Computable General Equilibrium Modelling

EKN 819

2021

Tukkiewerf Building

University of Pretoria

Presented by

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CGE modelling is a challenging field. It requires mastery of economic theory, meticulous preparation of data and familiarity with underlying accounting conventions, knowledge of econometric methods, and an understanding of solution algorithms and associated software for solving large equation systems. However, the most important requirement is the ability to communicate. CGE modelling is primarily about shedding light on real-world policy issues. For CGE analyses to be influential, modellers must explain their results in a way that is comprehensible and convincing to their fellow economists, and eventually to policy makers.

While CGE modelling is challenging, it is also rewarding. CGE models are used in almost every part of the world to generate insights into the effects of policies and other shocks in the areas of trade, taxation, public expenditure, social security, demography, immigration, technology, labour markets, environment, resources, infrastructure and major-project expenditures, disasters, and financial crises. CGE modelling is the only practical way of quantifying these effects on industries, occupations, regions and socioeconomic groups.

Peter B. Dixon & Dale W. Jorgenson

Course Overview

Computable general equilibrium (CGE) models are primarily designed for practical policy analysis. A key feature of CGE models is its ability to accommodate a high level of detail within a rigorous theoretical framework. Consumers of CGE analysis often include governments and large institutions who wish to understand the economy-wide effects of proposed policies or unanticipated economic shocks. CGE models help policymakers and stakeholders to identify the potential winners and losers of such events in order to strengthen their policy formulation and response framework.

The *Computable General Equilibrium Modelling* course brings together decades of CGE modelling and training experience in a comprehensive one-week package built around the University of Pretoria General Equilibrium Model (UPGEM) of South Africa. The course aims to teach participants to be **responsible CGE model users**. We understand that becoming a fully-fledged CGE modeller is a massive task beyond the scope of a single course. As such, our focus will be on developing participants' ability to run simulations and properly interpret results. The training model used for this course is a simplified version of the dynamic single-country UPGEM, developed in collaboration with the Centre of Policy Studies in Melbourne, Australia. The GEMPACK software platform will be used throughout the course. A copy of the UPGEM training model and an executable-image version of the GEMPACK software will be provided to all participants.

Training sessions are designed with the course and CGE methodology's practical focus in mind. Lectures on the theoretical and technical aspects of UPGEM and CGE modelling in general will be combined with practical hands-on sessions using GEMPACK. The second half of the course is devoted to practical exercises where participants will be required to set up simulation scenarios and interpret the results produced by UPGEM. Ultimately, this will enable participants to properly report on and present the results of a CGE simulation, drawing only on the model's theory, database and closure settings.

In order to successfully complete this course, participants must display competence in the key technical aspects of CGE modelling and be able to clearly interpret and communicate the results produced by UPGEM simulations. Participants who are interested in building a CGE model or database for their own institution or country will have the opportunity to consult with the UPGEM team after completion of the course. We wish you good luck with this course and hope that you enjoy it.

Instructors

Prof. Heinrich Bohlmann   heinrich.bohlmann@up.ac.za

Prof. Bohlmann holds a PhD in Economics from the Centre of Policy Studies in Melbourne, Australia. He joined the Department of Economics in 2011 and has been instrumental in developing dynamic CGE modelling expertise at the University of Pretoria. Prof. Bohlmann's research work has mainly focused on macroeconomic planning, energy and environmental policy, and labour market topics.

Prof. Jan van Heerden   jan.vanheerden@up.ac.za

Prof. Van Heerden holds a PhD in Economics from Rice University in Houston, USA. He has over 30 years of academic and professional experience as an economist and was the co-author of the first version of UPGEM with colleagues from the Centre of Policy Studies in Melbourne. Prof. Van Heerden's research work has mainly focused on regional economic planning, energy and environmental policy, and fiscal policy topics.

Assessments

Semester assignments and/or tests will count 40% towards your overall mark and the final examination the remaining 60%. The final exam may consist of two separate components: a theoretical exam and a take-home practical exam. An overall mark of 50% and a sub-minimum mark of 40% for each component are required to pass the course. Please note that there is no single textbook prescribed for EKN819. Assessments will be based on all materials and topics covered in class and key references listed under each section.

Additional Notes

Participants are encouraged to bring their own Windows-based laptop and mouse. Administrator rights for the laptop will be required to install the GEMPACK software. The course is laptop based and each student is responsible to provide his/her own laptop for the course. The UPGEM training model and database, GEMPACK software and licence file, and research papers distributed on the course disc are for the use of registered students only and may not be redistributed or used for commercial purposes. Those who wish to conduct CGE modelling for non-academic purposes should contact the Centre of Policy Studies at www.copsmodels.com to obtain a source-code version of GEMPACK.

Content and Schedule
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The *Computable General Equilibrium Modelling* course will be presented during the first semester of the Department's graduate programme. Students should be prepared to work extremely hard during the course. Teaching sessions will typically run from **09:00 to 12:00** on **Wednesdays**, but preparation for class and semester assignments will require many more hours each week. Initially, we will place more emphasis on the theoretical and technical aspects of CGE modelling, whereas later sessions will have a more practical focus. Key references listed under each session will be included on the course disc. Students are encouraged to consult as many of these references as possible in preparation for lectures.

Week 1-2 [3 and 10 March 2021] (HB)
- Installing and preparing GEMPACK and UPGEM
- Evolution of economy-wide analysis from input-output models to dynamic CGE models
- Overview of the basic tasks in CGE based analysis
- Structure of a typical CGE model database
- Data sources and requirements for building a typical CGE model database
- Practical exercise: Understanding S-U tables, I-O tables and SAM datasets
- Practical exercise: How are capital rentals determined within the SNA?
- Key references: Burfisher (2017), Dixon et al. (2013), Dixon & Rimmer (2013)

Week 3-4 [17 and 24 March 2021] (JVH)
- Overview of the GEMPACK suite of programs
- First look at the UPGEM database in ViewHar
- The percentage-change approach to implementing and solving CGE models
- Representation of CGE models in GEMPACK notation
- Practical exercise: who will be worst affected by a drought?
- Key references: Horridge et al. (2013), Roos et al. (2015), StatsSA (2016)

Week 5-6 [31 March and 7 April] (JVH)
- Basic functional forms used in the theoretical specification of CGE models
- Overview of the theoretical structure of dynamic UPGEM
- Reading and writing TABLO code in TABmate
- Practical exercise: Understanding the CES functional form and nesting structure
- First look at the UPGEM theory in TABmate
- Key references: Dixon et al. (1992), Harrison & Pearson (1996), Horridge et al. (2013)

Week 7 [14 April] (HB)
- Data exercise: Adding occupation and population group detail to the wage bill matrix
- Data exercise: Splitting land rentals from capital rentals
- Key references: Dixon & Rimmer (2002), Dixon et al. (2013)
- Basic model closures in CGE modelling
- Practical exercise: First simulation with UPGEM (Baseline & Sim 1A)
- Practical exercise: Using ViewSOL to look at simulation results
- Key references: Adams (2005), Dixon & Rimmer (2002; 2013), Giesecke & Schilling (2010)

Week 8-10 [21 April and 5 May][1] (JVH)
- Business-as-usual baseline projections
- Policy simulation design
- Strategies for interpreting and presenting CGE modelling results
- Practical exercise: Simulations with UPGEM
- Practical exercise: Using AnalyseGE to look at simulation results
- Special lecture: The impact of the 2014 platinum mining strike in South Africa
- Key references: Bohlmann et al. (2015), Dixon et al. (2013), Dixon & Rimmer (2013)

Week 11-14 [19 and 26 May; 2 and 9 June] (HB and JVH)
- UP Recess (10-14 May)
- Discussion of model extensions and other advanced modelling topics (HB)
- Practical exercises: Simulations with UPGEM (HB and JVH)
- Discussion of semester assignments and final examination
- Review and closing
- Key references: Adams et al. (2015), Dixon et al. (2013), Van Heerden et al. (2006), World Bank (2016)

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[1] Monday schedule followed on 28 April .

Appendix 1: Introduction to CGE Modelling

The University of Pretoria General Equilibrium Model (UPGEM) is used to conduct **economy-wide analysis** of exogenous changes or policy shocks on the South African economy. UPGEM is a flexible economic model that can be run in either comparative-static or recursive-dynamic mode. Various extensions to the basic model can be switched on when required, providing detail along regional dimensions, labour migration categories, electricity generation technologies and emissions data. In this appendix we describe the **standard recursive-dynamic version of UPGEM** similar to that used in Bohlmann et al. (2015). The ability of CGE models to recognise the many **inter-linkages** in the real economy, and account for **price-induced behaviour** and **resource constraints** in determining both the direct and indirect effects of an external shock on the economy over time, has made it one of the **preferred methodologies** for practical policy analysis around the world.

Four basic tasks distinguish CGE based analysis. **First** is the **theoretical derivation and description of the model**. The single-country dynamic version of UPGEM is based on the well-documented MONASH model described in Dixon & Rimmer (2002) and Dixon et al. (2013) developed by the Centre of Policy Studies (CoPS) in Melbourne, Australia. Following the CoPS-style of implementing a CGE model, based on the pioneering work of Johansen (1960), the general equilibrium core of UPGEM is made up of a linearised system of equations describing the theory underlying the behaviour of participants in the economy. It contains equations describing, amongst others: the nature of markets; intermediate demands for inputs to be used in the production of commodities; final demands for goods and services by households; demands for inputs to capital creation and the determination of investment; government demands for commodities; and foreign demand for exported goods.

The specifications in UPGEM recognise each industry as producing one or more commodities, using as inputs combinations of domestic and imported commodities, different types of labour, capital and land. The multi-input, multi-output production specification is kept manageable by a series of separability assumptions, as illustrated by Figure 3B in Appendix 3. Figure 3B also illustrates how the optional electricity generation composite nest would feature in the model if activated. The nested production structure reduces the number of estimated parameters required by the model. Optimising equations determining the commodity composition of industry output are derived subject to a CET function, whilst functions determining industry inputs are determined by a series of CES nests. At the top level of this nesting structure intermediate commodity composites and a primary-factor composite are combined using a Leontief or fixed-proportions production function. Consequently, they are all demanded in direct proportion to industry output or activity. Each commodity composite is a CES function of a domestic good and its imported equivalent. This incorporates Armington's assumption of imperfect substitutability for goods by place of production (Armington, 1969).

The primary-factor composite is a CES aggregate of composite labour, capital and, in the case of primary sector industries, land. Composite labour demand is itself a CES aggregate of the different types of labour distinguished in the model's database. In UPGEM, all industries share this common production structure, but input proportions and behavioural parameters vary between industries based on base year data and available econometric estimates, respectively.

The demand and supply equations in UPGEM are derived from the solutions to the optimisation problems which are assumed to underlie the behaviour of private sector agents in conventional neo-classical microeconomics. Each industry minimises cost subject to given input prices and a constant returns to scale production function. Zero pure profits are assumed for all industries. Households maximise a Klein-Rubin utility function subject to their budget constraint. Units of new industry-specific capital are constructed as cost-minimising combinations of domestic and imported commodities. The export demand for any locally produced commodity is inversely related to its foreign-currency price. Government consumption and the details of direct and indirect taxation are also recognised in the model.

The recursive-dynamic behaviour in UPGEM is specified through equations describing: physical capital accumulation; lagged adjustment processes in the labour market; and changes in the current account and net foreign liability positions. Capital accumulation is specified separately for each industry and linked to industry-specific net investment in the preceding period. Investment in each industry is positively related to its expected rate of return on capital, reflecting the price of capital rentals relative to the price of capital creation. For the government's fiscal accounts, a similar mechanism for financial asset/liability accumulation is specified. Changes in the public sector debt are related to the public sector debt incurred during a particular year and the interest payable on previous debt. Adjustments to the national net foreign liability position are related to the annual investment/savings imbalance, revaluations of assets and liabilities and remittance flows during the year. In policy simulations, the labour market follows a lagged adjustment path where wage rates respond over time to gaps between demand and supply for labour across each of the different occupation groups.

The **second** task is **calibration**, which incorporates the construction of a balanced database and evaluation of coefficients and parameters. As required for CoPS-style models, the initial levels solution of the model is provided by the base year data. The database, in combination with the model's theoretical specification, describes the main inter-linkages in the economy. The theory of the model is then, essentially, a set of equations that describe how the values in the model's database move through time and move in response to any given policy shock. The core UPGEM database is calibrated to the latest available set of supply-use or input-output tables published by Statistics South Africa. Additional data from sources such as the social accounting matrix published by Statistics South Africa and the South African Reserve Bank's quarterly bulletin are also typically required. An added benefit of dynamic CGE models is that they allow for updating of the model's database structure by including historical information in the baseline. Figure 3A in Appendix 3 provides a stylized representation and description of the core UPGEM database. Regional and other extensions of the core database are dependent on availability of data across the required dimensions.

The **third** task is **solving the model using a suitable closure**. Dynamic CGE models such as UPGEM are designed to quantify the effects of a policy change, or exogenous shock, to the economy, over a period of time. A good way to examine the impacts of an exogenous shock is to compute the differences between a scenario in which the shock has occurred – the policy simulation – and a counterfactual scenario in which the particular shock under examination did not occur – the baseline scenario. Results are then reported as percentage change deviations over time between the first 'baseline' simulation run and the second 'policy' simulation run. The model's closure settings, that is, the choice of exogenous versus

endogenous variables, can be considerably different between the two runs. In the baseline we exogenise those variables for which reliable forecast information exists, which typically includes macroeconomic variables, such as the components of GDP, population growth and various price indices forecast by various macroeconomic specialists. Industries, such as electricity, which have explicit build programmes and a regulated pricing structure, may also have its available forecast data included in the baseline run. In the policy run, all the naturally endogenous variables are indeed set as endogenous, because we are interested in the impact of the policy change on them. This setting represents a more natural model closure where the variable for which the equation was written is typically set as endogenous. Standard baseline forecast and policy closures appropriate for UPGEM are described in Dixon & Rimmer (2002: 262-274).

UPGEM is implemented and solved using the GEMPACK suite of programs described in Harrison & Pearson (1996). GEMPACK eliminates linearisation errors by implementing shocks in a series of small steps and updating the database between steps. GEMPACK has also been proven the most efficient software package for solving large CGE models, such as the dynamic version of UPGEM (Horridge et al. 2013).

The **fourth** and final task involves **proper interpretation of simulation results**, drawing only on values given in the database, the underlying theory and the model closure. This is ultimately the most important and challenging task and requires at least some knowledge of each of the first three tasks. To simplify this enormous task, we often use condensed back-of-the-envelope representations of the model combined with key aggregates in the database to understand and explain the macro-level results produced by the full model. Small-scale versions of the larger model also help non-CGE modellers to better understand the causal mechanisms that produce specific results, which ultimately build credibility in the modelling process. Readers interested in the finer details of the methodology, UPGEM and the CoPS-style of modelling in general are encouraged to consult the various references, in particular Dixon et al. (2013).

Appendix 2: Stylized Representation of UPGEM

Equations (E2.1–E2.21) in Table 2A describe the **key macroeconomic relationships in UPGEM**. This system of equations will hereafter be referred to as the BOTE-M model. Miniature or back-of-the-envelope models such as BOTE-M have become a popular method for presenting and explaining the core elements contained in large and often complex CGE models. The simple and compact nature of BOTE models is also useful when interpreting simulation results produced by the full model. Other examples of BOTE models can be found in Adams (2005), Giesecke & Schilling (2010) and Dixon & Rimmer (2013).

Table 2A: BOTE-M Stylized Representation of UPGEM

GDP	= C + I + **G** + (X-M)	(E2.1)
GDP	= **A***f(K, D)	(E2.2)
C	= **APC***HINC	(E2.3)
HINC	= GDP*f(TofT)*(1-**TQ**) – (**BTRW**.D)***TL** – (**NFLH*****R**)	(E2.4)
GINC	= GDP*f(TofT)***TQ** + (**BTRW**.D)***TL** – (**NFLG*****R**)	(E2.5)
GNDI	= HINC + GINC	(E2.6)
M	= f(GDP, TofT, **TWS**)	(E2.7)
TofT	= PX/**PM**	(E2.8)
PX	= f(X, **F_X**)	(E2.9)
PY	= f(**CPI**, TofT)	(E2.10)
I/**K**	= R_IK	(E2.11)
I	= f(RoR, **F_I**)	(E2.12)
RoR	= f(**K**/L, TofT, **A**)	(E2.13)
BTRW = f(**K**/L, TofT, **A**)		(E2.14)
ATRW	= **BTRW** *(1-**TL**)	(E2.15)
D	= L – **U**	(E2.16)
ΔK	= I – **DEP*****K**	(E2.17)
ΔNFLG	= **G** – GINC	(E2.18)
ΔNFLH	= I – (1-**APC**)*HINC + **REM**	(E2.19)
ΔATRW	= f(D, L)	(E2.20)
L	= f(ATRW, Δ**ΔL_PREF**)	(E2.21)

Table 2B: BOTE-M Description of Variables

A	Primary-factor augmenting technical change
APC	Average propensity to consume
ATRW, BTRW	After-tax real wage; Before-tax real wage
C	Real private household expenditure
CPI	Consumer price index
D, L, U	Labour demand; Labour supply; Unemployment
DEP	Depreciation rate
F_I, F_X	Shift in investment; Shift in export demand schedule
G	Real government expenditure
GDP	Real gross domestic product
GINC, HINC	Real government income; Real household income
GNDI	Gross national disposable income
I	Real investment expenditure
K	Capital stock
MPK, RoR	Marginal product of capital; Rate of return on capital
NFLG	Real net foreign liabilities of government
NFLH	Real net foreign liabilities of households
PX, PM	Foreign-currency export price; Foreign-currency import price
PY	GDP deflator
R	Interest rate on net foreign liabilities
REM	Net outward remittance flows
R_IK	Capital growth rate (I/K ratio)
TL, TQ	Labour tax rate; Production tax rate
TofT	Terms of trade
TWS	Cost-neutral import/domestic preference twist
X, M	Export volumes; Import volumes
ΔK	Change in capital stock between years t and t+1
ΔNFLG	Change in net foreign liabilities of government between t and t+1
ΔNFLH	Change in net foreign liabilities of households between t and t+1
ΔATRW	Change in after-tax real wage between t and t+1
ΔL_PREF	Change in labour supply preferences between t and t+1

In BOTE-M, equations (E2.1–E2.16) describe variables within any given year of a dynamic simulation. Equations (E2.17–E2.19) describe how key stock variables move through time, and hold between any two successive years of a dynamic simulation. Equations (E2.20–E2.21) describe the real wage adjustment mechanism applicable to policy simulations.

Equation (E2.1) is the well-known identity describing real gross domestic product (GDP) from the expenditure side. In South Africa, private households contribute around 60 per cent to GDP, with investment and government expenditure each contributing roughly 20 per cent. The balance of trade (X–M) typically shows a deficit. Equation (E2.2) describes an economy-wide constant returns to scale production function, relating real GDP from the supply side to inputs

of capital, labour, and primary-factor augmenting technical change. In South Africa, both compensation of employees (labour) and gross operating surplus (capital) contribute around 50 per cent of GDP at factor cost.

Equation (E2.3) relates private household expenditure (C) to household disposable income via the average propensity to consume. Equations (E2.4) and (E2.5) define real household disposable income (HINC) and government revenue (GINC), respectively. The first term in both these equations describe the value of real GDP which could be expressed as [(GDP*PY)/CPI]. PY measures the average price level of GDP, and as such contains no information on import prices. PY does however incorporate prices of domestically-produced exports. In contrast, the CPI contains no information on export prices, but does include import prices. We can therefore interpret [PY/CPI] as a function of the terms of trade and write this component as GDP*f(TofT). HINC is defined as total income available for household expenditure after taking into account tax and net foreign liability payments. GINC is defined as the sum of all production and labour taxes collected minus any interest payments by government on its foreign liabilities. Equation (E2.6) confirms that (E2.4) and (E2.5) exhaust all claims on gross national disposable income (GNDI).

Equation (2.7) relates imports (M) to the level of GDP, the terms of trade, and an import/domestic preferences twist variable. The terms of trade (TofT) is defined in (E2.8) as the foreign-currency price of domestically produced exports relative to the price of imports. Commodity exports in UPGEM are inversely related to foreign-currency prices via constant elasticity demand functions. This is summarised by (E2.9) which relates the foreign-currency price of exports (PX) to the volume of exports and an export-demand shift variable. This is consistent with the assumption of South Africa being an open economy facing downward-sloping demand curves for its exports. This allows us to incorporate appropriate export-demand elasticities for South African commodities in UPGEM. Import prices are typically exogenous as South Africa is considered a price-taker in the global import market.

In our determination of (E2.10) we write the percentage-change of the economy-wide output price as $P_Y = [S_A P_A + S_X(P_X – P_M)]$. Following the notation employed by many MONASH-style CGE modellers, S_A and S_X reflects the share of absorption and exports in the economy, while P_Y, P_A, P_X and P_M represent the percentage-change in the price of their respective upper-case variables. For BOTE-M we assume that the price of absorption is reflected in the CPI, and that trade is balanced. From here, we are able to write equation (E2.10) which relates the price of economy-wide output (PY) to the consumer price index and terms of trade. PY may also be interpreted as the GDP deflator.

Equation (E2.11) defines the investment-capital ratio (R_IK) whilst equation (E2.12) relates investment expenditure (I) to the rate of return on capital (RoR) and an investment-demand shift variable. In (E2.11), R_IK may also be used to determine the gross capital growth rate.

Since the production function in (E2.2) is constant returns to scale, marginal product functions are homogenous of degree zero and can thus be expressed as functions of the capital-labour (K/L) ratio and technical change (A). In our description of the capital and labour markets, we recognise that the marginal product of capital (MPK) is negatively related to the K/L ratio and the marginal product of labour (MPL) positively related to the K/L ratio. In determining (E2.13) we assume that the rate of return on capital (RoR) can be expressed as [Q/PI] with Q the

factor payment to capital and PI the price index for new investments. We then assume Q is determined by the value of the marginal product of capital, written as [MPK*PY]. With MPK a function of the K/L ratio and technical change, and [PY/PI] a function of the terms of trade, we are able to summarise this relationship through equation (E2.13).

In determining (E2.14) we assume that the before-tax real wage of consumers (BTRW) can be expressed as [W/CPI] with W the factor payment to labour and CPI the consumer price index. We assume W is determined by the value of the marginal product of labour, written as [MPL*PY]. In similar fashion to (E2.13), we are then able to write equation (E2.14) linking the BTRW to the K/L ratio, technical change and terms of trade effect. Equation (E2.15) defines the after-tax real wage (ATRW) and equation (E2.16) allows for unemployment (U).

Equations (E2.17–E2.19) relate movements in three key stock variables to relevant flow variables. Equation (E2.17) shows that changes in capital stock (ΔK) is calculated as the sum of new capital investments minus depreciation of old capital stock – this is the familiar capital accumulation mechanism taught in undergraduate macroeconomics. Equation (E2.18) relates changes in government's net foreign liability position (ΔNFLG) to the public sector deficit incurred during the year. Equation (E2.19) relates changes in the net foreign liability position of households (ΔNFLH) to the excess of investment over savings and net outward remittance flows from migrants.

Equations (E2.20) and (E2.21) capture the real wage adjustment and labour supply mechanism in policy simulations. Where a policy has elevated labour demand relative to labour supply, the after-tax real wage (ΔATRW) will increase over time relative to its baseline value. Under this specification, the local labour market does not clear in the short-run perturbed scenario. An appropriate parameter in the wage adjustment equation governs the lagged wage response to gaps between labour demand and supply. Labour demand (D) is determined as a function of the before-tax real wage in (E2.14) and labour supply (L) is determined in (E2.21) as a function of the after-tax real wage and any change in preferences of workers for offering their services to a local activity.

To complete our description of the BOTE-M model we have to consider an **appropriate closure** suitable for policy modelling for the system of equations described in Table 2A. In doing so we must distinguish between equations that describe economic relationships within any given year (E2.1–E2.16), equations that describe movements in stock variables between years (E2.17–E2.19) and equations describing the real wage adjustment mechanism (E2.20–E2.21). In our exposition of BOTE-M we consider a typical **short-run** recursive-dynamic modelling environment. The development and interpretation of different model closures are discussed more thoroughly in Dixon & Rimmer (2002).

Within any given year, K, NFLH and NFLG can be considered exogenous, with movements between years dependent on their respective flow variables. Similarly, our 'sticky' real wage adjustment mechanism allows us to effectively treat BTRW, and therefore ATRW, as fixed within any given year, with movement between years dependent on the interaction between labour demand and supply. Reflecting changes in policy or economic conditions that are considered extraneous to the model, we also set DEP, REM and ΔΔL_PREF exogenous. Recognising that (E2.17–E2.21) governs dynamics across years, our task of finding a suitable model closure narrows to evaluating (E2.1–E2.16). We assume that the labour market clears

within this set of equations and that the unemployment level is fixed or exogenous within any given year. These 16 equations contain 31 unknown variables. As the number of endogenous variables must correspond to the number of equations, 15 variables must therefore be treated as exogenous in order to close the model. In Table 2A we provide a quick reference to our choice of model closure by highlighting **exogenous variables in bold**. For these equations, a conventional year-on-year short-run closure would have GDP, C, I, X, M, D, L, ATRW, GINC, GNDI, HINC, TofT, PX, PY, RoR and R_IK determined endogenously, given exogenous values for A, APC, G, K, BTRW, TQ, TL, NFLG, NFLH, R, PM, U, F_X, F_I and TWS.

Under this closure, each equation can be readily associated with the determination of a specific endogenous variable. This also allows us to follow the GEMPACK convention of naming each equation for the variable it is considered to endogenously determine. For large models such as UPGEM this is particularly useful as it allows the software to find an initial automatic closure by treating all those variables named by an equation as endogenous, and setting all remaining variables as exogenous. This saves the user a great amount of time as most of the numerous technical change and shift variables, all naturally exogenous, would already be chosen and listed as such.

The short-run closure applied to (E2.1–E2.16) via our choice of exogenous variables reflects the standard macro assumptions of 'sticky' real wages and fixed capital stocks in primary-factor markets. Although ATRW is endogenous in our model closure, it can effectively be seen as fixed within any given year since both BTRW and TL are exogenous in the short-run. To simplify our analysis, we assume that labour markets clear within this set of equations. Hence, with BTRW and K fixed, and A, TQ and TL also exogenous, (E2.14) can be identified with the determination of D. Since U is exogenous, this allows (E2.16) to determine L. With K and A exogenous, (E2.2) then determines GDP. With GDP now determined, (E2.4–E2.6) calculates GNDI and its distribution between HINC and GINC. With HINC determined by (E2.4) and APC exogenous, (E2.3) determines C. Ignoring any movements in the TofT, (E2.7) determines M with GDP already determined and TWS exogenous. With L determined by (E2.14), and K and A exogenous, (E2.13) determines the RoR via the marginal product of capital. This determines I via (E2.12), which allows R_IK to be calculated via (E2.11). With GDP, C, I, G and M explained, (E2.1) determines X. With PM exogenous, this determines PX and TofT via (E2.8) and (E2.9), respectively. With BTRW and TL fixed in the short-run, (E2.15) simply determines ATRW. To allow the absolute price level to be determined, the CPI acts as the numeraire in our system of equations. With the CPI exogenous and TofT already determined, (E2.10) determines PY.

This exposition of the BOTE-M model describes the key macroeconomic relationships in UPGEM under a typical short-run recursive-dynamic environment, with every equation linked to the determination of a specific endogenous variable. Whilst BOTE-M is useful as a quick reference to UPGEM and providing insight for the interpretation of simulation results, it should not be considered as a replacement for the full model.

Appendix 3: UPGEM Database and Production Structure

Figure 3A: Core Database Structure

					ABSORPTION MATRIX (USE TABLE)			
		1	2	3	4	5	6	7
		PRODUCERS	INVESTORS	HOUSEHOLD	EXPORT	GENGOV	STOCKS	TOTAL
	SIZE	IND	IND	HOU	1	1	1	ALL USERS
BASIC FLOWS	COMxSRC	V1BAS	V2BAS	V3BAS	V4BAS	V5BAS	V6BAS	V0BAS BASIC
MARGINS	COMx SRCxMAR	V1MAR	V2MAR	V3MAR	V4MAR	V5MAR	N/A	V0MAR MARGINS
INDIRECT TAXES	COMxSRC	V1TAX	V2TAX	V3TAX	V4TAX	V5TAX	N/A	V0TAX TLSP
BAS + MAR + TAX = PUR VALUES	COM	V1PUR INTERMED USE	V2PUR INVESTMENT	V3PUR PRIV CONS	V4PUR EXPORTS	V5PUR PUB CONS	V6BAS STOCKS	TOTAL COM DEMAND
LABOUR COSTS	OCC	V1LAB						
PRODUCTION TAXES	1	V1PTX						
CAPITAL RENTALS	1	V1CAP						
V1PUR + V1PRIM = TOTAL COST	1	TOTAL IND COSTS						

		PRODUCTION MATRIX (SUPPLY TABLE)			
SIZE	IND	1	1	1	ALL SOURCES
COM	MAKE SUPPLY TABLE	V0IMP IMPORTS	V0MAR MARGINS	V0TAX TLSP	TOTAL COM SUPPLY
1	TOTAL IND SALES				

The UPGEM database structure presented in Figure 3A has two main parts: an absorption matrix; and a joint-production matrix. The absorption matrix simultaneously shows total industry costs and total commodity demand across all users. The production matrix simultaneously shows total industry sales and total commodity supply. **The main data source for the core UPGEM database is the supply-use table, or alternatively, the input-output table**, published by Statistics South Africa. For particular applications, the model and database may be extended to include external matrices on emissions, migration, etc.

The first row in the absorption matrix, V1BAS,…, V6BAS, shows flows in the base year of commodities to producers, investors, households, exports, government consumption and inventory accumulation. Each of these matrices has COMxSRC rows, one for each of COM commodities from SRC sources.
V1BAS and V2BAS each have IND columns where IND is the number of industries. The typical component of V1BAS is the value of good i from source s used by industry j as an input to current production, and the typical component of V2BAS is the value of (i,s) used to create capital for industry j. V3BAS to V6BAS typically each have one column, which refers to one representative household, one foreign buyer, one category of public demand and one category of inventory demand. These dimensions can be extended if necessary, for example, the single

representative household may be split according to HOU number of household categories based on detailed income or ethnic group information found in social accounting matrices.

All of the flows in V1BAS,…, V6BAS are valued at basic prices. The basic price of a domestically produced good is the price received by the producer (that is the price paid by users excluding sales taxes, transport costs and other margin costs). The basic price of an imported good is the landed-duty-paid price, i.e., the price at the port of entry just after the commodity has cleared customs.

Costs separating producers or ports of entry from users appear in the input-output data in the margin matrices and in the row of sales-tax matrices. The margin matrices, V1MAR,…, V5MAR, show the values of MAR margin commodities, typically trade and transport services, used in facilitating the flows identified in V1BAS,…, V5BAS. The sales tax matrices V1TAX,…, V5TAX show collections of indirect taxes (positive) or payments of subsidies (negative) associated with each of the flows in V1BAS,…, V5BAS.

Payments by industries for labour by occupation (OCC) or skill group are recorded in the matrix V1LAB, whilst payments by industries for the use of capital and land are recorded in the vectors V1CAP. The vector V1PTX shows collections of net taxes on production. We may also include a vector V1OCT (not shown in Figure A1) to capture other industry costs not elsewhere classified, where appropriate.

The remaining data items are MAKE and V0TAR (not shown in Figure 3A). V0TAR is a vector showing tariff revenue by imported commodity. The joint-product matrix, MAKE, has dimensions COMxIND and its typical component is the output of commodity c by industry i, valued in basic prices. The content of the MAKE matrix is equivalent to the supply table.

Together, the absorption and joint-production matrices satisfy **two balancing conditions**. First, the column sums of MAKE (values of industry outputs) are identical to the values of industry inputs. Hence, the j-th column sum of MAKE equals the j-th column sum of V1BAS, V1MAR, V1TAX, V1LAB, V1CAP and V1PTX. Second, the row sums of MAKE (basic values of outputs of domestic commodities) are identical to basic values of demands for domestic commodities. If i is a non-margin commodity, then the i-th row sum of MAKE is equal to the sum across the (i,"dom")-rows of V1BAS to V6BAS.

The presentation of Figure 3A also highlights certain national accounting conventions. The green cells indicate the value of total industry costs across all categories; and total industry sales of domestically produced commodities. As noted previously, the column totals for each industry in both the cost and sales matrices should match. This also reflects the zero pure profit condition. Similarly, the blue cells indicate the value of total commodity demand across all users; and total commodity supply from all sources, at purchasers prices. As a balancing condition, the row totals for each commodity in both the demand and supply matrices should match. This represents the market clearing condition.

One more cell deserves some explanation. The V1CAP cell is commonly interpreted as the cost of capital rentals, including land, to industries in the production process. Within the system of national accounting (SNA) this cell represents the gross operating surplus (including mixed income) of an industry. Gross operating surplus (GOS) is defined in the context of national

accounting as the balancing item in the generation of income account. GOS differs from profits shown in industry accounts for several reasons. Only a subset of total costs is subtracted from total sales to calculate GOS. Essentially, GOS is the value of gross output or sales less the cost of intermediate goods and services to give gross value added, and less compensation of employees. It is gross because it makes no allowance for consumption of fixed capital (CFC). By deducting CFC from GOS one calculates net operating surplus (NOS).

When adapted to the structure of the UPGEM database, the gross operating surplus (V1CAP) can be calculated as the value of total sales (MAKE) less intermediate input costs (V1PUR), less labour input costs (V1LAB), less production taxes (V1PTX) and less other costs not elsewhere classified (V1OCT). V1CAP therefore represents the value of capital rentals – the money earned by and payable to investors as compensation for the effort, risk and opportunity cost related to the provision and maintenance of capital stock. In a 'normal' year, capital rental flows in each industry should reflect the normal risk-adjusted rate of return expected by investors based on the value of the underlying capital. Annual capital rental flows, reflected by V1CAP, in combination with normal required rates of return, are often used to estimate underlying industry capital stock values in UPGEM.

Figure 3B: Industry Production Structure
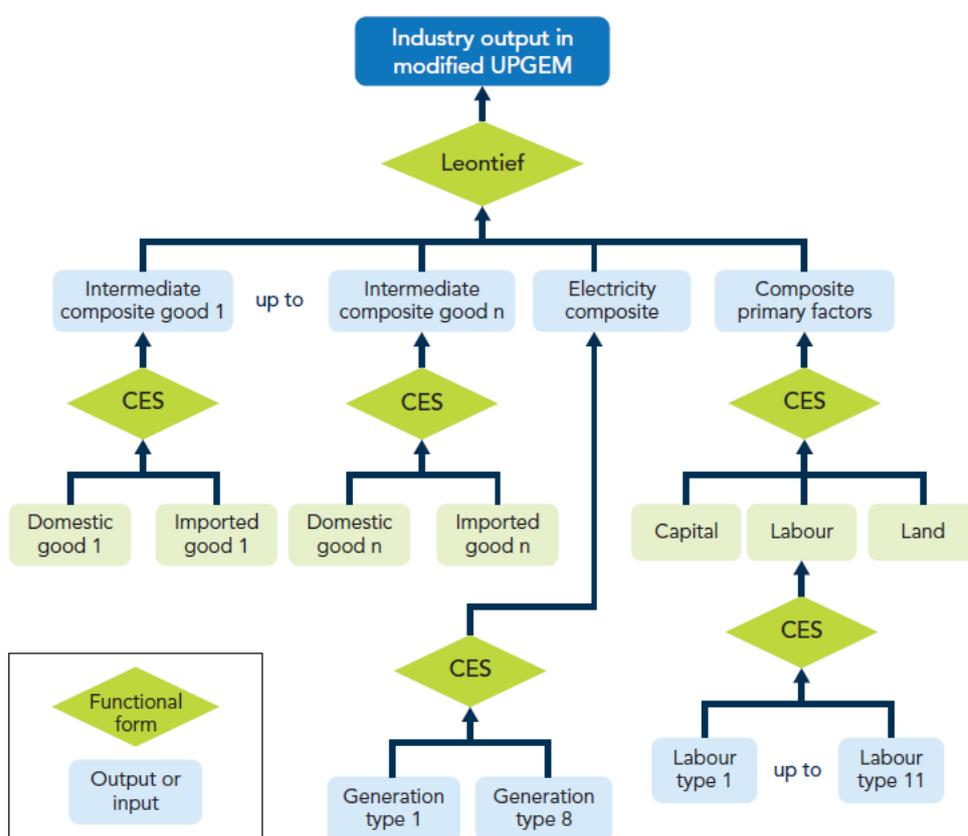
Industries in UPGEM-E are modelled to combine various composite intermediate goods, including an optional electricity composite, in fixed proportion to composite primary factor bundles. For each top-level composite in the production recipe, CES sub-nests allow price-induced substitution between imported and domestically produced versions of each good, electricity generation types (not included in the standard version of UPGEM), primary factors and labour types.

Appendix 4: Model Closures

Introduction

The choice of model closure is one of the most important aspects in economy-wide modelling. For detailed computable general equilibrium (CGE) models such as UPGEM the number of equations (m) and variables (n) can be very large. CGE models typically contain many more variables than equations. **To close the model and compute a solution, (n-m) variables must therefore be treated as exogenous**. Alternatively stated, the number of endogenous variables must equal the number of equations for the model closure to be valid. The selection of the (n-m) exogenous variables is largely **user-determined**, but should be chosen to **best describe the economic environment for which the simulation is run**. CGE models usually contain many non-linear equations and relationships. Following Johansen (1960), we simplify the implementation of our model by linearising the system of equations and writing the variables in terms of changes or percentage changes, as described in Appendix 5. This linear system is then solved by matrix manipulations, generously computed by the software package GEMPACK, giving the effects on the (m) endogenous variables of changes in the (n-m) exogenous variables.

The ability to have flexible model closures was one of the most useful extensions to Johansen's early framework. Led by the efforts of modellers at the Centre of Policy Studies and IMPACT Project since the mid-1970s, the closure flexibility initially described for the comparative-static ORANI model in Dixon *et al* (1982) was a significant step in allowing more practical and policy-oriented CGE modelling work. The development of modelling software packages such as GEMPACK (Harrison & Pearson, 1996) further improved the ease with which model closures could be found and altered. **The idea of flexible closures was fully extended in the dynamic MONASH model** with the development of four basic model closures (Dixon & Rimmer, 2002:233-277). These closures each had a distinct application in mind. Starting from a typical one-period long-run decomposition closure, swaps were applied

between endogenous and exogenous variables to find closures suitable for historical analysis, forecasting, and policy simulations.

In this appendix we use a stylized CGE model to illustrate and explain the features of the two main closures used in policy modelling: **the baseline forecast closure and the policy closure**. We start from a standard long-run closure and then apply various swaps to find our forecast and policy closures.

Back-of-the-Envelope Model

To gain a better understanding of the main features of each model closure, we follow the approach of Dixon & Rimmer (2002) by first developing a stylized back-of-the-envelope version of a typical CGE model as reference. This BOTE model, described in Table 4A, is a simplified version of BOTE-M described in Appendix 2. Despite its compact nature, it retains the ability to capture the main macro mechanisms in UPGEM.

Table 4A: BOTE Stylized Representation of UPGEM

GDP	$= C + I + G + (X – M)$	(E4.1)
GDP	$= A*f(K, L)$	(E4.2)
$C + G$	$= APC*GDP$	(E4.3)
C / G	$= R\_CG$	(E4.4)
M	$= f(GDP, TofT, TWS)$	(E4.5)
TofT	$= f(X, F\_X)$	(E4.6)
I	$= f(RoR, F\_I)$	(E4.7)
R_IK	$= I / K$	(E4.8)
CPI	$= f(PY, TofT)$	(E4.9)
RoR	$= f(K/L, TofT, A)$	(E4.10)
RW	$= f(K/L, TofT, A)$	(E4.11)

In the stylized model, (E4.1) is the well-known identity for real gross domestic product (GDP) from the expenditure side. In South Africa, the size of private household consumption (C) relative to GDP is around 60 per cent. Investment (I) and government expenditure (G) each contribute roughly 20 per cent to GDP. South Africa's balance of trade (X-M) typically shows a small deficit. Equation (E4.2) describes the production function for real GDP from the supply side, relating total output in the economy to inputs of capital (K) and labour (L), and a primary-factor productivity or technical change term (A). In South Africa, both compensation of employees and gross operating surplus contribute around 50 percent of GVA at factor cost. For the stylized model it is assumed that all sectors are competitive and all markets clear.

Equation (E4.3) relates the sum of private (C) and public (G) consumption to GDP via the average propensity to consume (APC), whilst equation (E4.4) defines the ratio of private to public consumption (R_CG). Equation (E4.5) relates imports (M) positively to the level of GDP, the terms of trade (TofT), and an import/domestic preference twist variable (TWS). In our

stylized model we make the simplifying assumption that there is only one imported good and one domestically produced good which is both consumed locally and exported. Ignoring tariffs and other taxes, the import-determining price can then be represented by the TofT, that is, the price of the domestically produced (and also exported) good relative to the price of the imported good. Equation (E4.6) relates the TofT to the volume of exports (X) and an export demand shift variable (F_X). This is consistent with the assumption of South Africa being an open economy facing downward-sloping demand curves for its exports, with F_X representing movements in those demand curves. With regard to imports, South Africa is considered a price-taker. Equation (E4.7) relates investment expenditure (I) to the rate of return on capital (RoR) and a demand shift variable (F_I) whilst equation (E4.8) defines the investment-capital ratio (R_IK). Equation (E4.9) relates the price of domestically consumed goods (CPI) to the price of the domestically produced good (PY) and the terms of trade effect (TofT).

In our description of the capital and labour markets, we recognise that the marginal product of capital (MPK) is negatively related to the K/L ratio and the marginal product of labour (MPL) positively related to the K/L ratio. In determining (E4.10) we assume that the rate of return on capital (RoR) can be expressed as [Q/PI] with Q the factor payment to capital and PI the price index for new investments. We then assume Q is determined by the value of the marginal product of capital, written as [MPK*PY]. With MPK a function of the K/L ratio and technical change (A), and [PY/PI] a function of the terms of trade (TofT), we are able to summarise this relationship through equation (E4.10). In determining (E4.11) we assume that the real wage (RW) can be expressed as [W/CPI] with W the factor payment to labour and CPI the consumer price index. We then assume W is determined by the value of the marginal product of labour, written as [MPL*PY]. In similar fashion to (E4.10), we are then able to write equation (E4.11) linking the RW to the K/L ratio, technical change (A) and terms of trade (TofT) effect. In deriving (E4.10) and (E4.11) we assume that domestic production is via a constant returns-to-scale production function of capital and labour inputs, and that the costs of employing capital and labour equal the values of the marginal products of capital and labour, respectively. This enables us to derive the 'back-of-the-envelope' equations used in Table 4B to better interpret movements in the capital and labour markets, and subsequently the K/L ratio.[2]

Table 4B: Marginal Products of Capital and Labour

$$\frac{\partial GDP}{\partial K} = MPK \approx \frac{1}{A} * \frac{Q}{P_y} \quad \text{or equivalently} \quad \frac{\partial GDP}{\partial K} = MPK \approx \frac{1}{A} * \frac{Q}{P_i} * \frac{P_i}{P_y} \quad (E4.12)$$

$$\frac{\partial GDP}{\partial L} = MPL \approx \frac{1}{A} * \frac{W}{P_y} \quad \text{or equivalently} \quad \frac{\partial GDP}{\partial L} = MPL \approx \frac{1}{A} * \frac{W}{P_c} * \frac{P_c}{P_y} \quad (E4.13)$$

In the capital (E4.12) and labour (E4.13) equations, Q and W are factor payments to capital and labour; Pi and Pc the price indexes for new investment and consumption goods; Py the

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[2] The two back-of-the-envelope equations in (E4.12) and (E4.13) are easily derived by maximising economy-wide profits, $P_Y.Y - (W.L + Q.K)$, subject to a Cobb-Douglas production function where $Y = A[L^{\beta}.K^{(1-\beta)}]$

price index for domestically produced goods; and MPK and MPL the marginal products of capital and labour, respectively. In this exposition, the term Pc is equivalent to the CPI.

In equation (E4.12), the (Q/Py) term can be split into two components or effects to enhance our analysis. The first term, (Q/Pi), can be interpreted as the rate of return on capital (RoR). The second term, (Pi/Py), similar to the (Pc/Py) term in (E4.13), can be interpreted as a decreasing function of the terms of trade (TofT). This is because both Pi and Pc include imports but not exports, whilst Py includes exports but not imports. The terms of trade effect is especially important in economies for which X and M are relatively large values. In equation (E4.13), the (W/Py) term can be interpreted as the real producer wage or cost of employing a unit of labour. To enhance our analysis, we again split this term into two effects. The first term, (W/Pc), can be interpreted as the real consumer wage, and the second, (Pc/Py), as the terms of trade effect.

Recognising that the marginal product of capital (MPK) is negatively related to the capital-labour ratio (K/L), equation (E4.12) can be used to show that the K/L ratio in (E4.10) is negatively related to the RoR, and positively related to the TofT and A. That is, as the relative amount of capital in the economy increases, and the MPK falls, we can expect a decline in the rate of return on capital investments. Similarly, with the marginal product of labour (MPL) positively related to the K/L ratio, equation (E4.13) can be used to show that the K/L ratio in (E4.11) is positively related to the RW. Alternatively, we can think of the labour-capital ratio (L/K) as being negatively related to the RW.

Developing the Decomposition Closure

The decomposition closure can best be described as a standard one-period long-run closure.[3] Although not explicitly used in standard policy modelling, the decomposition closure serves as a good base from where to develop other closures, such as the forecast and policy closure. By applying a number of standard macroeconomic assumptions, we are able to find a suitable long-run decomposition closure for the stylized model described in Table 4A.

From the supply-side of the economy, described by (E4.2), we can assume aggregate labour (L) fixed at its full employment level. This is consistent with the concept of a fixed non-accelerating inflation rate of unemployment (NAIRU) in the long-run. Real wages (RW) are considered flexible in the long-run as remuneration packages are periodically renegotiated. By absorbing any demand-side pressure via changes in real wages, the labour market is allowed to clear. Capital stock (K) is endogenous in the long-run to reflect changes in net investment. Rates of return on capital investments (RoR) are assumed to be exogenous. Many factors that influence RoR in the long-run, such as interest rates and risk premiums, are relatively stable over time and unlikely to be affected by policy decisions. Technical change

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[3] See Dixon & Rimmer (2002) for a complete description of the decomposition closure with illustrative examples of its practical application.

(A) is assumed to be exogenous in the long-run. We also expect the K/L ratio to stabilise at a level consistent with the economy's steady-state level of growth in the long-run.

With GDP now determined from the income side, we can use (E4.1) to analyse movements from the expenditure side of the economy. As suggested by macroeconomic evidence, we can expect private household consumption (C) to closely follow GDP in the long-run. Similarly, we can expect government (G) and import (M) demands to follow household consumption. This implies that both the average propensity to consume (APC) and the ratio of private to public consumption (R_CG) should be exogenous in the long-run. Investment (I) is endogenous via (E4.7), but bound to the gross capital growth trend in the base data, that is (I_{base}/K_{base}), since rates of return (RoR) are assumed to be constant in the long-run. This allows the overall investment to capital ratio (R_IK) to be determined via (E4.8). In (E5.1), the level of exports (X) is then determined as a residual to balance GDP from the expenditure side with GDP from the income side.

To accommodate the required move in X, the terms of trade (TofT) capturing export prices, is allowed to adjust. The consumer price index (CPI) is exogenous and acts as the numeraire in our stylized model. The numeraire determines the absolute price level in the economy.4 Other variables such as the import/domestic preference twist (TWS) and demand shift variables (F_X, F_I) are assumed to be exogenous in the long-run. In the context of equations (E4.12) and (E4.13) we can interpret any technical change affecting the marginal product of capital, to be reflected in a change to K with the RoR fixed in the long-run. Technical change affecting the marginal product of labour will be absorbed through a change to the real wage in the long-run under full employment.

The choice of endogenous and exogenous variables in the decomposition closure for the stylized model can therefore be outlined as in Table 4C. Under this closure, each equation of the stylized model can be readily associated with the determination of a specific endogenous variable. With RoR and L fixed in the long-run, and A also exogenous, (E4.10) and (E4.11) can be identified with the determination of K and RW. With RoR fixed, F_I exogenous and K already determined, this allows equations (E4.7) and (E4.8) to determine I and R_IK, respectively. With L and A exogenous, (E4.2) then determines GDP. C and G is determined via (E4.3) and (E4.4) given a fixed APC and R_CG. Ignoring any movements in TofT, (E4.5) determines M with GDP already determined and TWS exogenous. With GDP, C, I, G and M explained, equation (E4.1) determines X, which then allows (E4.6) to determine TofT. With the CPI set as the numeraire and TofT already determined, equation (E4.9) determines PY. Readers may also note that the number of variables listed as endogenous in Table 4C correspond to the number of equations in Table 4A.

For large-scale CGE models, GEMPACK may be used at first to generate a valid automatic closure by listing as exogenous all those variables not directly described or matched by an equation in the model code. When following the GEMPACK naming convention, each equation is typically named for the variable it is considered to endogenously determine. Although slightly unorthodox for a model in which the system of equations is solved simultaneously, this naming convention allows the software to find an automatic closure by treating all those variables

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[4] To determine the absolute price level in the economy requires one nominal price variable to be set as exogenous. Another popular choice as numeraire in CGE modelling is the nominal exchange rate.

named by an equation as endogenous, and setting all remaining variables as exogenous. Accordingly, the choice of (*n-m*) exogenous variables correspond very closely to the structure of the model. From the automatic closure, a small number of swap statements may be required to produce a model closure of sensible economic meaning, such as the long-run decomposition closure for the stylized model described in Table 4C. The main advantage to using the automatic closure as a starting point for developing any further model closures is that it saves the user a great amount of time as most of the numerous technical change, preference, and shift variables, all naturally exogenous, would already be chosen and listed as such.

After developing a valid and sensible model closure, users are encouraged to alter their closure through swap statements. Swap statements in the model closure file are a useful way of keeping track of any changes to the economic environment or assumptions imposed on the model. This involves a simple statement below the initial closure specification instructing GEMPACK to swap one variable for another on the exogenous list. With the decomposition closure as our starting point, we use this approach to develop a baseline forecast closure for our stylized model in the next section.

Table 4C: Decomposition Closure for the Stylized Model

Endogenous variables	Exogenous variables
gross domestic product (GDP)	employment (L)
real wages (RW)	rates of return (RoR)
capital stock (K)	average propensity to consume (APC)
household consumption (C)	investment demand shifter (F_I)
investment expenditure (I)	private to public consumption (R_CG)
government expenditure (G)	import/domestic preference twist (TWS)
imports (M)	technical change (A)
exports (X)	export demand shifter (F_X)
terms of trade (TofT)	consumer price index (CPI)
investment to capital stock (R_IK)	
domestic price index (PY)	

Developing the Forecast Closure

The forecast closure is used in simulations to produce a believable business-as-usual or baseline picture of the future evolution of the economy. One of the main purposes of CGE models is to provide projections of the impact of economic policy changes on a wide variety of economic variables. To accomplish this, a baseline forecast scenario incorporating available forecast data is first simulated. A perturbed scenario incorporating the relevant policy shocks is then run and compared to the outcome of the baseline scenario, with deviations usually reported as percentage changes. When conducting policy simulations stretching over multiple periods, the importance of a suitable baseline forecast becomes apparent. The effects of policies imposed on economies with structures likely to be relevant in the future are often different from the effects imposed on economies with the structures of the present.

For the baseline forecast closure, we try to exogenise everything that we think we know about the future. The choice of (*n-m*) exogenous variables is therefore usually based on the availability of reliable macroeconomic forecast data, with little regard to causation. In the application of the forecast simulation we exogenise many naturally endogenous variables such as the components of GDP from the expenditure side, the consumer price index, population growth, etc. to take advantage of forecasts prepared by various analysts and macroeconomic specialists. To allow these variables to be exogenous requires us to endogenise many naturally exogenous variables such as the average propensity to consume, the positions of foreign demand curves, the positions of domestic export supply curves, etc. Some technical change and preference variables may also be exogenous and given shocks that are informed by trends derived from historical simulations or available time-series data.

Baseline forecast simulations for recursive-dynamic CGE models are performed as a sequence of annual solutions. The forecast closure is therefore short-run in nature. Because we are dealing with annual solutions, start-of-year *t+1* stock variables such as capital or net foreign liabilities are completely determined by end-of-year *t* stock variables in the baseline solution. While these start-of-year stock variables can be thought of as exogenous in the computation for any year *t*, they should be thought of as endogenous for the sequence of annual solutions which make up the forecast simulation. For example, changes in net investment together with changes in capital stocks in year *t* determine the change in end-of-year capital stocks for year t, which subsequently determine the change in start-of-year capital stocks for year *t+1*. In UPGEM, this capital accumulation mechanism can be expressed as follows:

Table 4D: Capital Accumulation Mechanism in UPGEM

$$K_{t+1} = K_t * [1 - DEP_t] + I_t \qquad\qquad\qquad \text{(E4.14)}$$

$$I_t = f(ERoR_t) \qquad\qquad\qquad \text{(E4.15)}$$

As shown in equation (E4.14), end-of-year capital stock (K_{t+1}) is calculated as the net value of start-of-year capital stock (K_t), taking into account the rate of depreciation (DEP_t) on old capital stock, plus new capital investments made during the year (I_t). In equation (E4.15) we relate investment decisions to expected rates of return (ERoR). Thus, investment this period only affects the growth rate of capital in the next period. This 'gestation' period for new investments allows us to keep start-of-year capital stock exogenous and end-of-year capital stock endogenous when developing year-on-year short-run closures for the UPGEM model.

In terms of the closure for the stylized model, we have to create a suitable short-run forecast environment and exogenise the macro variables for which reliable annual forecast data typically exists. This requires a number of swaps to our current decomposition closure. The first step involves moving from a one-period long-run closure to a short-run closure. With regard to the factor market, we assume real wages (RW) to be 'sticky' and aggregate labour (L) to be flexible in the short-run. Aggregate capital stock (K) is fixed in short-run with rates of return (RoR) allowed to adjust. The required swaps to our model closure described earlier in Table 4C therefore are:

Table 4E: Swap Statements to Produce Forecast Closure (Step 1)

Previously exogenous variables	Newly exogenous variables
employment (L) ………………………………………………..	real wages (RW)
rates of return (RoR) ………………………………………	capital stock (K)

The second step involves incorporating available macroeconomic forecast data for household consumption (C), investment expenditure (I), government expenditure (G) and imports (M) into our baseline projection. Most of these macro variables that determine GDP from the expenditure side in (E4.1) are typically forecast by specialist agencies, allowing us to integrate these values into our projection. In order for these variables to become exogenous and adopt their forecasted values, we have to endogenise some naturally exogenous variables. In this step we endogenise the average propensity to consume (APC), investment demand shifter (F_I), ratio of private to public consumption (R_CG) and the import/domestic preference twist (TWS) variables via the system of equations in (E4.1–E4.11). The additional swaps are:

Table 4F: Swap Statements to Produce Forecast Closure (Step 2)

Previously exogenous variables	Newly exogenous variables
average propensity to consume (APC)	household consumption (C)
investment demand shifter (F_I)	investment expenditure (I)
	government expenditure (G)
private to public consumption (R_CG)	imports (M)
import/domestic preference twist (TWS)	

The third and final step in producing our forecast closure for the stylized model involves incorporating available forecast data for exports (X). In the first step, we exogenised real wages (RW) and capital stock (K). This allowed us to create a suitable short-run environment and include any reliable forecast data on these variables. In the second step, we exogenised C, I, G and M from the expenditure-side of the model to further accommodate available forecast data on these variables. To also exogenise exports (X), we must determine which variable to endogenise.

Figure 4A illustrates demand and supply curves for exports as functions of the terms of trade $(TofT)$, with the current market clearing position indicated by point A. This shows an equilibrium export quantity of X_E and export price of TofT_E. From here we can achieve a desired level of exports at X_F by shifting either the export demand curve from $(X1_D)$ to $(X2_D)$ giving point O1, or the export supply curve from $(X1_S)$ to $(X2_S)$ giving point O2.

Figure 4A: Demand and Supply of Exports
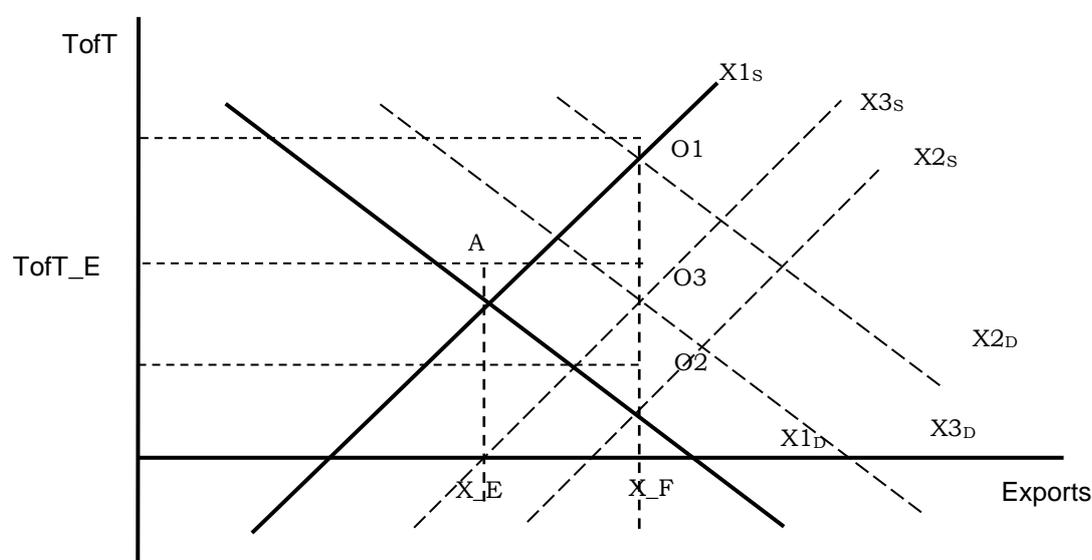
Generating these shifts requires us to define and understand the corresponding demand and supply curves in Figure 4A. The nature of the downward-sloping export demand curve is clear,

it is simply (E4.6) from Table 4A. In this equation, demand for domestically-produced exports is expressed as $[X_D = f(TofT, F\_X)]$. Defining the upward-sloping export supply curve is more complicated. To explain the nature of this supply curve, we start by writing the identity contained in (E4.1) as $[X = GDP - (C + I + G - M)]$. With C, I, G and M exogenous, this equation expresses X as a function of GDP. From equation (E4.2) we know that GDP is determined via the production function as $[GDP = A * f(K, L)]$ with labour (L) determined via equation (E4.11) as $[L = f(RW, K, TofT, A)]$. Substituting this expression into (E4.2), we find $[GDP = f(RW, K, TofT, A)]$, which allows us to write exports (X) in (E4.1) as $[X_S = f(RW, K, TofT, A) - (C + I + G - M)]$. For given values of RW, K and A, this defines the export supply (X_S) curve in Figure 4A that traces out the relationship between X and $TofT$.

However, we have yet to prove that this expression for the X_S curve has a positive slope. Following our earlier discussion of equation (E4.13) in Table 4B, we recognise that $\left[\frac{\partial L}{\partial TofT}\right] \geq 0$. That is, holding RW, K and A constant, an increase in the price of exports would lead to an increase in employment in the short-run. We derive this result via the terms of trade effect in equation (E4.13). Since P_y does not include import prices, and P_c does not include export prices, any movement in $\left[\frac{P_c}{P_y}\right]$ must therefore be a function of the terms of trade. An increase in export prices, reflected via P_y, will then lead to a reduction in $\left[\frac{P_c}{P_y}\right]$. With real wages $\left[\frac{W}{P_c}\right]$ and technical change $\left[\frac{1}{A}\right]$ fixed, the fall in $\left[\frac{P_c}{P_y}\right]$ will lead to a fall in the marginal product of labour (MPL) according to (E4.13). If we assume optimising conditions are to be maintained, any fall in the MPL in the short-run must then be accompanied by an increase in L with RW fixed. This confirms that $\left[\frac{\partial L}{\partial TofT}\right] \geq 0$ in the short-run. An increase in L will then contribute to an increase in GDP via (E4.2). For given values of $(C + I + G - M)$, this will also increase GDP in (E4.1), and as a result X. According to the relationships determined in this exposition, it confirms that our earlier expression for export supply, $[X_S = f(RW, K, TofT, A) - (C + I + G - M)]$, has a positive slope.

Having established the underlying nature of the downward-sloping export demand and upward-sloping export supply curve, we return to Figure 4A. We now know how to shift both curves within the framework of equations (E4.1–E4.11). From the export demand side, we can allow X_D to move by endogenising the export demand shifter (F_X). The export demand shifter captures the willingness to pay of foreigners for domestically-produced exports. This setting will accommodate a shift from $X1_D$ to $X2_D$, indicated by point 01 in Figure 4A. The model then determines the market clearing $TofT$ consistent with this point and desired export quantity of X_F. From the supply side, we can allow X_S to move by introducing movements in $[f(RW, K, TofT, A)]$. With RW and K fixed, this leaves endogenisation of A as our only legitimate option. This setting will accommodate a shift from $X1_S$ to $X2_S$, indicated by point 02 in Figure 4A. The model then again determines the market clearing $TofT$ consistent with this point and X_F.

As suggested by Figure 4A, either option would require a relatively large change in the price of exports or $TofT$. The practical results produced by both these options, that is, moving to

either point 01 or point 02 in our example from Figure 4A, is not considered satisfactory within our application of the forecast simulation in UPGEM.

A third alternative for imposing an exogenous forecast value for exports must therefore be considered. It is often the case with variables such as the $TofT$ that we do not have sufficient data to make reliable forecasts. However, historical evidence suggests that movements in the $TofT$ are relatively stable over the long-term. To prevent large changes in the $TofT$, and maintain our exogenous setting of exports at X_F, we should therefore combine both options described earlier. That is, we must endogenise both technical change (A) and the export-demand shifter (F_X). This will allow us to exogenise the quantity of exports at X_E and fix the $TofT$ at TofT_E. The assumption of no change in the $TofT$ over the simulation period can in itself be considered as a forecast, albeit a rather conservative one. The additional swaps required to complete the move from our model closure described earlier in Table 4C to the forecast closure are:

Table 4G: Swap Statements to Produce Forecast Closure (Step 3)

Previously exogenous variables			**Newly exogenous variables**
technical	change	(A)	exports (X)
…………………….………………			
export	demand	shifter	(F_X)	terms of trade (TofT)
…………..……………			

After step 3 we now have virtually all macroeconomic variables for which forecast data usually exist chosen as exogenous. In the full UPGEM we have many more variables and therefore flexibility in how we apply our swaps and which variables we can set as exogenous. It is often found that with the introduction of labour/capital preference twist and other shift variables in the full model, we can also exogenise employment, even with real wages exogenous, to incorporate available forecast and trend data.

Care should always be taken in developing these closures as not any combination of (*n-m*) variables will be legitimate. For example, at least one monetary variable should be included on the exogenous list in order to determine the absolute price level in the model. In terms of the forecast closure, we also have to be aware of the potential over-determination of GDP when incorporating exogenous forecasts for employment (L). With the exogenisation of C, I, G, X and M, GDP from the expenditure side is tied down. If technical change (A) were also entirely exogenous, then GDP from the supply side would also be tied down by the exogenous setting of aggregate employment (L) and capital (K). In UPGEM then, to allow GDP from the supply side to adjust to equal GDP from the expenditure side, overall primary-factor augmenting technical change must be endogenised in the forecast closure.

Given the proposed swaps to the model closure, the choice of endogenous and exogenous variables in the forecast closure for the stylized model can be summarised as in Table 4H.

Table 4H: Forecast Closure for the Stylized Model

Endogenous variables	Exogenous variables
gross domestic product (GDP)	real wages (RW)
employment (L)	capital stock (K)
rates of return (RoR)	household consumption (C)
average propensity to consume (APC)	investment expenditure (I)
investment demand shifter (F_I)	government expenditure (G)
private to public consumption (R_CG)	imports (M)
import/domestic preference twist (TWS)	exports (X)
technical change (A)	terms of trade (TofT)
export demand shifter (F_X)	consumer price index (CPI)
investment to capital stock (R_IK)	
domestic price index (PY)	

Developing the Policy Closure

The policy closure is used in simulations analysing the impact of an exogenous shock to the economy, beyond the natural baseline scenario. Policy simulations are performed as a sequence of annual solutions in recursive-dynamic models. As a result, the closure reflects many short-run features. The policy closure is more orthodox than the forecast closure used to construct a baseline for the economy. In this closure, naturally endogenous variables such as the macro components of GDP are again set to be endogenous. The impact on macro variables is usually of particular interest to policymakers and must therefore be allowed to respond to the policy change under consideration. UPGEM also has the ability to produce disaggregated results on an industry level, giving an indication of the possible winners and losers of a proposed policy. From the supply side of the economy, employment is set as endogenous, with real wages 'sticky' in the short-run. Variables usually unaffected by policy decisions in the short-run are set as exogenous. Correspondingly, naturally exogenous variables such as the positions of foreign demand curves or the average propensity to consume are set as such, and take on values generated in the baseline forecast. That is, there will be no deviation between the policy and forecast simulation values for these variables.

In the full UPGEM model we often impose budget neutrality for policy simulations involving government, that is, we allow no change to public sector debt relative to the baseline as a result of the policy. To achieve this result, we typically let consumption taxes adjust to compensate for the change in government finances.

In terms of the closure for the stylized model, our choice of (*n-m*) exogenous variables is similar to that of the decomposition closure. The only difference lies in the short-run nature of the policy closure versus the long-run nature of the decomposition closure. The more orthodox nature of the policy closure, as compared to the forecast closure, is also evident. Variables which are exogenous in the policy run will take on values imposed or generated in the forecast simulation. Policy shocks imposed on any of these exogenous variables reflect changes away from their forecasted values and generate changes in the model's endogenous variables away from their respective forecasted values. This deviation away from the baseline forecast may then be interpreted as the impact of the particular policy under consideration. Typically, the choice of variables required to carry the policy shocks is unique to every study.

From the expenditure side of the economy, we endogenise household consumption (C) and government expenditure (G) via the consumption function described in (E4.3) and (E4.4). Investment (I) is endogenised via (E4.7). With the balance of trade (X–M) endogenous, we exogenise the export demand shifter (F_X) and import/domestic preference twist variable (TWS). With exports (X) determined via (E4.1), the TofT is endogenised via (E4.6).

With the policy closure applied within a short-run environment, we exogenise start-of-year capital stock and activate the 'sticky' wage adjustment mechanism. This allows us to endogenise aggregate employment (L) and rates of return on capital (RoR) via (E4.10) and (E4.11). Technical change (A) is usually set as exogenous in policy simulations. Under these assumptions, technical change and the real wage rate together determine the rate of return on capital via the factor-price frontier. The factor-price frontier is the relationship between the marginal product of labour (MPL), which is tied down by the real wage (RW), and the marginal product of capital (MPK), which determines the rate of return on capital (RoR). This relationship is determined by the equations and assumptions underlying (E4.2) and (E4.10–

E4.13) described earlier. With labour being paid according to its marginal product, employment is determined by the real wage rate. Capital and technical change is then determined via the marginal-product function for labour. For a general understanding of the policy closure, we can therefore think of capital stock (K), real wages (RW), and technical change (A) as being exogenous in our stylized model.

We have now established a number of possible swaps that will allow us to move from a forecast closure to a typical short-run oriented policy closure. The following swaps from our stylized model closure in Table 4H may therefore be made:

Table 4I: Swap Statements to Produce Policy Closure

Previously exogenous variables	Newly exogenous variables
household consumption (C) ………………………….…	average propensity to consume (APC)
investment expenditure (I) ………………………….…..	investment demand shifter (F_I)
government expenditure (G) …………………………….	private to public consumption (R_CG)
imports (M) ……………………………………………………………	import/domestic preference twist (TWS)
exports (X) ……………………………………………………………..	technical change (A)
terms of trade (TofT) …………………………………………	export demand shifter (F_X)

Given the proposed swaps to the model closure, the choice of endogenous and exogenous variables in the policy closure for the stylized model can be summarised as in Table 4J.

Table 4J: Policy Closure for the Stylized Model

Endogenous variables	Exogenous variables
gross domestic product (GDP)	real wages (RW)
employment (L)	capital stock (K)
rates of return (RoR)	average propensity to consume (APC)
household consumption (C)	investment demand shifter (F_I)
investment expenditure (I)	private to public consumption (R_CG)
government expenditure (G)	import/domestic preference twist (TWS)
imports (M)	technical change (A)
exports (X)	export demand shifter (F_X)
terms of trade (TofT)	consumer price index (CPI)
investment to capital stock (R_IK)	
domestic price index (PY)	

Concluding Remarks

With the Johansen/Euler type solution strategy implemented through GEMPACK, moving between different types of model closures has been made easy. Once a sensible model closure has been developed, alterations to the choice of (*n-m*) exogenous variables can be made through swap statements. However, care should always be taken in maintaining the conditions for a legitimate closure. Although closures may be developed for specific simulation purposes, the economic meaning of the assumptions imposed via the closure should always be interpretable. Using a stylized model for this purpose is well advised.

The **different notions of exogeneity** in decomposition and policy closures on the one hand compared with forecast closures on the other **reflect different objectives**. In forecast simulations we use all available data and information about the future to generate a believable baseline for the economy. In policy simulations we attempt to explain the effects of specific shocks to the economy. For this purpose we need to make realistic assumptions about which variables are likely to be unaffected and which will be responsive to the shocks under consideration (Dixon & Rimmer, 2002:276).

It is also important to acknowledge the difference in describing a closure based on the stylized model in Table 4A, which can be thought of as a small comparative-static CGE model, and a closure based on a detailed recursive-dynamic model such as UPGEM. In UPGEM we have to consider the various adjustment mechanisms and stock/flow relationships of a recursive-dynamic model. Additional cost-neutral preference twists and shift variables also allow greater

flexibility in the setup of our model closures. However, the underlying nature of each closure remains the same, regardless of which model we use as reference.

Appendix 5: The Percentage-Change Approach

Introduction to the Percentage-Change Approach

For a detailed CGE model, the number of equations (m) and variables (n) can be very large. Many of the equations in a CGE model may also be non-linear in its levels form. To avoid the computational problems involved in solving a large non-linear system, **Johansen (1960) approximated his model by a system of linear equations in changes or percentage changes of the variables**. This linear system was then solved by matrix manipulations, giving the approximate effects on the (m) endogenous variables of changes in the ($n - m$) exogenous variables. **In GEMPACK we follow this Johansen-style percentage-change approach in the implementation and solving of the model**.

Therefore, instead of writing the system of equations in the model as

$$Y = f(X_1, X_2) \qquad\qquad\qquad \text{(EA.1)}$$

where *Y* is output and *X₁* and *X₂* inputs, a Johansen-style model uses the linear percentage-change form

$$y - \varepsilon_1 x_1 - \varepsilon_2 x_2 = 0 \qquad\qquad\qquad \text{(EA.2)}$$

where ε_i is the elasticity of output with respect to inputs of factor i, and y and x_i are the percentage changes in *Y* and *Xᵢ* respectively. In matrix notation, a Johansen-style model can be represented by

$$Av = 0 \qquad\qquad\qquad \text{(EA.3)}$$

where A is a matrix of coefficients and v is the vector of percentage changes in the model's variables.

In this representation A is a $(m \times n)$ matrix, with the number of variables (*n*) usually considerably more than the number of equations (*m*). To close the model and compute a solution as shown in equation (EA.5) we therefore have to choose an appropriate combination of m endogenous and $(n-m)$exogenous variables. We can express the changes or percentage changes in our endogenous variables as linear functions of the changes or percentage changes in our exogenous and predetermined variables by rearranging equation (EA.3) in the form

$$A_1 y + A_2 x = 0 \qquad\qquad\qquad \text{(EA.4)}$$

where y is the vector of percentage changes in those variables chosen to be endogenous and x is the vector of percentage changes in the predetermined variables and those chosen to be exogenous. *A₁* and *A₂* are matrices formed by selecting appropriate columns of *A*. Therefore, to compute the effects of changes in any of the exogenous variables on the endogenous variables, we need to solve (EA.5) via matrix inversion.

$$y = -A_1^{-1} . A_2 x \qquad\qquad\qquad \text{(EA.5)}$$

A system of equations in the form of (EA.4) is much **easier to solve and interpret** than when written in its underlying, and often complex, non-linear form. The choice of different parameter values, in particular, becomes much easier to interpret under the percentage-change form. We illustrate this point later in the appendix by deriving the percentage-change form of the Cobb-Douglas and CES function and contrasting the results to their respective levels form. However, because *A* is assumed constant, these approximate solutions produced by (EA.5) are only accurate for small changes in the vector of exogenous variables otherwise unacceptably large linearisation error may occur. Eliminating the linearisation error in solving large-scale CGE models has become a relatively easy task thanks to GEMPACK and modern-day computing power as users are now able to enjoy features such as multi-step solution and Richardson's extrapolation without much cost in terms of time.

Linearisation Error

In the following section we use a simple example to illustrate how the linearisation error introduced with the Johansen-approach is eliminated using techniques available to GEMPACK users. We do this by contrasting the true result produced by a levels-form equation to the approximate result of a linearised-form equation.

We start by using equation (EA.6).

$$X = Y.Z \tag{EA.6}$$

From here we can write

$$(X + \Delta X) = (Y + \Delta Y).(Z + \Delta Z) \tag{EA.7}$$

$$(X + \Delta X) = Y.Z + Y.\Delta Z + \Delta Y.Z + \Delta Y.\Delta Z \tag{EA.8}$$

Since we know that $X = Y.Z$ we can subtract X on both sides to find the expression for total change in X shown in (EA.9).

$$\Delta X = Y.\Delta Z + \Delta Y.Z + \Delta Y \Delta Z \tag{EA.9}$$

To find the percentage change in X we divide across by X and multiply with 100.

$$\frac{\Delta X}{X} * 100 = \frac{Y.\Delta Z}{Y.Z} * 100 + \frac{\Delta Y.Z}{Y.Z} * 100 + \frac{\Delta Y.\Delta Z}{X} * 100 \tag{EA.10}$$

From here we follow the GEMPACK convention of writing the percentage-change form of a particular variable in lower-case symbols. For example, the percentage-change in the variable represented by the upper-case symbol X will be represented by the lower-case symbol x determined as $x = \frac{dX}{X} * 100$ or alternatively $dX = \frac{X.x}{100}$. Equation (EA.10) can therefore be written as (EA.11) following this notational convention.

$$x = \frac{Y.\Delta Z}{Y.Z} * 100 + \frac{\Delta Y.Z}{Y.Z} * 100 + \frac{\Delta Y.\Delta Z}{X} * 100 \tag{EA.11}$$

Applying similar notation to express the percentage-change in Y and Z allows us to write the true percentage change in X as

$$x = y + z + \frac{\Delta Y.\Delta Z}{X} * 100 \tag{EA.12}$$

This shows that the true percentage change in X is equal to the percentage change in Y, plus the percentage change in Z, plus a second-order term $\frac{\Delta Y.\Delta Z}{X} * 100$. We can confirm this through a simple numerical example. Assume that in equation (EA.6) *Y=20* and *Z=5* to give us an initial solution for X of *20*5 = 100*. If we impose an exogenous increase of 20 per cent in the value of both *Y* and *Z*, the new value for *X* would be 20(1.20)*5(1.20) = 24*6 = 144.
Substituting these values into equation (EA.12) confirms that the true increase in *X* is 44 per cent.

$$y + z + \frac{\Delta Y . \Delta Z}{X} * 100 = x$$

$$20 + 20 + \left(\frac{4*1}{100}\right) * 100 = 44 \qquad \text{(EA.13)}$$

In GEMPACK, we use the total differential of an equation to find the approximate percentage change in the dependent variable as a result of a change in one of the independent variables. By applying this derivative-based method to the same equation and example as before, we are able to illustrate and contrast the approximate result produced by the linearised-form equation with the true result produced in (EA.13) via equation (EA.12).

We again use equation (EA.6).

$$X = Y . Z$$

From here we can write the total differential of X for a given change in Y and Z as

$$dX = \frac{\partial X}{\partial Y} dY + \frac{\partial X}{\partial Z} dZ \qquad \text{(EA.14)}$$

$$dX = 1(Y^{1-1}.Z)dY + 1(Y.Z^{1-1})dZ \qquad \text{(EA.15)}$$

$$dX = Z.dY + Y.dZ \qquad \text{(EA.16)}$$

Remember that we can express the percentage-change of a variable as $x = \frac{dX}{X} * 100$ or alternatively $dX = \frac{X.x}{100}$. Applying this notational convention again allows us to find equation (EA.18).

$$\frac{x.X}{100} = \left(\frac{y.Y}{100}\right)Z + \left(\frac{z.Z}{100}\right)Y \qquad \text{(EA.17)}$$

$$x.X = y.YZ + z.ZY \qquad \text{(EA.18)}$$

Since we know that $X = Y.Z$ we can divide both sides with X to find the expression for total approximate percentage change in X shown in equation (EA.19).

$$x = y + z \qquad \text{(EA.19)}$$

In comparing the true change in X from equation (EA.12) with the approximate change in X derived from equation (EA.19), we can see that the linearisation error when using the percentage-change form of an equation is equivalent to the second-order term $\frac{\Delta Y . \Delta Z}{X} * 100$ shown in equation (EA.12). This suggests that the larger the change in Y or Z, the greater the proportional error in x. Conversely, this also suggests that as changes to the independent variables, in this case Y and Z, tend closer to zero, the answer for x generated via equation (EA.19) will become a better approximation of the true answer generated through equation (EA.12). That is, as the term ($\Delta Y . \Delta Z$) becomes smaller and smaller, the linearisation error would tend closer to zero.

This observation forms the basis to how the **linearisation error is eliminated in GEMPACK**. When breaking large changes in *Y* or *Z* into a number of steps, for each sub-change in *Y* or *Z*, we use the linear approximation to derive the consequent sub-change in *X*. Then, using the new values of *Y* and *Z*, we recomputed the coefficient matrices equivalent to A_1 and A_2 in equation (EA.5). Essentially, this updates the sales and cost shares imbedded in the *A* matrix. This process is repeated for each step. We can show that, given sensible restrictions on the derivatives of the model coefficients, we can obtain a solution as accurate as we like by dividing the process into sufficiently many steps. This technique, known as the Euler method, is the simplest of several related techniques of numerical integration, that is, the process of using differential equations to move from one solution to another. GEMPACK offers users the choice of several such techniques. The accuracy of these multi-step solution techniques can further be improved by using Richardson's extrapolation.

The Cobb-Douglas Function

In the next part of this appendix we use the standard constrained optimisation problem facing consumers to give a complete exposition on how to derive the percentage-change equation of a simple Cobb-Douglas function.

Features of the Cobb-Douglas function include i) homogeneous of degree $(\alpha + \beta)$ and in the special case of $(\alpha + \beta) = 1$ it is linearly homogeneous, ii) its isoquants are negatively sloped and strictly convex for positive values of inputs X_1 and X_2, iii) since the function is homogenous, it is also homothetic, i.e. consumer's preferences or budget shares depend only on the ratio of prices of goods, not income, and iv) like Leontief, Cobb-Douglas functions are easy to parameterize.

The function's linear homogeneity, that is $(\alpha + \beta) = 1$, is easily seen from the fact that, using a standard Cobb-Douglas production function given as $Y = f(K, L) = A(K^\alpha . L^\beta)$, and changing inputs *K* and *L* to *zK* and *zL* respectively, the output will be changed to $A(zK^\alpha . zL^\beta) = z^{(\alpha+\beta)} . A(K^\alpha . L^\beta) = zY$. Note that the function is not linear however. The generalized Cobb-Douglas production function is characterized by a constant, unitary elasticity of substitution $(\sigma = 1)$. The derivation of this result in no way relies upon the assumption that $(\alpha + \beta) = 1$. Thus, the elasticity of substitution of the production function $Y = A(K^\alpha . L^\beta)$ will be unitary even if $(\alpha + \beta) \neq 1$. The elasticity of substitution can be interpreted, under competitive conditions, as the sensitivity of the K/L ratio for a given change in the real wage to rental of capital ratio.

In the following optimisation problem, choose amounts of goods X_1 and X_2 to consume that will maximise utility $U = (X_1, X_2) = X_1{}^\alpha . X_2{}^\beta$ with α and β positive fractions that sum to 1 indicating the share of income allocated to X_1 and X_2, respectively, subject to the budget constraint $X_1 . P_1 + X_2 . P_2 = M$.

First set up the Lagrange function

$$L(X_1, X_2, \lambda) = X_1{}^\alpha . X_2{}^\beta + \lambda(M - X_1 . P_1 - X_2 . P_2)$$

which we can write as

$$L = X_1{}^\alpha . X_2{}^\beta + \lambda M - \lambda X_1 . P_1 - \lambda X_2 . P_2 \tag{EA.20}$$

Taking the partial derivatives of L with respect to X_1, X_2 and λ in (EA.20) yields the following first-order conditions in (EA.21–EA.23).

$$\frac{\partial L}{\partial X_1} = \alpha . X_1{}^{\alpha-1} . X_2{}^\beta - \lambda P_1 = 0 \tag{EA.21}$$

$$\frac{\partial L}{\partial X_2} = \beta . X_1{}^\alpha . X_2{}^{\beta-1} - \lambda P_2 = 0 \tag{EA.22}$$

$$\frac{\partial L}{\partial \lambda} = M - X_1 . P_1 - X_2 . P_2 = 0 \tag{EA.23}$$

Divide (EA.21) with (EA.22) to eliminate λ, simplify and write in terms of $X_1 . P_1$ to find equation (EA.26).

$$\frac{\lambda P_1}{\lambda P_2} = \frac{\alpha . X_1{}^{\alpha-1} . X_2{}^\beta}{\beta . X_1{}^\alpha . X_2{}^{\beta-1}} \tag{EA.24}$$

$$\frac{P_1}{P_2} = \frac{\alpha . X_2}{\beta . X_1} \tag{EA.25}$$

$$X_1 . P_1 = \frac{\alpha}{\beta} . X_2 . P_2 \tag{EA.26}$$

Now substitute (EA.26) into (EA.23) and simplify.

$$M = X_1 . P_1 + X_2 . P_2 \text{ with } (X_1 . P_1) = \frac{\alpha}{\beta} . X_2 . P_2 \text{ can therefore be written as}$$

$$M = \left(\frac{\alpha}{\beta} . X_2 . P_2\right) + X_2 . P_2 \tag{EA.27}$$

$$M = (X_2 . P_2)\left(\frac{\alpha}{\beta} + 1\right) \tag{EA.28}$$

$$M = (X_2 . P_2)\left(\frac{\alpha}{\beta} + \frac{\beta}{\beta}\right) \tag{EA.29}$$

Remember that $(\alpha + \beta) = 1$ which allows us find (EA.30).

$$M = (X_2.P_2)\left(\frac{1}{\beta}\right) \tag{EA.30}$$

From here, the demand equation for good X_2 can be written as shown in (EA.31).

$$X_2 = \beta.\frac{M}{P_2} \tag{EA.31}$$

In the case of $(\alpha + \beta) \neq 1$ the demand equation for good X_2 can be written as

$$X_2 = \frac{\beta}{\alpha+\beta}.\frac{M}{P_2} \tag{EA.32}$$

Given that $(\alpha + \beta) = 1$ we can similarly show that the demand equation for good X_1 can be written as

$$X_1 = \alpha.\frac{M}{P_1} \text{ since } (1-\beta) = \alpha \tag{EA.33}$$

By writing the demand equations shown in (EA.32) and (EA.33) in terms of α and β we can easily interpret these parameters as the budget shares, or share of expenditure on good X_1 and X_2, respectively, by agents in their total income M. This is shown by (EA.34) and (EA.35).

$$\beta = \frac{X_2.P_2}{M} \tag{EA.34}$$

$$\alpha = \frac{X_1.P_1}{M} \tag{EA.35}$$

To write the demand equation for a Cobb-Douglas function in percentage-change form as shown in (EA.40) requires us to take the total differential of the general demand function derived from equations (EA.32) and (EA.33), which can be written as

$$X_i = \beta_i.\frac{M}{P_i} \text{ with } \sum_i \beta_i = 1$$

From this general levels form demand equation $X_i = \beta_i.\frac{M}{P_i}$ we can write $X_i = \beta_i.M.P_i^{-1}$.

The total differential of this expression is

$$dX_i = (\beta_i.P_i^{-1})dM + \left[-1(\beta_i.M.P_i^{-2})\right]dP_i \tag{EA.36}$$

which can be simplified and written as

$$dX_i = (\beta_i.P_i^{-1})dM - (\beta_i.M.P_i^{-2})dP_i \tag{EA.37}$$

Remember that the percentage change of a variable can be written as $x = \frac{dX}{X} * 100$ or alternatively $dX = X.x * 0.01$.

$$X_i.x_i * 0.01 = \left(\beta_i.P_i^{-1}\right)M.m * 0.01 - \left(\beta_i.M.P_i^{-2}\right)P_i.p_i * 0.01 \qquad \text{(EA.38)}$$

Since we know that $X_i = \beta_i.M.P_i^{-1}$ equation (EA.38) can be simplified to

$$X_i.x_i * 0.01 = X_i.m * 0.01 - X_i.p_i * 0.01 \qquad \text{(EA.39)}$$

From here, the **percentage-change form of the demand equation for a Cobb-Douglas function** can be written as shown in (EA.40) by dividing both sides with $X_i * 0.01$.

$$x_i = m - p_i \qquad \text{(EA.40)}$$

From the percentage-change form in (EA.40) it is easy to show that the Cobb-Douglas function exhibits unitary income elasticities. Income elasticities of demand can be defined as $\frac{dX_i}{dM}.\frac{M}{X_i}$ or alternatively $\frac{x_i}{m}$, and holding prices constant ($p_i = 0$) it must therefore be that $x_i = m$. The income elasticity defined as above must also be equal to one for all x_i given $m \neq 0$. It is also clear from (EA.40) that the Cobb-Douglas function exhibits own-price elasticities equal to one and cross-price elasticities equal to zero as a change in p_k will have no effect on x_i.

The CES Function

In the next part of this appendix we use the standard constrained optimisation problem facing producers to give a complete exposition on how to derive the percentage-change equation of a Constant Elasticity of Substitution (CES) function.

Features of the CES function include i) it is homogeneous of degree 1 and therefore exhibits constant returns to scale, ii) its isoquants are negatively sloped and strictly convex for positive values of X_1 and X_2, and iii) since the function is homogenous, it is also homothetic, i.e.

consumer's preferences or budget shares depend only on the ratio of prices of goods, not income, and iv) it relaxes the imposed substitution elasticities in the Leontief ($\sigma = 0$) and Cobb-Douglas ($\sigma = 1$) functions.

In the following optimisation problem, choose inputs X_i (with i = 1…n) to minimise the cost $\sum_i P_i.X_i$ of producing given output Z subject to the CES production function.

$$Z = \beta(\textstyle\sum_i \delta_i X_i^{-\rho})^{-1/\rho}$$

In this equation, X_i represents types of inputs, and β, δ and ρ are three parameters, with β the efficiency or technology parameter, δ_i the distribution parameter indicating the relative factor shares in the product, and ρ the substitution parameter which determines the value of the (constant) elasticity of substitution defined as $\sigma = \frac{1}{1+\rho}$.

First set up the Lagrange function

$$L = P_1X_1 + P_2X_2+\ldots+P_iX_i+\ldots+P_kX_k+\ldots+P_nX_n + \lambda Z$$
$$-\lambda\beta[(\delta_1X_1{}^{-\rho}) + (\delta_2X_2{}^{-\rho})+\ldots+(\delta_iX_i{}^{-\rho})+\ldots+(\delta_kX_k{}^{-\rho})+\ldots+(\delta_nX_n{}^{-\rho})]^{-1/\rho}$$

which we can write as

$$L = \textstyle\sum_i P_iX_i + \lambda\big[Z - \beta(\sum_i \delta_i X_i^{-\rho})^{-1/\rho}\big] \tag{EA.41}$$

Taking the partial derivatives of L with respect to X_i, X_k and λ yields the following first-order conditions in (EA.42–EA.44).

$$\frac{\partial L}{\partial X_i} = P_i - \lambda\beta.\left(-\frac{1}{\rho}\right)(\textstyle\sum_i \delta_i X_i^{-\rho})^{-(1/\rho)-1}.\left(-\rho\delta_i X_i^{-\rho-1}\right) = 0 \tag{EA.42}$$

$$\frac{\partial L}{\partial X_k} = P_k - \lambda\beta.\left(-\frac{1}{\rho}\right)(\textstyle\sum_i \delta_i X_i^{-\rho})^{-(1/\rho)-1}.\left(-\rho\delta_k X_k^{-\rho-1}\right) = 0 \tag{EA.43}$$

$$\frac{\partial L}{\partial \lambda} = Z - \beta(\textstyle\sum_i \delta_i X_i^{-\rho})^{-1/\rho} = 0 \tag{EA.44}$$

Divide (EA.43) with (EA.42) to eliminate λ, simplify and write in terms of $X_i^{-\rho}$ to find equation (EA.49).

$$\frac{P_k}{P_i} = \frac{\lambda\beta.\left(-\frac{1}{\rho}\right)(\sum_i \delta_i X_i^{-\rho})^{-(1/\rho)-1}.\left(-\rho\delta_k X_k^{-\rho-1}\right)}{\lambda\beta.\left(-\frac{1}{\rho}\right)(\sum_i \delta_i X_i^{-\rho})^{-(1/\rho)-1}.\left(-\rho\delta_i X_i^{-\rho-1}\right)} \tag{EA.45}$$

$$\frac{P_k}{P_i} = \frac{\delta_k X_k^{-\rho-1}}{\delta_i X_i^{-\rho-1}} \quad \text{which can be written as} \quad \frac{P_k}{P_i} = \frac{\delta_k}{\delta_i}\left(\frac{X_i}{X_k}\right)^{1+\rho} \tag{EA.46}$$

$$\left(\frac{P_k}{P_i}\right)^{\frac{1}{1+\rho}} = \left(\frac{\delta_k}{\delta_i}\right)^{\frac{1}{1+\rho}}\left(\frac{X_i}{X_k}\right) \tag{EA.47}$$

Remember that $\left(\frac{\delta_k}{\delta_i}\right)^{\frac{1}{1+\rho}}$ can be written as $\frac{\delta_k^{\frac{1}{1+\rho}}}{\delta_i^{\frac{1}{1+\rho}}}$ which allows us to find (EA.49).

$$\left(\frac{\delta_i.P_k}{\delta_k.P_i}\right)^{\frac{1}{1+\rho}} . X_k = X_i \tag{EA.48}$$

$$X_i^{-\rho} = \left(\frac{\delta_i.P_k}{\delta_k.P_i}\right)^{-\frac{\rho}{(1+\rho)}} . X_k^{-\rho} \tag{EA.49}$$

Now substitute (EA.49) into (EA.44) and simplify.

$$Z = \beta \left[\sum_i \delta_i \left\{\left(\frac{\delta_i.P_k}{\delta_k.P_i}\right)^{-\frac{\rho}{(1+\rho)}} . X_k^{-\rho}\right\}\right]^{-1/\rho} \tag{EA.50}$$

which can be written as (EA.51) since $X_k^{-\rho.\left(-\frac{1}{\rho}\right)} = X_k^1$

$$Z = X_k\beta \left[\sum_i \delta_i \left\{\left(\frac{\delta_k.P_i}{\delta_i.P_k}\right)^{\frac{\rho}{(1+\rho)}}\right\}\right]^{-1/\rho} \tag{EA.51}$$

From here we can find the input demand equation by writing (EA.51) in terms of X_k.

$$X_k = \frac{Z}{\beta\left[\sum_i \delta_i \left\{\left(\frac{\delta_k.P_i}{\delta_i.P_k}\right)^{\frac{\rho}{(1+\rho)}}\right\}\right]^{-1/\rho}} \tag{EA.52}$$

$$X_k = Z.\beta \left[\sum_i \delta_i \left\{\left(\frac{\delta_k.P_i}{\delta_i.P_k}\right)^{\frac{\rho}{(1+\rho)}}\right\}\right]^{1/\rho} \tag{EA.53}$$

From (EA.53) we can use some algebraic manipulation to find equation (EA.58).

$$X_k = Z.\beta \left[\left(\frac{\delta_k}{P_k}\right)^{\frac{\rho}{(1+\rho)}} . \sum_i \delta_i \left(\frac{P_i}{\delta_i}\right)^{\frac{\rho}{(1+\rho)}}\right]^{1/\rho} \tag{EA.54}$$

$$X_k = Z.\beta.\left(\frac{\delta_k}{P_k}\right)^{\frac{1}{(1+\rho)}} \left[\sum_i \delta_i \left(\frac{P_i}{\delta_i}\right)^{\frac{\rho}{(1+\rho)}}\right]^{1/\rho} \tag{EA.55}$$

$$X_k = Z.\beta.\left(\frac{\delta_k}{P_k}\right)^{\frac{1}{(1+\rho)}} \left[\sum_i \delta_i . \left(\frac{1}{\delta_i^{\frac{\rho}{(1+\rho)}}}\right) . \left(\frac{P_i^{\frac{\rho}{(1+\rho)}}}{1}\right)\right]^{1/\rho} \tag{EA.56}$$

$$X_k = Z.\beta.\left(\frac{\delta_k}{P_k}\right)^{\frac{1}{(1+\rho)}} \left[\sum_i \delta_i^1 . \delta^{-\frac{\rho}{(1+\rho)}} . P_i^{\frac{\rho}{(1+\rho)}}\right]^{1/\rho} \tag{EA.57}$$

$$X_k = Z.\beta.\left(\frac{\delta_k}{P_k}\right)^{\frac{1}{(1+\rho)}} \left[\sum_i \delta_i^{1-\frac{\rho}{(1+\rho)}} . P_i^{\frac{\rho}{(1+\rho)}}\right]^{1/\rho} \tag{EA.58}$$

Remember that $1 - \frac{\rho}{(1+\rho)}$ can be written as $\frac{(1+\rho)}{(1+\rho)} - \frac{\rho}{(1+\rho)}$ which becomes $\frac{1}{(1+\rho)}$. This allows us to find equation (EA.59).

$$X_k = Z.\beta.\delta_k^{\frac{1}{(1+\rho)}}.P_k^{-\frac{1}{(1+\rho)}}\left[\sum_i \delta_i^{\frac{1}{(1+\rho)}}.P_i^{\frac{\rho}{(1+\rho)}}\right]^{1/\rho} \tag{EA.59}$$

Some more algebraic manipulation of (EA.59) yields the input demand equation for X_k as shown in equation (EA.60). The form shown in (EA.60) represents the non-linear levels-form of a CES equation.

$$X_k = Z.\beta.\delta_k^{\frac{1}{(1+\rho)}}\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}} \tag{EA.60}$$

with $P_{ave} = \left(\sum_i \delta_i^{\frac{1}{(1+\rho)}}.P_i^{\frac{\rho}{(1+\rho)}}\right)^{\frac{1+\rho}{\rho}}$ or alternatively $\frac{1}{P_{ave}} = \left(\sum_i \delta_i^{\frac{1}{(1+\rho)}}.P_i^{\frac{\rho}{(1+\rho)}}\right)^{-\frac{1+\rho}{\rho}}$

since $\left[\left(\sum_i \delta_i^{\frac{1}{(1+\rho)}}.P_i^{\frac{\rho}{(1+\rho)}}\right)^{-\frac{1+\rho}{\rho}}\right]^{-\frac{1}{(1+\rho)}}$ must be equal to $\left[\sum_i \delta_i^{\frac{1}{(1+\rho)}}.P_i^{\frac{\rho}{(1+\rho)}}\right]^{1/\rho}$

To find the exponent used for P_{ave} in the expression above we had to find an exponent which multiplied with $-\frac{1}{(1+\rho)}$ would yield $\frac{1}{\rho}$ from equation (EA.59).

The demand equation derived from the CES optimisation problem given earlier can now be converted to percentage-change form as shown in (EA.65). In its exposition, we continue to use the convention of writing percentage-change variables in lower case, that is, $x = \frac{dX}{X} * 100$ or alternatively $dX = \frac{X.x}{100}$.

To write the demand equation for a CES function in percentage-change form as shown in (EA.64) requires us to take the total differential of the demand function derived earlier in equation (EA.60).

$$dX_k = \left(1.Z^0.\beta.\delta_k^{\frac{1}{(1+\rho)}}\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}}\right)dZ + \left(Z.\beta.\delta_k^{\frac{1}{(1+\rho)}}\left(-\frac{1}{1+\rho}\right)\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}-1}\right)d\frac{P_k}{P_{ave}}$$

from where we can write

$$X_k.x_k * 0.01 = \left(\beta.\delta^{\frac{1}{(1+\rho)}}\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}}\right)Z.z * 0.01 +$$

$$\left(Z.\beta.\delta_k^{\frac{1}{(1+\rho)}}\left(-\frac{1}{1+\rho}\right)\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}-1}\right)d\frac{P_k}{P_{ave}}$$

$$\tag{EA.61}$$

To write $d\left(\frac{P_k}{P_{ave}}\right)$ in percentage-change form requires use of the quotient rule where $d\left(\frac{B}{A}\right) = \frac{A.B' - B.A'}{A^2}$

Following the quotient rule we can write $d\frac{P_k}{P_{ave}}$ as $\left(\frac{P_{ave}.P_k' - P_k.P_{ave}'}{P_{ave}^2}\right)$ which can then be further simplified to find equation (EA.64).

$$\left[\frac{P_{ave}.(P_k.p_k*0.01) - P_k.(P_{ave}.p_{ave}*0.01)}{P_{ave}^2}\right] \tag{EA.62}$$

$$\left(\frac{(P_k.p_k*0.01) - (P_k.p_{ave}*0.01)}{P_{ave}}\right) \tag{EA.63}$$

$$\left[\left(\frac{P_k*0.01}{P_{ave}}\right)(p_k - p_{ave})\right] \tag{EA.64}$$

In (EA.61) we can also split the $\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}-1}$ component into $\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}}.\left[\frac{P_{ave}}{P_k}\right]^1$ from where we can write

$$X_k.x_k*0.01 = \left(\beta.\delta_k^{\frac{1}{(1+\rho)}}\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}}\right)Z.z*0.01$$

$$+ \left(Z.\beta.\delta_k^{\frac{1}{(1+\rho)}}\left(-\frac{1}{1+\rho}\right)\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}}.\left[\frac{P_{ave}}{P_k}\right]^1\right)\left(\frac{P_k.0,01}{P_{ave}}\right)(p_k - p_{ave})$$

Since we know that $X_k = Z.\beta.\delta_k^{\frac{1}{(1+\rho)}}\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}}$ this equation can be simplified to

$$X_k.x_k*0.01 = X_k.z*0.01 + \left(X_k.\left[\frac{P_{ave}}{P_k}\right]\right)\left(\frac{P_k*0.01}{P_{ave}}\right)(p_k - p_{ave}) \tag{EA.65}$$

From here, the **percentage-change form of the demand equation for a CES function** can be written as (EA.66).

$$x_k = z - \sigma(p_k - p_{ave}) \tag{EA.66}$$

with $p_{ave} = \sum_i S_i p_i$ and where $\sigma = \frac{1}{1+\rho}$ and $S_i = \frac{\delta_i^{\frac{1}{(1+\rho)}}.P_i^{\frac{\rho}{(1+\rho)}}}{\sum_k \delta_k^{\frac{1}{(1+\rho)}}.P_k^{\frac{\rho}{(1+\rho)}}}$

To find the meaning of S_i we return to the levels-form of the equation shown in (EA.60) and multiply both sides with P_k and simplify to find (EA.68).

$$P_k.X_k = Z.\beta.\delta_k^{\frac{1}{(1+\rho)}}.(P_{ave})^{\frac{1}{(1+\rho)}}.(P_k)^{-\frac{1}{(1+\rho)}}.(P_k)^1 \tag{EA.67}$$

$$P_k.X_k = Z.\beta.\delta_k^{\frac{1}{(1+\rho)}}.(P_{ave})^{\frac{1}{(1+\rho)}}.(P_k)^{\frac{\rho}{(1+\rho)}} \tag{EA.68}$$

Remember that $\left[\frac{P_k}{P_{ave}}\right]^{-\frac{1}{(1+\rho)}} = (P_{ave})^{\frac{1}{(1+\rho)}}.(P_k)^{-\frac{1}{(1+\rho)}}$ and that $-\frac{1}{(1+\rho)} + \frac{(1+\rho)}{(1+\rho)} = \frac{\rho}{(1+\rho)}$

From (EA.68) we can show that the ratio of $\frac{P_i X_i}{P_k X_k}$ yields the expression in (EA.69).

$$\frac{P_i.X_i}{P_k.X_k} = \frac{\delta_i^{\frac{1}{(1+\rho)}}.(P_i)^{\frac{\rho}{(1+\rho)}}}{\delta_k^{\frac{1}{(1+\rho)}}.(P_k)^{\frac{\rho}{(1+\rho)}}} \tag{EA.69}$$

From here we can find equation (EA.70) which shows that the economic meaning of S_i can be interpreted as the cost share of input X_i in producing output Z.

$$\frac{P_i.X_i}{\sum_k P_k.X_k} = \frac{\delta_i^{\frac{1}{(1+\rho)}}.(P_i)^{\frac{\rho}{(1+\rho)}}}{\sum_k \delta_k^{\frac{1}{(1+\rho)}}.P_k^{\frac{\rho}{(1+\rho)}}} = S_i \tag{EA.70}$$

To further clarify the interpretation of the substitution parameter ρ, which determines the value of the (constant) elasticity of substitution defined as $\sigma = \frac{1}{1+\rho}$ in the CES production function $Z = \beta(\sum_i \delta_i X_i^{-\rho})^{-1/\rho}$, we can rewrite σ in terms of $-\rho$.

With $\sigma = \frac{1}{1+\rho}$ we multiply both sides with $(1+\rho)$ to find (EA.71)

$$\frac{\sigma(1+\rho)}{1} = \frac{1(1+\rho)}{(1+\rho)} \tag{EA.71}$$

which can be simplified to find

$$\rho = \frac{1-\sigma}{\sigma} \text{ or alternatively } -\rho = \frac{\sigma-1}{\sigma} \tag{EA.72}$$

Therefore, the CES production function $Z = \beta(\sum_i \delta_i X_i^{-\rho})^{-1/\rho}$ could also be written as $Z = \beta\left(\sum_i \delta_i X_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{-\sigma}{1-\sigma}}$ or alternatively $Z = \beta\left(\sum_i \delta_i X_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{EA.73}$

The Klein-Rubin Function

In the final part of this appendix we briefly describe the Klein-Rubin utility function used in the determination of household demand in standard CoPS models such as ORANI-G.

The Klein–Rubin utility function is used in the top nest of household demand. The Klein–Rubin function (also known as the Stone–Geary or Linear Expenditure System) is non-homothetic which means that rising income will cause the preferences of consumers (budget shares) to change, even with price ratios fixed. That is, the marginal rate of substitution changes as income increases.

Total demand or consumption of household goods consists of a combination of subsistence goods and luxury goods, which can be written as $X_i = X_i{}^{sub} + X_i{}^{lux}$. From here we are able to derive an algebraic equation of Klein-Rubin household demand by solving the optimisation problem of choosing amounts of $X_i{}^{sub}$ and $X_i{}^{lux}$ (with i = 1…n) that will maximise utility $U = U(X_1 \ldots X_i) = \sum_i S_i{}^{lux} * ln(X_i - X_i{}^{sub})$ subject to the consumer's budget constraint $\sum_i X_i . P_i = M$.

Using a simple algebraic exposition, we are able to derive an expression for household demand equivalent to that used in standard CoPS models. We start again by defining total household demand as the sum of subsistence and luxury consumption.

$$X_i = X_i{}^{sub} + X_i{}^{lux} \qquad\qquad \text{(EA.74)}$$

Assuming that all income is spent, we can express luxury spending as

$$M^{lux} = M - \sum_i P_i . X_i{}^{sub} \qquad\qquad \text{(EA.75)}$$

With luxury spending on good *i* a fixed proportion of total supernumerary income in the Klein-Rubin function, we can show that

$$P_i . X_i{}^{lux} = S_i{}^{lux} . M^{lux} \qquad\qquad \text{(EA.76)}$$

which can be written as

$$X_i{}^{lux} = \left[\frac{S_i{}^{lux}}{P_i}\right] * M^{lux} \qquad\qquad \text{(EA.77)}$$

From here we can substitute (EA.75) into (EA.77) to find equation (EA.78)

$$X_i{}^{lux} = \left[\frac{S_i{}^{lux}}{P_i}\right] * \left(M - \sum_i P_i . X_i{}^{sub}\right) \qquad\qquad \text{(EA.78)}$$

which can then be substituted into (EA.74) to find equation (EA.79).

$$X_i = X_i{}^{sub} + \left[\frac{S_i{}^{lux}}{P_i}\right] * \left(M - \sum_i P_i . X_i{}^{sub}\right) \tag{EA.79}$$

Equation (EA.79) shows the general form of a Klein-Rubin household demand equation, with expenditure on each good *i* a linear function of income. A complete exposition on deriving the percentage-change form of the Klein-Rubin household demand equation is contained in Dixon et al (1982:96-103).

Appendix 6: Explaining Twist Variables in UPGEM

In UPGEM, input demand equations by industries are derived subject to a CES aggregation function with substitution elasticities (σ) between primary factors typically set at between 0.3 and 0.5. Setting $\sigma \neq 1$ in the linearised input demand equations allows cost-neutral preference twists ($TWLK$) accommodating exogenous historical data or forecasts in the primary-factor market to be introduced and converted into technical or taste changes.[5] Equations (E6.1) and (E6.2) show these linearised demand equations as they appear in the UPGEM code.

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[5] This method of implementing cost-neutral preference twists eliminates a problem arising with these variables when set as exogenous in policy simulations. The same method is used to implement cost-neutral import/domestic preference twists.

$$cap = z - \sigma S_L(p_K - p_L) + S_L TWLK \qquad\qquad\text{(E6.1)}$$

$$lab = z - \sigma S_K(p_L - p_K) - S_K TWLK \qquad\qquad\text{(E6.2)}$$

Following the CoPS-style notation used in UPGEM, *cap* and *lab* represent the percentage change in industry demands for capital (K) and labour (L), respectively. We note that in the absence of any change in output *(z)* and relative factor prices *(p_K – p_L)*, this representation gives *[cap – lab = TWLK]* and *[S_K*cap + S_L*lab = 0]*. Thus, if the K/L ratio in UPGEM increases by 10 per cent beyond what is explained by relative factor price movements, then $TWLK$ will equal 10. **The twist is therefore equivalent to movements in a_K and a_L, the technical change associated with capital and labour, respectively, which satisfy equations (E6.3) and (E6.4)**.

$$S_L TWLK = a_K - \sigma S_L(a_K - a_L) \qquad\qquad\text{(E6.3)}$$

$$-S_K TWLK = a_L - \sigma S_K(a_L - a_K) \qquad\qquad\text{(E6.4)}$$

We can further show that by implementing the twist via the technical change variables, we are in effect assuming that $\left[a_K = \frac{S_L}{1-\sigma} * TWLK\right]$ and $\left[a_L = -\frac{S_K}{1-\sigma} * TWLK\right]$. With σ below one, a positive $TWLK$ value is equivalent to a cost-neutral capital-using technical change combined with a labour-saving technical change.

To provide additional insight into the composition of our K/L movements, we use the two back-of-the-envelope equations shown in (E6.5) and (E6.6).[6] Assuming competitive conditions, the marginal products of capital and labour would equal their respective factor payments. In our exposition of (E6.5) we recognise that the marginal product of capital (*F_K*) is negatively related to the K/L ratio. In (E6.6) we recognise that the marginal product of labour (*F_L*) is positively related to the K/L ratio. In determining (E6.5) we assume that the rate of return on capital can be expressed as (*Q/P_i*) with *Q* the factor payment to capital and *P_i* the price index for new investments. We then assume *Q* is determined by the value of the marginal product of capital, written as (F_K*P_y). With (F_K*P_y) a function of the K/L ratio, and technical change (*A*) and (P_y/P_i) a function of the terms of trade (TofT), we are able to summarise this relationship through equation (E5). In similar fashion, we are able to write (E6.6) linking the real wage (*W/P_c*) to the K/L ratio, technical change, and the terms of trade effect.

$$F_K\, A \left[\frac{\left(\frac{K}{A_K}\right)}{\left(\frac{L}{A_L}\right)}\right] \approx \frac{Q}{P_i} * \frac{P_i}{P_y} \qquad\text{(E6.5)} \qquad\qquad F_L\, A \left[\frac{\left(\frac{K}{A_K}\right)}{\left(\frac{L}{A_L}\right)}\right] \approx \frac{W}{P_c} * \frac{P_c}{P_y} \qquad\text{(E6.6)}$$

Given the implementation of our labour/capital twist via the technical change variables, we are now able to use our BOTE equations to help interpret the results for $TWLK$. In (E6.5), a preference twist affecting capital will therefore be transmitted via the technical change variable

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[6] The two BOTE equations in (E6.5) and (E6.6) are easily derived by maximising economy-wide profits, $P_y.Y$ – (W.L + Q.K), subject to a Cobb-Douglas production function where Y = A[$L^\beta.K^{(1-\beta)}$]

(A_K). In (E6.6), a preference twist affecting labour will similarly be transmitted via the technical change variable (A_L).

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