$$

where $\underline{n} := 1 + q + q^2 + \ldots + q^{n-1}$, $\underline{n}! := \underline{1} \cdot \underline{2} \cdots \underline{n}$

2. Yang-Baxter Deformation

An operator $V : \mathcal{H} \to \mathcal{H}$ is called a Hecke operator if there exists $q \geq -1$ such that

$$V^2 = (q-1)V + qI \ .$$

We get T a selfadjoint Yang-Baxter-Hecke operator.

Bożejko proved that for each n

$$(P_T^{(n)})^2 = \underline{n}! P_T^{(n)} = \underline{n}!(P_T^{(n)})^* \geq 0 \, ,$$

where $\underline{n} := 1 + q + q^2 + \ldots + q^{n-1}$, $\underline{n}! := \underline{1} \cdot \underline{2} \cdots \underline{n}$ and

$$\| P_T^{(n)} \| = \underline{n}!$$

2. Yang-Baxter Deformation

Define a pre-inner product for $\xi \in \mathcal{H}^{\otimes n}$, $\eta \in \mathcal{H}^{\otimes m}$, by

$$\langle \xi, \eta \rangle_T := \delta_{n,m} \langle \xi, P_T^{(n)} \eta \rangle \ ,$$

By "routine" arguments, the T-deformed Fock space is

$$\mathcal{F}_T(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_T^n$$

for $\mathcal{H}_T^0 = \mathbb{C}$ and $\mathcal{H}_T^1 = \mathcal{H}$.
$\Omega := (1, 0, 0, \dots)$ is the vacuum.

2. Yang-Baxter Deformation

Define a pre-inner product for $\xi \in \mathcal{H}^{\otimes n}$, $\eta \in \mathcal{H}^{\otimes m}$, by

$$\langle \xi, \eta \rangle_T := \delta_{n,m} \langle \xi, P_T^{(n)} \eta \rangle \ ,$$

By "routine" arguments, the T-deformed Fock space is

$$\mathcal{F}_T(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_T^n$$

for $\mathcal{H}_T^0 = \mathbb{C}$ and $\mathcal{H}_T^1 = \mathcal{H}$.
$\Omega := (1, 0, 0, \dots)$ is the vacuum.
For each $f \in \mathcal{H}$, $n \in \mathbb{N}$, the creation operator is given by

$$a^\dagger(f)\xi := f \otimes \xi \, , \ \ \xi \in \mathcal{H}_T^n \, .$$

Its conjugate $a(f)$ s.t. $a(f)\Omega = 0$ is called the annihilator.

2. Yang-Baxter Deformation

From now on $\mathcal{H} := \ell^2(J)$ for some index-set J with cardinality the Hilbertian dimension of $\mathcal{H}$.

2. Yang-Baxter Deformation

From now on $\mathcal{H} := \ell^2(J)$ for some index-set J with cardinality the Hilbertian dimension of $\mathcal{H}$.
If e_j, $j \in J$, is the generic element of the canonical basis, we get

$$a_j := a(e_j)\ , \quad a_j^\dagger := a^\dagger(e_j)\ .$$

2. Yang-Baxter Deformation

From now on $\mathcal{H} := \ell^2(J)$ for some index-set J with cardinality the Hilbertian dimension of $\mathcal{H}$.
If e_j, $j \in J$, is the generic element of the canonical basis, we get

$$a_j := a(e_j) \ , \quad a_j^\dagger := a^\dagger(e_j) \ .$$

Define t by

$$T(e_i \otimes e_j) := \sum_{k,l \in J} t_{ij}^{kl} e_k \otimes e_l,$$

and get the commutation rule for creation and annihilations operators:

$$a_i a_j^\dagger - \sum_{k,l \in J} t_{jl}^{ik} a_k^\dagger a_l = \delta_{ij} I \ , \quad i,j \in J.$$

Examples

Examples

- **Bosons or CCR relations**
  Take $\sigma : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ the flip map $(\sigma(x \otimes y) := y \otimes x)$.
  The Symmetric Fock spaces (and the CCR algebra) are obtained with $T_{\text{Bose}} = \sigma$ and

$$P^{(n)}_{\text{Bose}} = \sum_{\pi \in \mathbb{P}_n} \pi \, .$$

Examples

- **Bosons or CCR relations**
  Take $\sigma : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ the flip map
  $(\sigma(x \otimes y) := y \otimes x)$.
  The Symmetric Fock spaces (and the CCR algebra) are obtained with $T_{\text{Bose}} = \sigma$ and

$$P^{(n)}_{\text{Bose}} = \sum_{\pi \in \mathbb{P}_n} \pi \, .$$

- **Fermions or CAR relations**
  In this case $T_{\text{Fermi}} = -\sigma$, and

$$P^{(n)}_{\text{Fermi}} = \sum_{\pi \in \mathbb{P}_n} \epsilon(\pi)\pi \, ,$$

where $\epsilon(\pi)$ denotes the sign of the permutation π.

- **Monotone**
  Here one has

$$T_m(e_i \otimes e_j) := \begin{cases} 0 & \text{if } i < j\,, \\ -(e_i \otimes e_j) & \text{if } i \geq j\,. \end{cases}$$

and $P_m^{(n)}$ is the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto the linear span of $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} | i_1 < i_2 < \cdots < i_n\}$.

- **Monotone**
  Here one has

$$T_m(e_i \otimes e_j) := \begin{cases} 0 & \text{if } i < j\,, \\ -(e_i \otimes e_j) & \text{if } i \geq j\,. \end{cases}$$

  and $P_m^{(n)}$ is the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto the linear span of $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} | i_1 < i_2 < \cdots < i_n\}$.
- **Boolean**
  Here $T = -I$ and, consequently, $P^{(n)} = 0$ if $n \geq 2$.

3. Ergodic Properties in Yang-Baxter-Hecke Case

Our aim: finding ergodic properties for $(\mathfrak{A}, \alpha)$, where α is the shift and $\mathfrak{A}$ is a C^*-algebras generated by creation and annihilation operators on $\mathcal{F}_T(\mathcal{H})$. We have basically 2 obstructions:

Our aim: finding ergodic properties for $(\mathfrak{A}, \alpha)$, where α is the shift and $\mathfrak{A}$ is a C^*-algebras generated by creation and annihilation operators on $\mathcal{F}_T(\mathcal{H})$. We have basically 2 obstructions:

- One has

$$\left\| a_i \lceil_{\mathcal{H}_T^n} \right\|_T = \left\| a_i^\dagger \lceil_{\mathcal{H}_T^n} \right\|_T \leq \left\| R^{(n+1)} \right\|^{\frac{1}{2}},$$

  i.e. a_i and $a_i^\dagger$ are not necessarily bounded

Our aim: finding ergodic properties for $(\mathfrak{A}, \alpha)$, where α is the shift and $\mathfrak{A}$ is a C^*-algebras generated by creation and annihilation operators on $\mathcal{F}_T(\mathcal{H})$. We have basically 2 obstructions:

- One has

$$\left\| a_i \lceil_{\mathcal{H}_T^n} \right\|_T = \left\| a_i^\dagger \lceil_{\mathcal{H}_T^n} \right\|_T \leq \left\| R^{(n+1)} \right\|^{\frac{1}{2}} ,$$

  i.e. a_i and $a_i^\dagger$ are not necessarily bounded

  This is the case of Canonical Commutation Relation (CCR) algebra (Bosons).

- In
$$a_i a_j^\dagger - \sum_{k,l\in J} t_{jl}^{ik} a_k^\dagger a_l = \delta_{ij} I \,, \quad i,j \in J.$$
we may have infinite sums.

- In
$$a_i a_j^\dagger - \sum_{k,l\in J} t_{jl}^{ik} a_k^\dagger a_l = \delta_{ij} I\,, \quad i,j \in J.$$
we may have infinite sums.

  This case appears, e.g., for Boolean and Monotone C^*–algebras...
  But we can reach our goal even in this situation.

3. Ergodic Properties in Yang-Baxter-Hecke Case

To pursue the goal, we need to make another condition:

$$M_T := \sup_{n\in\mathbb{N}} \|R^{(n)}\| < \infty \, .$$

Then one has that, for each $f \in \mathcal{H}$

$$\|a_i\|_T = \|a_i^\dagger\|_T \leq \sqrt{M_T} \, .$$

To pursue the goal, we need to make another condition:

$$M_T := \sup_{n\in\mathbb{N}} \|R^{(n)}\| < \infty \,.$$

Then one has that, for each $f \in \mathcal{H}$

$$\|a_i\|_T = \|a_i^\dagger\|_T \leq \sqrt{M_T} \,.$$

The uniform boundedness above is satisfied in many cases of interests (e.g. Boolean and Monotone).

3. Ergodic Properties in Yang-Baxter-Hecke Case

To pursue the goal, we need to make another condition:

$$M_T := \sup_{n\in\mathbb{N}} \|R^{(n)}\| < \infty\,.$$

Then one has that, for each $f \in \mathcal{H}$

$$\|a_i\|_T = \|a_i^\dagger\|_T \leq \sqrt{M_T}\,.$$

The uniform boundedness above is satisfied in many cases of interests (e.g. Boolean and Monotone).
Remark The condition above is only sufficient for the boundedness of the annihilators!
For Fermions creators and annihilators are bounded **but** the operators $R^{(n)}$ are not *uniformly* bounded.

Take:

3. Ergodic Properties in Yang-Baxter-Hecke Case

Take:

- $$\mathfrak{R}_T := \overline{* - alg\{a_i \mid i \in \mathbb{Z}\}}^{\|\cdot\|}$$

- $$\mathfrak{G}_T := \overline{* - alg\{a_i + a_i^\dagger \mid i \in \mathbb{Z}\}}^{\|\cdot\|}$$

3. Ergodic Properties in Yang-Baxter-Hecke Case

Take:

- $$\mathfrak{R}_T := \overline{* - alg\{a_i \mid i \in \mathbb{Z}\}}^{\|\cdot\|}$$
- $$\mathfrak{G}_T := \overline{* - alg\{a_i + a_i^\dagger \mid i \in \mathbb{Z}\}}^{\|\cdot\|}$$

We suppose the unitary $e_i \mapsto e_{i+1}$, $i \in \mathbb{Z}$ acts as Bogoliubov automorphisms α^n, $n \in \mathbb{Z}$ on $\mathfrak{R}_T$ by

$$\alpha(a_i) := a_{i+1} \ , i \in \mathbb{Z}$$

Then it acts also on $\mathfrak{G}_T$ by restriction.

3. Ergodic Properties in Yang-Baxter-Hecke Case

Problem: the C^*-dynamical systems $(\mathfrak{R}_T, \alpha)$ and $(\mathfrak{G}_T, \alpha)$ enjoy some strong ergodic property?

Problem: the C^*-dynamical systems $(\mathfrak{R}_T, \alpha)$ and $(\mathfrak{G}_T, \alpha)$ enjoy some strong ergodic property?

What we already know:
For a Yang-Baxter operator T with $\|T\| < 1$, Dykema-Fidaleo proved that these C^*-algebras are uniquely mixing for the shift with the vacuum expectation as the only invariant state.

Problem: the C^*-dynamical systems $(\mathfrak{R}_T, \alpha)$ and $(\mathfrak{G}_T, \alpha)$ enjoy some strong ergodic property?

What we already know:
For a Yang-Baxter operator T with $\|T\| < 1$, Dykema-Fidaleo proved that these C^*-algebras are uniquely mixing for the shift with the vacuum expectation as the only invariant state.
But for a Yang-Baxter-Hecke operator: $\|T\| \geq 1$...

Theorem (C. -Fidaleo -Lu)

Let T be a Yang-Baxter-Hecke selfadjoint operator on $\mathcal{H} = \ell^2(\mathbb{Z})$. Suppose that

3. Ergodic Properties in Yang-Baxter-Hecke Case

Theorem (C. -Fidaleo -Lu)

Let T be a Yang-Baxter-Hecke selfadjoint operator on $\mathcal{H} = \ell^2(\mathbb{Z})$. Suppose that

- *the sum below is finite:*

$$a_i a_j^\dagger - \sum_{k,l} t_{jl}^{ik} a_k^\dagger a_l = \delta_{ij} I$$

3. Ergodic Properties in Yang-Baxter-Hecke Case

Theorem (C. -Fidaleo -Lu)

Let T be a Yang-Baxter-Hecke selfadjoint operator on $\mathcal{H} = \ell^2(\mathbb{Z})$. Suppose that

- *the sum below is finite:*

$$a_i a_j^\dagger - \sum_{k,l} t_{jl}^{ik} a_k^\dagger a_l = \delta_{ij} I$$

- $M_T < +\infty$.

Theorem (C. -Fidaleo -Lu)

Let T be a Yang-Baxter-Hecke selfadjoint operator on $\mathcal{H} = \ell^2(\mathbb{Z})$. Suppose that

- *the sum below is finite:*

$$a_i a_j^\dagger - \sum_{k,l} t_{jl}^{ik} a_k^\dagger a_l = \delta_{ij} I$$

- $M_T < +\infty$.
- *the group $\mathbb{Z}$ acts as a group of automorphisms on $\mathfrak{R}_T$*

3. Ergodic Properties in Yang-Baxter-Hecke Case

Theorem (C. -Fidaleo -Lu)

Let T be a Yang-Baxter-Hecke selfadjoint operator on $\mathcal{H} = \ell^2(\mathbb{Z})$. Suppose that

- *the sum below is finite:*

$$a_i a_j^\dagger - \sum_{k,l} t_{jl}^{ik} a_k^\dagger a_l = \delta_{ij} I$$

- $M_T < +\infty$.
- *the group $\mathbb{Z}$ acts as a group of automorphisms on $\mathfrak{R}_T$*

Then the dynamical system $(\mathfrak{R}_T, \alpha)$ is uniquely mixing with $\langle \cdot \Omega, \Omega \rangle$ as the unique invariant state.

3. Ergodic Properties in Yang-Baxter-Hecke Case

Theorem (C. -Fidaleo -Lu)

Let T be a Yang-Baxter-Hecke selfadjoint operator on $\mathcal{H} = \ell^2(\mathbb{Z})$. Suppose that

- *the sum below is finite:*

$$a_i a_j^\dagger - \sum_{k,l} t_{jl}^{ik} a_k^\dagger a_l = \delta_{ij} I$$

- $M_T < +\infty$.
- *the group $\mathbb{Z}$ acts as a group of automorphisms on $\mathfrak{R}_T$*

Then the dynamical system $(\mathfrak{R}_T, \alpha)$ is uniquely mixing with $\langle \cdot \Omega, \Omega \rangle$ as the unique invariant state.

The result above holds for $(\mathfrak{G}_T, \alpha)$ by restriction.

4. Stationary states on Monotone C^*-algebra

For $k \geq 1$

$$I_k := \{(i_1, i_2, \ldots, i_k) \mid i_1 < i_2 < \cdots < i_k, i_j \in \mathbb{Z}\},$$

and for $k = 0$, we take $I_0 := \{\emptyset\}$, $\emptyset$ being the empty sequence.

The Hilbert space $\mathcal{H}_k := \ell^2(I_k)$ is the k-particles space.
The 0-particle space is identified with $\mathbb{C}$.

The monotone Fock space is $\mathcal{F}_m = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$.

4. Stationary states on Monotone C^*-algebra

Given an increasing sequence $(i_1, i_2, \ldots, i_k)$ of integers, we take $e_{(i_1,i_2,\ldots,i_k)}$ a vector of the canonical basis of $\ell^2(I_k)$.

The monotone creation and annihilation operators are, for any $i \in \mathbb{Z}$,

$$a_i^\dagger(e_{(i_1,i_2,\ldots,i_k)}) := \begin{cases} e_{(i,i_1,i_2,\ldots,i_k)} & \text{if } i < i_1 \,, \\ 0 & \text{otherwise} \end{cases}$$

$$a_i(e_{(i_1,i_2,\ldots,i_k)}) := \begin{cases} e_{(i_2,\ldots,i_k)} & \text{if } k \geq 1 \quad \text{and} \quad i = i_1 \,, \\ 0 & \text{otherwise} \end{cases}$$

Note $\|a_i^\dagger\| = \|a_i\| = 1$.

4. Stationary states on Monotone C^*-algebra

$a_i^\dagger$ and a_i are mutually adjoint and satisfy the following relations

4. Stationary states on Monotone C^*-algebra

$a_i^\dagger$ and a_i are mutually adjoint and satisfy the following relations

$$\begin{aligned} a_i^\dagger a_j^\dagger &= a_j a_i = 0 \quad && \text{if } i \geq j\,, \\ a_i a_j^\dagger &= 0 && \text{if } i \neq j\,. \end{aligned}$$

4. Stationary states on Monotone C^*-algebra

$a_i^\dagger$ and a_i are mutually adjoint and satisfy the following relations

$$\begin{aligned} a_i^\dagger a_j^\dagger &= a_j a_i = 0 && \text{if } i \geq j\,, \\ a_i a_j^\dagger &= 0 && \text{if } i \neq j\,. \end{aligned}$$

In addition, the following commutation relation

$$a_i a_i^\dagger = I - \sum_{k \leq i} a_k^\dagger a_k$$

is also satisfied.

The C^*-algebra $\mathfrak{R}_m$ and its subalgebra $\mathfrak{G}_m$ acting on $\mathcal{F}_m$, are the unital C^*-algebras generated by the annihilators $\{a_i \mid i \in \mathbb{Z}\}$, and the selfadjoint part of annihilators $\{s_i \mid i \in \mathbb{Z}\}$ respectively, with $s_i := a_i + a_i^+$.

The C^*-algebra $\mathfrak{R}_m$ and its subalgebra $\mathfrak{G}_m$ acting on $\mathcal{F}_m$, are the unital C^*-algebras generated by the annihilators $\{a_i \mid i \in \mathbb{Z}\}$, and the selfadjoint part of annihilators $\{s_i \mid i \in \mathbb{Z}\}$ respectively, with $s_i := a_i + a_i^+$.

Fact The shift α acts on both of them and the sum in "commutation relations" is infinite... In fact

..."problem"
$(\mathfrak{R}_m, \alpha)$ is **not** uniquely ergodic (so not uniquely mixing) w.r.t. the fixed point subalgebra.

..."problem"

$(\mathfrak{R}_m, \alpha)$ is **not** uniquely ergodic (so not uniquely mixing) w.r.t. the fixed point subalgebra.

Let $i \in \mathbb{Z}$. For $n \to +\infty$, $\alpha^n(a_i a_i^\dagger) \downarrow P_\Omega$ and so, **strongly**,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(a_i a_i^\dagger) = P_\Omega$$

..."problem"
$(\mathfrak{R}_m, \alpha)$ is **not** uniquely ergodic (so not uniquely mixing) w.r.t. the fixed point subalgebra.
Let $i \in \mathbb{Z}$. For $n \to +\infty$, $\alpha^n(a_i a_i^\dagger) \downarrow P_\Omega$ and so, **strongly**,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(a_i a_i^\dagger) = P_\Omega$$

And the convergence is **not** in norm, since

$$1 \geq \left\| \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(a_i a_i^\dagger) - P_\Omega \right\| \geq \left\| \left(\frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(a_i a_i^\dagger) - P_\Omega \right) e_{(i+n)} \right\| = 1\,.$$

4. Stationary states on Monotone C^*-algebra

Main goal now: classification of shift invariant states. How?

4. Stationary states on Monotone C^*-algebra

Main goal now: classification of shift invariant states. How?

A possibility: find the structure of $\mathfrak{R}_m$ and infer in some way the result.

Main goal now: classification of shift invariant states. How?

A possibility: find the structure of $\mathfrak{R}_m$ and infer in some way the result.

We gave standard reduced forms for words in in the $*$-algebra generated by monotone commutation relations (and further we obtained also a basis)

4. Stationary states on Monotone C^*-algebra

Main goal now: classification of shift invariant states. How?

A possibility: find the structure of $\mathfrak{R}_m$ and infer in some way the result.
We gave standard reduced forms for words in in the $*$-algebra generated by monotone commutation relations (and further we obtained also a basis)

Proposition

The the fixed-point subalgebra $\mathfrak{R}_m^{\mathbb{Z}}$ w.r.t. the action of the shift α is trivial: $\mathfrak{R}_m^{\mathbb{Z}} = \mathbb{C}I$.

4. Stationary states on Monotone C^*-algebra

Then there is more than a single stationary state. How many?

Then there is more than a single stationary state. How many?

Fact

$$\mathfrak{R}_m = \mathfrak{A}_m + \mathbb{C}I.$$

...$\mathcal{S}(\mathfrak{R}_m)$ is the one-point compactification of all the positive functionals on $\mathfrak{A}_m$ with norm less than or equal to 1.

Then there is more than a single stationary state. How many?

Fact

$$\mathfrak{R}_m = \mathfrak{A}_m + \mathbb{C}I.$$

...$\mathcal{S}(\mathfrak{R}_m)$ is the one-point compactification of all the positive functionals on $\mathfrak{A}_m$ with norm less than or equal to 1.

The *state at infinity*

$$\omega_\infty(X + cI) := c\,, \quad X \in \mathfrak{A}_m\,, c \in \mathbb{C}\,,$$

provides such a "point at infinity". It is shift-invariant.

Then there is more than a single stationary state. How many?

Fact

$$\mathfrak{R}_m = \mathfrak{A}_m + \mathbb{C}I.$$

...$\mathcal{S}(\mathfrak{R}_m)$ is the one-point compactification of all the positive functionals on $\mathfrak{A}_m$ with norm less than or equal to 1.

The *state at infinity*

$$\omega_\infty(X + cI) := c\,, \quad X \in \mathfrak{A}_m\,, c \in \mathbb{C}\,,$$

provides such a "point at infinity". It is shift-invariant.
The monotone vacuum

$$\omega(Y) := \langle Y\Omega, \Omega\rangle\,, \quad Y \in \mathfrak{R}_m\,,$$

is shift-invariant

The structure of such invariant states

The structure of such invariant states

Theorem (C.-Fidaleo-Lu)

The weak *-compact set of shift-invariant states on $\mathfrak{R}_m$ is given by

$$\mathcal{S}_{\mathbb{Z}}(\mathfrak{R}_m) = \{(1-\gamma)\omega_\infty + \gamma\omega \mid \gamma \in [0,1]\}\,.$$

4. Stationary states on Monotone C^*-algebra

The structure of such invariant states

Theorem (C.-Fidaleo-Lu)

The weak *-compact set of shift-invariant states on $\mathfrak{R}_m$ is given by

$$\mathcal{S}_{\mathbb{Z}}(\mathfrak{R}_m) = \{(1-\gamma)\omega_\infty + \gamma\omega \mid \gamma \in [0,1]\}\,.$$

For the subalgebra of selfadjoints:

Theorem (C.-Fidaleo-Lu)

We have $\mathfrak{S}_m = \mathfrak{R}_m$.

5. The case of Boolean C^*-algebra

Let $\mathcal{H}$ be a complex Hilbert space. Recall that the Boolean Fock space over $\mathcal{H}$ is given by $\Gamma(\mathcal{H}) := \mathbb{C} \oplus \mathcal{H}$, where the vacuum vector Ω is $(1, 0)$.

5. The case of Boolean C^*-algebra

Let $\mathcal{H}$ be a complex Hilbert space. Recall that the Boolean Fock space over $\mathcal{H}$ is given by $\Gamma(\mathcal{H}) := \mathbb{C} \oplus \mathcal{H}$, where the vacuum vector Ω is $(1, 0)$.
On $\Gamma(\mathcal{H})$ the creation and annihilation operators, for $f \in \mathcal{H}$ are

$$a^\dagger(f)(\alpha \oplus g) := 0 \oplus \alpha f, \quad a(f)(\alpha \oplus g) := \langle g, f \rangle_{\mathcal{H}} \oplus 0, \quad \alpha \in \mathbb{C},\ g \in \mathcal{H}.$$

They are mutually adjoint, and satisfy the following relations

$$a(f)a^\dagger(g) = \langle g, f \rangle P_\Omega = \langle g, f \rangle (I - \sum_{j \in J} a^\dagger(e_j)a(e_j)),$$
$$a(f)a(g) = a^\dagger(f)a^\dagger(g) = 0, \quad f, g \in \mathcal{H}$$

for any orthonormal basis $\{e_j \mid j \in J\}$ of the involved Hilbert space.

5. The case of Boolean C^*-algebra

Put $\mathcal{H} = \ell^2(\mathbb{Z})$, $\mathfrak{R}_{\mathrm{Boole}}$ the C^*-algebra generated by boolean annihilators and $\mathfrak{G}_{\mathrm{Boole}}$ that generated by position operators.

If $\mathcal{K}(\Gamma(\ell^2(\mathbb{Z}))$ denotes the compact linear operators on the Boolean Fock, one has

Proposition (C.-Fidaleo)

$$\mathcal{K}(\Gamma(\ell^2(\mathbb{Z})) + \mathbb{C}I = \mathfrak{R}_{Boole} = \mathfrak{G}_{Boole} = \mathfrak{b}\,.$$

The shift, as well as the permutations $\mathbb{P}_{\mathbb{Z}}$ naturally act on $\mathfrak{b}$ as Bogoliubov automorphisms.

Notice that here too the sum is infinite as soon as $\dim(\mathcal{H}) = \infty$. But, one reaches the goal also in this case. In fact

5. The case of Boolean C^*-algebra

Notice that here too the sum is infinite as soon as $\dim(\mathcal{H}) = \infty$. But, one reaches the goal also in this case. In fact

Denote $\mathfrak{b}^{\mathbb{P}_\mathbb{Z}}$, $\mathfrak{b}^{\mathbb{Z}}$ the fixed-point subalgebras w.r.t. the actions of the permutations and the shift, that is the exchangeable and the invariant C^*-subalgebra, respectively. We get

Proposition (C.-Fidaleo-Lu)

For the fixed-point subalgebras $\mathfrak{b}^{\mathbb{P}_\mathbb{Z}}$, $\mathfrak{b}^{\mathbb{Z}}$, one has

$$\mathfrak{b}^{\mathbb{P}_\mathbb{Z}} = \mathfrak{b}^{\mathbb{Z}} = \mathbb{C}P_\Omega \oplus \mathbb{C}P_\Omega^\perp ,$$

where $P_\Omega = a_i a_i^\dagger$ denotes the orthogonal projection onto the linear span of Ω.

5. The case of Boolean C^*-algebra

Denote E the conditional expectation onto $\mathfrak{b}^{\mathbb{Z}}$ given by

$$E(A+bI) := \langle A\Omega, \Omega\rangle P_\Omega + bI\,, \quad A \in \mathcal{K}(\ell^2(\mathbb{Z}))\,, b \in \mathbb{C}\,.$$

It is invariant both for the action of the shift and the permutations.

5. The case of Boolean C^*-algebra

Denote E the conditional expectation onto $\mathfrak{b}^{\mathbb{Z}}$ given by

$$E(A+bI):=\langle A\Omega,\Omega\rangle P_\Omega+bI\,,\quad A\in\mathcal{K}(\ell^2(\mathbb{Z}))\,,b\in\mathbb{C}\,.$$

It is invariant both for the action of the shift and the permutations.

Proposition (C.-Fidaleo-Lu)

The C^*-dynamical system $(\mathfrak{b},\alpha)$ is $E^{\mathbb{Z}}$-mixing with $E=E^{\mathbb{Z}}$ is the unique invariant conditional expectation onto the fixed-point subalgebra.

5. The case of Boolean C^*-algebra

Denote E the conditional expectation onto $\mathfrak{b}^{\mathbb{Z}}$ given by

$$E(A + bI) := \langle A\Omega, \Omega\rangle P_\Omega + bI\,, \quad A \in \mathcal{K}(\ell^2(\mathbb{Z}))\,, b \in \mathbb{C}\,.$$

It is invariant both for the action of the shift and the permutations.

Proposition (C.-Fidaleo-Lu)

The C^*-dynamical system $(\mathfrak{b}, \alpha)$ is $E^{\mathbb{Z}}$-mixing with $E = E^{\mathbb{Z}}$ is the unique invariant conditional expectation onto the fixed-point subalgebra.

So, even if the fixed-point subalgebra is non trivial the $E^{\mathbb{Z}}$-mixing property for the shift holds. It is the first case in our setting we find a strong mixing condition with a unique **non trivial** invariant: $E = E^{\mathbb{Z}}$.

5. The case of Boolean C^*-algebra

For the symmetric and stationary states, one has

Proposition (C.-Fidaleo-Lu)

$$\mathcal{S}_{\mathbb{Z}}(\mathfrak{b}) = \mathcal{S}_{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{b}) = \{\gamma\omega_\Omega + (1-\gamma)\omega_\infty \mid \gamma \in [0,1]\}\,,$$

where

$$\omega_\infty(A + bI) := b\,, \quad A \in \mathcal{K}(\Gamma(\ell^2(\mathbb{Z})))\,, b \in \mathbb{C}\,.$$

and ω_Ω is the vacuum state.

5. The case of Boolean C^*-algebra

For the symmetric and stationary states, one has

Proposition (C.-Fidaleo-Lu)

$$\mathcal{S}_{\mathbb{Z}}(\mathfrak{b}) = \mathcal{S}_{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{b}) = \{\gamma\omega_\Omega + (1-\gamma)\omega_\infty \mid \gamma \in [0,1]\}\,,$$

where

$$\omega_\infty(A + bI) := b\,, \quad A \in \mathcal{K}(\Gamma(\ell^2(\mathbb{Z})))\,, b \in \mathbb{C}\,.$$

and ω_Ω is the vacuum state.

Hence, as in the monotone case, we have a "segment", although the situation is completely different...

Conclusions

For C^*-dynamical systems considered, we found:

Conclusions

For C^*-dynamical systems considered, we found:

	$\mathfrak{A}^{\mathbb{Z}}$	ergodicity	stat/symm
Fermi	$\mathbb{C}\mathbf{1}$	NO	CCH product states
Monotone	$\mathbb{C}\mathbf{1}$	NO	$\simeq$ segment / "No"
Boolean	$\mathbb{C}P_{\#} \oplus \mathbb{C}P_{\#}^{\perp}$	YES	$\simeq$ segment

Conclusions

For C^*-dynamical systems considered, we found:

	$\mathfrak{A}^{\mathbb{Z}}$	ergodicity	stat/symm
Fermi	$\mathbb{C}\mathbf{1}$	NO	CCH product states
Monotone	$\mathbb{C}\mathbf{1}$	NO	$\simeq$ segment / "No"
Boolean	$\mathbb{C}P_{\#} \oplus \mathbb{C}P_{\#}^{\perp}$	YES	$\simeq$ segment

Other cases already studied not covered in the talk

For CCR (better Weyl algebra) and tensor product: not uniquely mixing. The symmetric are the closed convex hull of product states.

REFERENCES

- Bożejko M. *Deformed Fock spaces, Hecke operators and monotone Fock space of Muraki*, Dem. Math. **XLV** (2012), 399-413.
- Crismale V., Fidaleo F. *De Finetti theorem on the CAR algebra*, Comm. Math. Phys. **315** (2012), 135–152.
- Crismale V., Fidaleo F. *Exchangeable stochastic processes and symmetric states in quantum probability*, Ann. Mat. Pura Appl. **194** (2015), 969–993.
- Crismale V., Fidaleo F., Lu Y.G. *Ergodic theorems in quantum probability: an application to the monotone stochastic processes*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), Vol. XVII (2017), 113-141.
- Crismale V., Fidaleo F., *Symmetryes and ergodic properties in quantum probability*, Colloq. Math., to appear
- ...work in progress...