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FORECASTING WITH SECOND-ORDER APPROXIMATIONS AND MARKOV SWITCHING DSGE MODELS

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Abstract

This paper compares the out-of-sample forecasting performance of first- and second-order perturbation approximations for DSGE models that incorporate Markov-switching behaviour in the policy reaction function and the volatility of shocks. These results are compared to those of a model that does not incorporate any regime-switching. The results suggest that second-order approximations provide an improved forecasting performance in models that do not allow for regime-switching, while for the MS-DSGE models, a first-order approximation would appear to provide better out-of-sample properties. In addition, we find that over short-horizons, the MS-DSGE models provide superior forecasting results when compared to those models that do not allow for regime-switching (at both perturbation orders).

JEL Classifications: C13, C32, E37.

Keywords: Regime-switching, second-order approximation, non-linear MS-DSGE estimation, forecasting.

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1 Introduction

Dynamic Stochastic General Equilibrium (DSGE) models are frequently used by academics and researchers at public institutions for policy-analysis and forecasting purposes.[1] Most of these models make use of a framework that is based on linear first-order perturbations, where it is assumed that the sample of data that is used in the estimation is not subject to any regime-switching behaviour.[2] This has lead a certain amount of critique from those who note that these linear models may fail to capture many of the non-linear features that are present in macroeconomic data.[3]

This paper seeks to contribute towards this literature by comparing the forecasting performance of models that make use of first- and second-order approximations in a manner that is similar to that of Pichler (2008). Fernández-Villaverde *et al.* (2016) and Schmitt-Grohé *&* Uribe (2004) have noted that the use of higher-order perturbation techniques could provide improvements in terms of the accuracy of the model solution, where they could also be used to capture important features relating to asset pricing or welfare effects.

We then extend this analysis to consider the use of higher-order perturbation approximations in models that allow for regime-switching behaviour. The use of Markov-switching DSGE (MS-DSGE) models is considered in Liu *et al.* (2009), Liu *et al.* (2011), Liu *&* Mumtaz (2011) and Foerster *et al.* (2016), where it is noted that these models allow for the analysis of more complex dynamic features that may be present in the data. To complete this analysis we make use of the methodology that is described in Ivashchenko (2016) for the estimation of filtering of MS-DSGE models that make use of a second-order approximation.

The underlying model incorporates several New Keynesian features, with the addition of partial indexation on previous inflation. Nominal rigidities are introduced into the price setting mechanism of the firm and investment adjustment costs, where we make use of Rotemberg (1982) pricing. Two variants of the MS-DSGE model are considered, where the first considers the possibility of regime-switching in the policy rule and the second considers switching in the volatility of the shocks. All the models are then estimated over recursive data samples for the United States economy and the forecasts are evaluated on the basis of the root-mean squared-error and log-predictive score (where we use both Gaussian and mixed-Gaussian distributions).

After evaluating the model over an extensive out-of-sample period, the results suggest that while the use of higher-order approximations may provide a superior out-of-sample fit of the data in a model that is restricted to a single regime, this is not the case for those models that allow for regime-switching. This would suggest that when we allow for the possibility of Markov-switching, a single-order approximation of the model solution may be sufficiently accurate.[4]

The structure of our work is as follows: section 2 presents the model structure, which includes details of the regime-switching behaviour. Section 3 provides details of the data and the methodology for the evaluation of the forecasts. Section 4 discusses the results of

[1]Lindé (2018) provides a recent summary of the use of DSGE models within academic and policy-making institutions, while da Silva (2018) and Tovar (2009) make note of their use within central banks. Several other authors, including Blanchard (2016) and Reis (2018), suggest that while we need to improve upon the existing framework, it would be wrong to suggest that this framework should be discarded.

[2]Christiano *et al.* (2018) note that use of first-order approximations is motivated by the fact that these models appear to provide an accurate characterisation of the effects of small shocks (during the post-war period in the United States). In addition, these techniques also allow researchers to make use of linear filters and estimation methodologies that are not subject to computational restrictions of non-linear counterparts.

[3]In the critique of Stiglitz (2018), he suggests that the use of linear approximations in such a macroeconomic model would be inappropriate as it would not provide an accurate description of the effects of large shocks. As an alternative to high-order perturbation techniques, Fernández-Villaverde *&* Levintal (2017) describe the use of three projection methods that may be used to solve calibrated versions of a nonlinear DSGE model.

[4]Since we allow for Markov-switching in the volatilities of the shocks the model could distinguish between the effects of relatively large and small shocks. Hence, this feature of the model would not necessarily preclude one for investigating the effects of large shocks that may take us far away from the domain over which the single-regime fist-order approximation is valid.

the different specifications and section 5 concludes.

2 Model

The model takes the form of a closed-economy New Keynesian structure where household utility is dependent on consumption, leisure and real monetary balances. The utility function is subject to a budget constraint that incorporates holdings of money and bonds, as well as capital and consumption goods. The amount of labour hours and dividends are also included in the budget constraint, along with the level of investment.

The firm that is responsible for the production of finished goods makes use of a constant returns-to-scale technology when transforming the output of firms that are involved in the production of intermediate goods. A Cobb-Douglas production constraint is used for the intermediate producer that faces monopolistic competition and nominal rigidities are introduced into the pricing mechanism through the mechanism of Rotemberg (1982). The central bank makes use of a Taylor (1993) rule that is measured in terms of deviations from steady-state values.

The structure of this model is in many ways similar to that of Pichler (2008), however we also allow for partial indexation on previous inflation, following the specification in Lindé (2005). The equilibrium conditions that describe the optimal conditions of the respective agents are summarised in equations (1) - (13).

After solving for the optimal conditions for the household, the preference for leisure (χ_h) may be expressed as follows:

$$\log(\chi_h) = z_{a,t} + w_t - c_t\tau \tag{1}$$

where $z_{a,t}$ is the aggregate demand shock, w_t represents wages, consumption expenditure is c_t and τ denotes the preference for consumption in the household utility function. The intertemporal relationship for the decision to consume during the current or future period takes the form:

$$\exp(z_{a,t} - c_t\tau - i_t) = \beta\mathbb{E}_t \exp(z_{a,t+1} - c_{t+1}\tau - \pi_{t+1}) \tag{2}$$

where i_t represents the nominal interest rate, β is the discount factor, and $\pi_t + 1$ refers to inflation. Real monetary balances (m_t) are included, such that the preference of the household for real monetary balances (χ_m) is given as:

$$\log(\chi_m) = z_{a,t} + m_t + \log[1 - \exp(i_t)] - c_t\tau \tag{3}$$

Since it is assumed that the household is the owner of capital goods (k_t), which are incorporated in the budget constraint, the intertemporal nature for the household's choice of capital goods may be described as follows:

$$\begin{aligned} \exp(z_{a,t} - c_t\tau)\left[1 - \delta + \phi_k \exp(x_t - k_{t-1})\right] \ldots \\ -\beta\mathbb{E}_t \exp(z_{a,t+1} - c_{t+1}\tau + q_{t+1}) + 1 \ldots \\ -\delta - \frac{\phi_k\left[\exp(x_{t+1} - k_t) - \delta\right]^2}{2} \ldots \\ +\phi_k\left[\exp(x_{t+1} - k_t) - \delta\right]\left[\exp(x_{t+1} - k_t) + 1 - \delta\right] &= 0 \end{aligned} \tag{4}$$

where x_t refers to investment, which adds to the stock of capital goods, q_t is the real factor price for capital goods, ϕ_k is the adjustment cost for changing the capital stock, and σ_k is the rate of depreciation.

Firms involved in intermediate production face a Cobb-Douglas production function, such that the level of output (y_t) is a function of the available capital, hours of labour, and technology.

$$y_t = k_{t-1}\alpha + (z_{y,t} + h_t)(1-\alpha) \tag{5}$$

where, α represents the share of capital in output and $z_{y,t}$ refers to technology. This firm would then select the amount of goods to produce that will maximise it's total market value, subject to the predetermined input costs and the relative amount of inflation that is present in each period.

$$\begin{aligned} \exp(y_t) &= \exp(w_t + h_t) + \exp(q_t + k_{t-1}) \ldots \\ &+\phi_\pi \left(\exp\left\{\pi_t - [\bar{\pi}(1-\upsilon_\pi) + \pi_{t-1}\upsilon_\pi]\right\} - 1\right)^2 \exp(y_t) + \exp(d_t) \end{aligned} \tag{6}$$

where $\bar{\pi}$ is the steady-state for inflation, υ_π is the degree of indexation, and d_t represents the dividend payments that are made by the firm to the household. After solving the problem of the firm that produces intermediate goods, we are able to derive an expression that equates the prices for the factors of production:

$$\log(\alpha) + w_t + h_t = \log(1-\alpha) + q_t + k_{t-1} \tag{7}$$

It is assumed that the firm faces quadratic adjustment costs with a degree of indexation. Hence, using the mechanism of Rotemberg (1982), which considers the cost of adjusting the prices of all finished goods, we are able to express the rigidities in the price setting mechanism as follows:

$$\begin{aligned} &\exp(z_{a,t} - c_t\tau)\left\{\frac{1-\theta+\theta\exp(w_t+h_t-y_t)}{(1-\alpha)} \ldots\right. \\ &\left. -\phi_\pi \left[\exp\left(\pi_t - [\bar{\pi}(1-\upsilon_\pi) + \pi_{t-1}\upsilon_\pi]\right) - 1\right] \exp\left(\pi_t - [\bar{\pi}(1-\upsilon_\pi) + \pi_{t-1}\upsilon_\pi]\right)\right\} \ldots \\ &+\beta\phi_p \left[\exp\left(\pi_{t+1} - [\bar{\pi}(1-\upsilon_\pi) + \pi_t\upsilon_\pi]\right) - 1\right] \ldots \\ &\exp\left(\pi_{t+1} - [\bar{\pi}(1-\upsilon_\pi) + \pi_t\upsilon_\pi] + z_{a,t+1} - c_{t+1}\tau + y_t\right) = 0 \end{aligned} \tag{8}$$

where θ represents the elasticity of substitution between any two intermediate inputs.

The monetary policy rule for the central bank would then employ the Taylor (1993) principle, where $\varepsilon_{i,t}$ is the monetary policy shock to short-term interest rates and σ_i is the volatility that is associated with this shock.

$$i_t - \bar{i} = \gamma_i(i_{t-1} - \bar{i}) + \gamma_y(y_t - \bar{y}) + \gamma_\pi(\pi_t - \bar{\pi}) + \sigma_i\varepsilon_{i,t} \tag{9}$$

The budget constraint for the household could then be summarised as:

$$\begin{aligned} &\exp(c_t) + \exp(x_t) + \frac{\phi_k \exp(k_t)\left(\exp(x_t - k_{t-1}) - \delta\right)^2}{2} \ldots \\ &+\frac{\phi_\pi \exp(y_t)\left(\exp\left(\pi_t - (\bar{\pi}(1-\upsilon_\pi) + \pi_{t-1}\upsilon_\pi)\right) - 1\right)^2}{2} = \exp(y_t) \end{aligned} \tag{10}$$

while the expression for the flow of capital would take the form:

$$\exp(k_t) = (1-\delta)\exp(k_{t-1}) + \exp(x_t) \tag{11}$$

To ensure that the shocks are allowed to be relatively persistent in the model, they are specified as autoregressive processes:

$$
\begin{aligned}
z_{a,t} &= \eta_{1,a} z_{a,t-1} + \sigma_a \varepsilon_{a,t} && (12)\\
z_{y,t} &= \eta_{1,y} z_{y,t-1} + (1-\eta_{1,y})\,\bar{z} + \sigma_y \varepsilon_{y,t} && (13)
\end{aligned}
$$

where $\eta_{1,a}$ and $\eta_{1,y}$ measure the degree of persistence, while the independent innovations take the form $\varepsilon_{a,t} \sim \mathcal{N}(0,1)$ and $\varepsilon_{y,t} \sim \mathcal{N}(0,1)$.

Three versions of the model are used in the subsequent analysis. The first does not employ any regime-switching behaviour. The second variant allows for Markov-switching in the volatilities of the independent shocks. In this case we allow for two possible regimes within which values of volatility parameters σ_a, σ_i, σ_y can switch. The third version allows for the possibility of Markov-switching in the monetary policy rule, where there are two regimes within which monetary policy parameters γ_i, γ_π, γ_y can switch. This provides us with a total of six models, as we make use of both first- and second-order approximations for each of these models.

The parameters in the models are estimated with maximum likelihood techniques for both orders of perturbation. The RISE toolbox is used for the estimation of all the first-order models, which employs the filter of Kim (1994) and the solution method that is described in Maih (2015).[5] For the second-order models, we employ the Markov-switching quadratic Kalman filter (MSQKF) that is described in Ivashchenko (2016).[6]

3 Data and evaluation

The data for the observed variables in the model pertains to measures for output, prices and nominal interest rates in the United States. To be consistent with the model, the data for the observed variables would need to be stationary. Hence, we make use of the quarterly growth rate of real gross domestic product (GDP) for the measure of output as this is a important variable in many forecasting exercises. In addition, we make use of a broad measure for prices, which is measured by the quarterly change in the GDP deflator. To measure interest rates, we follow Wu *&* Xia (2016) and make use of the shadow rates for the Federal Funds Rate, which represent the rates that are applicable for the majority of the time points during each quarter. Since the mean values for these variables differ from zero over the respective subsamples, we incorporate additional constants in the observation equations (14) - (16) to ensure that the variable in the model takes on the a zero steady-state value.

$$
\begin{aligned}
i_{obs,t} &= i_t - \bar{i} + i_{obs} && (14)\\
\pi_{obs,t} &= \pi_t - \bar{\pi} + \pi_{obs} && (15)\\
y_{obs,t} &= y_t - \bar{y} + y_{obs} && (16)
\end{aligned}
$$

The full sample of data that is used in the subsequent analysis covers the period 1978q3 to 2017q3, which provides us with 153 observations after we exclude the first four observations that are used in the pre-sample period. The data is then divided into the initial training and testing sub-samples, where the initial in-sample period that is used for parameter estimation spans until 2006q3. Thereafter, the initial forecast is generated for the 1- to 8-step ahead horizon, over the period 2006q4 - 2008q4. Once we have stored these results, the in-sample estimation period is extended to 2006q4 for the forecasts that are generated for the period 2007q1 - 2009q1.

To evaluate the forecasting performance of the individual models we make use of three measures of forecasting accuracy. The first measure is traditional root-mean squared-error

[5]This RISE toolbox can be downloaded from: `https://github.com/jmaih/RISE_toolbox`.

[6]This filter is not presently incorporated in the RISE toolbox.

(RMSE) that is computed according to formula in equation (17), for respective forecasting horizon (μ). The second measure is the mean log predictive score (LPS) which considers the log-likelihood of the data and the respective forecast density. This measure can be computed for individual variables or for all observed variables according to the formula in equation (18), which assumes that the forecast density is Gaussian (LPSG). In the presence of large sample sizes, the model with the highest expected log predictive score (which would be equivalent to the lowest value for the Kullback-Leibler information criteria) will provide the most accurate forecast.

In the case of Markov-switching models, the forecast density would need to be approximated by a mixed-Gaussian density (LPSGM). To calculate this statistic we make use of the formula in equation (19), where obs_t is the vector of observed variables, while $\mathbb{E}_t(\cdot)$, refers to the expectations operator, conditional on information that is available at time t. Similarly, $\mathbb{E}_{t,s}(\cdot)$ is expectations operator this is conditional on information that is available at time t, which would include information relating to current regime, where $p_t(\psi_t = s)$ is the probability of being in regime $s = \{1, 2\}$. Similarly, $V_t(\cdot)$ and $V_{t,s}(\cdot)$ relate to the corresponding variances, conditional on information that is available at time t and about regime s.

$$RMSE = \left(\frac{\sum_{t=1}^{n} \left[obs_{t+\mu} - \mathbb{E}_t \left(obs_{t+\mu} \right) \right]^2}{n} \right)^{\frac{1}{2}} \tag{17}$$

$$\begin{aligned} LPSG = \sum_{t=1}^{n} \Bigg\{ & \frac{- \left(obs_{t+\mu} - \mathbb{E}_t \left[obs_{t+\mu} \right] \right)' \left(V_t \left[obs_{t+\mu} \right] \right)^{-1} \left(obs_{t+\mu} - \mathbb{E}_t \left[obs_{t+\mu} \right] \right)}{2} \ldots \\ & + \log \left(\frac{\left| V_t \left(obs_{t+\mu} \right) \right|^{-\frac{1}{2}}}{(2\pi)^{m/2}} \right) \Bigg\} / n \end{aligned} \tag{18}$$

$$\begin{aligned} LPSGM = \sum_{t=1}^{n} \log \Bigg\{ & \exp \Bigg[\frac{- \left(obs_{t+\mu} - \mathbb{E}_{t,1} \left[obs_{t+\mu} \right] \right)' \left(V_{t,1} \left[obs_{t+\mu} \right] \right)^{-1} \left(obs_{t+\mu} - \mathbb{E}_{t,1} \left[obs_{t+\mu} \right] \right)}{2} \ldots \\ & + \log \left(\frac{\left| V_{t,1} \left[obs_{t+\mu} \right] \right|^{-\frac{1}{2}}}{(2\pi)^{m/2}} \right) + \log \left(p_t \left[\psi_{t+\mu} = 1 \right] \right) \Bigg] \ldots \\ & + \exp \Bigg[\frac{- \left(obs_{t+\mu} - \mathbb{E}_{t,2} \left[obs_{t+\mu} \right] \right)' \left(V_{t,2} \left[obs_{t+\mu} \right] \right)^{-1} \left(obs_{t+\mu} - \mathbb{E}_{t,2} \left[obs_{t+\mu} \right] \right)}{2} \ldots \\ & + \log \left(\frac{\left| V_{t,2} \left(obs_{t+\mu} \right) \right|^{-\frac{1}{2}}}{(2\pi)^{m/2}} \right) + \log \left(p_t \left[\psi_{t+\mu} = 2 \right] \right) \Bigg] \Bigg\} / n \end{aligned} \tag{19}$$

4 Results

The RMSEs for the out-of-sample evaluation statistics are presented in Table 1, for the model that has no regime-switching (NOS) behaviour, switching in the volatilities (VOL) of the shocks, and switching in the monetary policy rule (POL). These results suggest that the forecasts for output that were produced by the model with Markov-switching in the policy function provide the most desirable RMSEs, when compared to the forecasting results for output that were generated by the other two models. In addition, the forecasting results for long-term interest rates that were provided by the model that does not allow for regime-switching (and makes use of a first-order approximation) produce slightly smaller RMSEs. However, the model that does not make use of regime-switching provides less desirable forecasts for inflation and output, when compared to both of the Markov-switching versions of the model (for each order of perturbation).

Table 1: RMSE out of sample forecasts

			$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
First-order	NOS	r_t	0.14%	0.22%	0.29%	0.34%	0.37%	0.41%	0.44%	0.48%
		π_t	0.25%	0.26%	0.27%	0.3%	0.32%	0.34%	0.37%	0.39%
		y_t	0.79%	0.8%	0.8%	0.81%	0.82%	0.82%	0.8%	0.8%
	VOL	r_t	0.12%	0.19%	0.26%	0.33%	0.39%	0.44%	0.48%	0.52%
		π_t	0.24%	0.24%	0.22%	0.23%	0.23%	0.24%	0.25%	0.25%
		y_t	0.89%	0.89%	0.88%	0.88%	0.89%	0.89%	0.86%	0.87%
	POL	r_t	0.12%	0.2%	0.29%	0.37%	0.44%	0.51%	0.57%	0.61%
		π_t	0.22%	0.22%	0.22%	0.23%	0.24%	0.25%	0.27%	0.28%
		y_t	0.69%	0.69%	0.69%	0.7%	0.7%	0.71%	0.69%	0.7%
Second-order	NOS	r_t	0.14%	0.23%	0.31%	0.39%	0.44%	0.5%	0.54%	0.57%
		π_t	0.26%	0.26%	0.26%	0.25%	0.26%	0.26%	0.28%	0.28%
		y_t	0.85%	0.87%	0.83%	0.84%	0.85%	0.86%	0.83%	0.84%
	VOL	r_t	0.12%	0.21%	0.29%	0.37%	0.43%	0.48%	0.53%	0.56%
		π_t	0.26%	0.28%	0.29%	0.3%	0.31%	0.3%	0.31%	0.33%
		y_t	0.84%	0.82%	0.82%	0.83%	0.85%	0.85%	0.82%	0.83%
	POL	r_t	0.11%	0.18%	0.26%	0.33%	0.39%	0.45%	0.5%	0.53%
		π_t	0.22%	0.23%	0.23%	0.27%	0.3%	0.33%	0.36%	0.4%
		y_t	0.68%	0.69%	0.69%	0.7%	0.71%	0.71%	0.69%	0.7%

When comparing the forecasting performance of the models that use either first- or second-order approximations, we note that in the case of the model that does not employ regime-switching, the model that employs the first-order approximation provides improved predictions for interest rates over the medium- and long-term. In addition, this model would also appear to provide slightly better forecasts of output. However, when we consider the results for inflation, we note that second-order approximation provides much more accurate out-of-sample predictions over the medium- and long-term. In the case of the model that allows for switching in the volatility of shocks, the model that makes use of a first-order approximation provides slightly better estimates for interest rates and inflation, while the second-order approximation provides more accurate forecasts for output. Then lastly, the first-order approximation of the model that allows for switching in the monetary policy function provides better predictions for medium- to long-term inflation, while the version of this model that employs a second-order approximation provides better forecasts for interest rates.

Table 2: Log predictive score (LPS) - out of sample

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
NOS [1st]	r_t	5.05	4.67	4.43	4.26	4.14	4.02	3.91	3.79
	π_t	4.49	4.47	4.46	4.35	4.27	4.2	4.07	3.95
	y_t	3.4	3.37	3.37	3.35	3.34	3.33	3.37	3.36
	LPSG	12.95	12.52	12.26	11.99	11.81	11.64	11.48	11.28
	LPSGM	12.95	12.52	12.26	11.99	11.81	11.64	11.48	11.28
VOL [1st]	r_t	5.29	4.76	4.48	4.24	4.08	3.93	3.83	3.73
	π_t	4.52	4.62	4.71	4.67	4.63	4.6	4.58	4.56
	y_t	3.41	3.42	3.42	3.25	3.13	3.19	3.36	3.3
	LPSG	13.15	12.7	12.48	12.05	11.73	11.66	11.72	11.56
	LPSGM	13.25	12.67	12.45	12.23	12	11.83	11.74	11.57
POL [1st]	r_t	5.2	4.76	4.46	4.23	4.06	3.91	3.8	3.72
	π_t	4.69	4.63	4.61	4.57	4.54	4.51	4.47	4.43
	y_t	3.56	3.54	3.54	3.53	3.52	3.51	3.55	3.54
	LPSG	13.53	12.99	12.67	12.39	12.17	12	11.89	11.76
	LPSGM	13.6	13.08	12.78	12.44	12.18	11.98	11.84	11.72
NOS [2nd]	r_t	5.06	4.63	4.34	4.12	3.98	3.86	3.77	3.72
	π_t	4.52	4.52	4.51	4.5	4.47	4.45	4.4	4.39
	y_t	3.3	3.27	3.31	3.29	3.27	3.27	3.32	3.3
	LPSG	12.92	12.51	12.28	12.04	11.86	11.73	11.66	11.56
	LPSGM	12.92	12.51	12.28	12.04	11.86	11.73	11.66	11.56
VOL [2nd]	r_t	5.33	4.74	4.42	4.16	4.01	3.86	3.75	3.66
	π_t	4.49	4.48	4.47	4.42	4.37	4.35	4.3	4.28
	y_t	3.51	3.5	3.48	3.31	3.2	3.22	3.34	3.31
	LPSG	13.27	12.6	12.21	11.7	11.39	11.29	11.27	11.14
	LPSGM	13.34	12.58	12.17	11.87	11.63	11.4	11.27	11.11
POL [2nd]	r_t	4.57	4.21	4	3.84	3.73	3.63	3.55	3.49
	π_t	4.27	4.07	3.96	3.86	3.77	3.7	3.64	3.6
	y_t	3.55	3.53	3.53	3.51	3.5	3.5	3.53	3.52
	LPSG	12.7	12.16	11.84	11.52	11.26	11.04	10.87	10.7
	LPSGM	13.6	12.94	12.5	12.09	11.78	11.6	11.49	11.31

The results for the log predictive scores are summarised in Table 2. These evaluation statistics suggest that the quality of the density forecasts would appear to provide similar results to those of the point forecasts. For example, in the case of the first-order approximations, the model with switching in the policy function provides superior results for output and short-term inflation, while the model with switching in the volatility of shocks provides better predictions for short-term interest rates and long-term inflation. This leaves the model that does employ regime as the one that provide the best long-term interest rate forecasts.

In addition, when considering the results for the second-order approximations, the model that allows for switching in the monetary policy rule is no longer superior, when looking to forecast interest rates and short-term inflation. In addition, the model that does not employ any switching now outperforms the other models with regard to the predictions for inflation forecasts and long-term interest rates. The aggregate quality of these predictions, as measured by the LPSG and LPSGM statistics, suggest that the when comparing the first-order approximation models, the version with switching in the policy rule is clearly the best performing model. However, for the second-order approximations, the model without switching outperforms other models over longer horizons, while the models that employ regime-switching perform better over shorter horizons. In general, we also note that predictions from models that utilise second-order approximations are better than those that provide predictions from first-order approximations (in absence of switching), which is largely due to the difference in the forecasts for inflation.

Table 3: Significance test for equal forecasting ability: First- and second-order approximations

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
NOS	RMSE r_t	67.42%	61.96%	32.2%	37.76%	43.73%	26.12%	31.36%	16.2%
	RMSE π_t	67.42%	61.96%	56.12%	73.36%	86.59%	97.34%	92.83%	97.65%
	RMSE y_t	0%	0%	0.01%	0%	0%	0%	0%	0%
	LPS r_t	97.56%	61.96%	14%	17.44%	21.48%	62.54%	43.57%	74.43%
	LPS π_t	2.44%	0.27%	0.14%	10.55%	21.48%	50%	68.64%	90.61%
	LPS y_t	8.71%	3.3%	4.42%	2.98%	4.03%	1.19%	3.65%	1%
	LPSG	55.98%	38.04%	22.04%	17.44%	56.27%	50%	68.64%	90.61%
	LPSGM	55.98%	38.04%	22.04%	17.44%	56.27%	50%	68.64%	90.61%
VOL	RMSE r_t	8.71%	18.02%	8.21%	17.44%	4.03%	62.54%	56.43%	74.43%
	RMSE π_t	14.56%	0.27%	0.98%	2.98%	0.01%	1.19%	0.01%	0%
	RMSE y_t	97.56%	100%	100%	100%	100%	100%	100%	100%
	LPS r_t	99.52%	96.7%	91.79%	82.56%	68.21%	83.16%	56.43%	62.86%
	LPS π_t	0.18%	0.03%	0%	0%	0%	0%	0.02%	0.04%
	LPS y_t	91.29%	96.7%	77.96%	89.45%	95.97%	83.16%	87.21%	90.61%
	LPSG	22.57%	6.31%	0%	0%	0%	0%	0%	0.04%
	LPSGM	8.71%	1.58%	0.98%	0.22%	0.11%	0.01%	0.02%	0.04%
POL	RMSE r_t	44.02%	61.96%	86%	98.62%	99.97%	99.95%	99.31%	99.62%
	RMSE π_t	22.57%	6.31%	14%	10.55%	1.92%	1.19%	3.65%	1%
	RMSE y_t	32.58%	50%	56.12%	73.36%	68.21%	73.88%	56.43%	50%
	LPS r_t	0%	0%	0.01%	0.02%	0.32%	2.66%	3.65%	4.94%
	LPS π_t	0.01%	0%	0%	0%	0%	0%	0%	0%
	LPS y_t	1.13%	0.69%	0.4%	0.58%	0.83%	1.19%	0.25%	0.38%
	LPSG	0%	0%	0%	0.01%	0%	0%	0%	0%
	LPSGM	55.98%	0.27%	0.14%	0.02%	0.01%	0.05%	0.25%	0.38%

To consider the extent to which the forecasts that were obtained from first- and second-order approximations provide a significant improvement, we make use of the method of (Clarke, 2007). Table 3 contains the respective p-values for this test, where the null hypothesis, H_0, is that models have equal forecasting ability, while the alternative hypothesis H_1, is that second-order approximation of the model outperforms the model that employs a first-order approximation. These results would suggest that the differences are significant in most cases, where the most superior model would depend on combinations of variable and forecasting horizons.

There are a few cases where the statistics provide different results. For example, the long-term forecasts for interest rates that are provided by the model with switching in the policy function, provide opposing results for the RMSE and LPS, where the RMSE results suggest superiority of the second-order model, while the LPS results suggest that the first-order approximation provides superior results. When considering the aggregate measures for the LPSG and LPSGM it should be noted that second-order approximation for the model that does not incorporate switching has a significant advantage over the first-order approximation of this model. However, this is not necessarily the case for both of the models that employ switching, particularly over the medium- to long-term.

Table 4 contains the results of a subsequent test, where we consider whether the inclusion of Markov-switching would make a significant difference to the forecasting performance of the respective models, where we use the method of Clarke (2007) once again. In this case the alternative hypothesis, H_1, would suggest that the inclusion of Markov-switching would provide significant improvements to the forecasting ability of the model. The results for the first-order approximation suggest that the difference in forecasting ability is significant when we compare the results of the model that makes use of monetary policy switching against the results of the model that does make use of any regime-switching. However, when we compare the results of the model that allows for Markov-switching in the volatility of the shocks, there is no significant improvements in the forecasting ability.

When we consider the results of the models that make use of second-order approximations in further detail, we note that the model that allows for switching behaviour in the monetary policy function displays a number of interesting properties. In this case the LPSGM results suggest that this model performance is significantly better over the short-term horizon, while LPSG suggests that the model without switching provides significant improvements over the long-term horizon. Thus, the model that allows for switching in the monetary policy rule produces an improved forecasting density, while its mean-variance forecast is inferior. In case of the model that allows for switching in the volatilities of the shocks, the situation is much simpler: the inclusion of this switching behaviour significantly improves upon the short-term forecasts, while there is an insignificant decrease in the ability of the model to forecast over the long-term. Thus, the relatively sharp likelihood function for second-order approximation would facilitate the identification of the parameters in a more accurate manner, which may also assist with the characterisation of more complicated dynamics. In addition, it may also suggest that the inclusion of additional regime-switching behaviour may be less important, when compared to the case of the first-order approximation.

Table 4: Significance test for equal forecasting ability: Markov-switching

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
VOL vs NOS [1st]	RMSE r_t	99.52%	99.31%	97.82%	97.02%	95.97%	90.02%	79.12%	74.43%
	RMSE π_t	77.43%	96.7%	99.6%	99.99%	100%	100%	100%	100%
	RMSE y_t	0.02%	0%	0%	0%	0%	0%	0%	0%
	LPS r_t	99.94%	99.31%	97.82%	97.02%	92.31%	83.16%	79.12%	62.86%
	LPS π_t	95.19%	99.31%	99.99%	100%	100%	100%	100%	100%
	LPS y_t	85.44%	88.9%	67.8%	37.76%	4.03%	0.47%	1.68%	1%
	LPSG	98.87%	96.7%	91.79%	99.42%	98.08%	98.81%	99.31%	99.62%
	LPSGM	97.56%	93.69%	67.8%	82.56%	92.31%	83.16%	92.83%	90.61%
POL vs NOS [1st]	RMSE r_t	99.52%	99.31%	91.79%	94.14%	68.21%	62.54%	31.36%	25.57%
	RMSE π_t	91.29%	88.9%	77.96%	98.62%	99.17%	99.99%	100%	99.96%
	RMSE y_t	99.82%	99.31%	99.6%	99.78%	99.68%	99.53%	99.31%	97.65%
	LPS r_t	100%	99.9%	86%	89.45%	68.21%	37.46%	31.36%	37.14%
	LPS π_t	22.57%	18.02%	43.88%	37.76%	31.79%	73.88%	68.64%	90.61%
	LPS y_t	100%	100%	100%	100%	100%	100%	100%	100%
	LPSG	100%	99.99%	99.95%	99.78%	99.97%	99.95%	99.75%	99.62%
	LPSGM	100%	99.99%	99.86%	99.99%	99.99%	100%	99.75%	99.87%
VOL vs NOS [2nd]	RMSE r_t	99.94%	99.73%	99.86%	82.56%	31.79%	50%	43.57%	50%
	RMSE π_t	8.71%	50%	14%	50%	21.48%	37.46%	68.64%	25.57%
	RMSE y_t	85.44%	98.42%	32.2%	62.24%	78.52%	83.16%	87.21%	90.61%
	LPS r_t	99.99%	99.97%	99.6%	99.78%	98.08%	99.53%	96.35%	90.61%
	LPS π_t	77.43%	88.9%	86%	50%	68.21%	50%	56.43%	50%
	LPS y_t	98.87%	99.9%	95.58%	50%	0.83%	1.19%	0.25%	0%
	LPSG	99.99%	99.73%	86%	73.36%	43.73%	37.46%	43.57%	25.57%
	LPSGM	100%	98.42%	86%	73.36%	43.73%	16.84%	31.36%	25.57%
POL vs NOS [2nd]	RMSE r_t	99.98%	99.9%	97.82%	89.45%	56.27%	90.02%	98.32%	97.65%
	RMSE π_t	67.42%	88.9%	22.04%	26.64%	4.03%	1.19%	0.25%	0.13%
	RMSE y_t	99.99%	99.97%	99.95%	99.99%	99.99%	99.99%	99.92%	99.87%
	LPS r_t	0%	0.27%	0.4%	0.58%	0.83%	2.66%	7.17%	4.94%
	LPS π_t	0.06%	0%	0%	0%	0%	0%	0%	0%
	LPS y_t	100%	100%	100%	100%	100%	100%	100%	100%
	LPSG	8.71%	6.31%	0.98%	2.98%	0.11%	0.17%	0.02%	0.04%
	LPSGM	99.99%	100%	99.95%	97.02%	68.21%	62.54%	43.57%	16.2%

Table 5: Parameters values

	NOS		VOL		POL		Pichler (2008)	
	1st	2nd	1st	2nd	1st	2nd	1st	2nd
σ_s	0.1167	0.0716	0.0798	0.0763	0.1509	0.1652	0.1061	0.0292
σ_i	0.0019	0.002	0.0009	0.0009	0.0043	0.0043	0.0014	0.0017
σ_y	0.6308	0.0176	1.891	1.3343	0.0443	0.0768	100	0.0083
α	0.7789	0	0.8949	0.8953	0.8144	0.8454	0.8347	0.36
σ_K	0.0522	0.0491	0.1	0.099	0.0852	0.1	0.0019	0.025
θ	2.3164	6.294	2	2.0022	6.9617	6.16	2.0004	6
ϕ_k	99.9999	1.2658	100	100	33.2747	29.1348	100	10
χ_h	9.9972	7.3313	10	7.4068	0.2143	0.2336	9.9947	*
χ_m	2.8746	0	8.6944	1.7745	0.8768	0.8768	4.2802	*
β	0.9455	0.9833	0.9	0.9017	0.9686	0.9684	0.9882	0.9931
ϕ_π	329.0216	76.4397	1000	897.7809	3.3443	2.2285	16.0592	84.6699
$\bar{\pi}$	1.907	0.0092	1.712	1.9704	1.4927	1.4351	2	0.008662
$\bar{z}$	3.4511	28.3427	5.796	24.7356	11.8958	12.0549	185.7394	7082.321
$\eta_{1,a}$	0.9928	1	0.992	0.9967	0.9469	0.9613	0.8895	0.929
$\eta_{1,y}$	-0.2037	0.8741	0.0054	-0.2078	0.951	0.9673	0.9961	0.9687
γ_i	0.7725	0.78	0.907	0.908	0.7903	0.6656	0.8602	0.7517
γ_y	0.0061	0.0126	0.0048	0.0057	0.0066	0.0022	-0.0007	0.0245
γ_p	0.2817	0.3859	0.1834	0.1836	1.9776	2.0122	0.518	0.3281
τ	10	9.9977	10	9.9995	9.8966	9.9999	9.9991	2.745
υ_p	0.8121	0	0.5919	0.8407	0	0	0	NA
i_{obs}	0.0383	0.0208	0.0062	0.0127	-0.0069	0.0139	-0.023	NA
π_{obs}	0.0463	0.0093	0.0054	0.0048	0.003	0.0065	-0.0033	NA
y_{obs}	0.0112	0.0082	0.0078	0.0079	0.0029	0.0062	0.0022	NA
$\sigma_a(\psi = 2)$	NA	NA	0.2692	0.2898	NA	NA	NA	NA
$\sigma_i(\psi = 2)$	NA	NA	0.0036	0.0037	NA	NA	NA	NA
$\sigma_y(\psi = 2)$	NA	NA	2.5602	1.7382	NA	NA	NA	NA
$p(\psi_{t+1} = 2 \vert \psi_t = 1)$	NA	NA	0.0297	0.0234	0.02	0.02	NA	NA
$p(\psi_{t+1} = 1 \vert \psi_t = 2)$	NA	NA	0.1202	0.1594	0.039	0.0293	NA	NA
$\gamma_i(\psi = 2)$	NA	NA	NA	NA	0.2844	0.2128	NA	NA
$\gamma_y(\psi = 2)$	NA	NA	NA	NA	-0.021	-0.0307	NA	NA
$\gamma_\pi(\psi = 2)$	NA	NA	NA	NA	1.0103	1.1105	NA	NA

To consider the differences in the respective models from a different perspective, we compare the parameter estimates that were generated for the full sample in Table 5.[7] The log-likelihood value for model that makes use of a first-order approximation without any regime-switching behaviour is 2005.63. If we fix values of parameters to the values in Pichler (2008), the log-likelihood would be equal about -3*1010. A similar exercise with second order approximation gives the values 1947.23 and -7*108, respectively. In other words, it would appear as if the estimated parameter values in Pichler (2008) are not robust to the modifications that have been applied to the model, which is perfectly understandable when considering the nature of these modifications.[8]

Another important contrast relates to the difference between the parameter estimates for the first- and second-order approximations, where we have observe relatively large differences, while in Pichler (2008) these are relatively small. The most likely source of this discrepancy may be due to the different filters that were used in the analysis, where Pichler (2008) makes uses of a particle filter with 120,000 particles, which according to Andreasen (2013) could be relatively inaccurate when compared to the Central Difference Kalman Filter (CDKF). In addition, subsequent research has shown that the QKF (and MSQKF) filters may provide slight improvements over the CDKF filter, particularly when there are large deviations from the steady-state (Ivashchenko, 2014). As a result, the particle filter may not capture the movements that are relatively far away from what would be provided by a first-order approximation.

[7] These estimates deviate from those that are contained in Pichler (2008) due to: different the sample periods, different observed variables (i.e. we use output growth instead of the output gap), different observation equations (i.e. we use additional parameters for demeaning), absence of measurement errors (i.e. Pichler (2008) made use of relatively large measurement errors), different model structure (i.e. particularly with regards to the role of indexation) and a different filtering methodology.

[8] Hence, this would imply that we are introducing new evidence and are not merely replicating the work of Pichler (2008).

Finally, to ensure that the results of this analysis are relatively robust, we compare the forecast ability of these models with similar forecasts that are provided by VAR and AR models. The results of this analysis is contained in Tables 6 and 7 of the appendix, where we note that the DSGE models have superior predictive ability with a few notable exceptions. For example, reduced-form models provide better forecasts for output and they would also appear to outperform the second-order approximation of the model that allows for regime-switching in the monetary policy function.

5 Conclusion

This paper considers the out-of-sample forecasting performance of two MS-DSGE models that make use of different perturbation orders for the model solution. These results are then compared to the results of models that do not make use of regime-switching behaviour. Our results suggest that in the case of the model that does not employ any switching, the model that is solved with a second-order approximation performs better than the model that makes use of a first-order approximation. However, the opposite is true for models that employ Markov-switching. It is also worth noting that while these results summarise the overall performance of the model, this would not imply that the models that provide the best aggregate performance would generate superior forecasts for each of the individual variables over different horizons.

In addition to these results, we also consider the effect of introducing Markov-switching behaviour on the out-of-sample forecasting performance of the respective models. Our findings suggest that when using a second-order approximation, the introduction of switching in the monetary policy rule would significantly improve the predictive ability of the model. However, the performance of the model that makes use of switching in the volatility of the shocks does not provide any significant improvement over the model that does not employ regime-switching. We also note that the introduction of Markov-switching to the monetary policy rule would improve the forecasting ability of the model over the short-term forecasting horizon for both perturbation orders.

To explain these results, we suggest that since the model that does not employ regime-switching generates a relatively narrow density for the likelihood function when we make use of a second-order approximation, the parameters would be more accurately identified. This would enable the model to describe the more complicated dynamic features that are present in the data. However, when the model incorporates Markov-switching behaviour, which would also facilitate the description of more complex dynamics, the incremental advantage of making use of a higher order perturbation order is no longer present. These results would be of particular interest to those who make use of DSGE models for forecasting purposes.

To incorporate many of the benefits of the individual models that have been discussed in this paper one could potentially make use of a pruned higher-order approximation for MS-DSGE models, which would be an interesting topic for future research.

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Table 6: Significance test of equal forecasting quality of models and VAR

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
NOS [1st]	RMSE r_t	44.02%	96.7%	97.82%	98.62%	99.17%	99.99%	100%	100%
	RMSE π_t	67.42%	61.96%	22.04%	2.98%	7.69%	0.05%	0.02%	0.38%
	RMSE y_t	14.56%	0.27%	14%	10.55%	7.69%	5.41%	7.17%	4.94%
	LPS r_t	67.42%	99.9%	99.99%	99.98%	100%	100%	100%	99.96%
	LPS π_t	98.87%	99.31%	99.86%	97.02%	95.97%	73.88%	87.21%	74.43%
	LPS y_t	0%	0%	0.05%	0.02%	0.11%	0.01%	0.08%	0.04%
	LPSG	1.13%	50%	91.79%	82.56%	95.97%	98.81%	96.35%	99%
	LPSGM	1.13%	50%	91.79%	82.56%	95.97%	98.81%	96.35%	99%
VOL [1st]	RMSE r_t	77.43%	88.9%	97.82%	94.14%	99.68%	99.95%	99.98%	99.96%
	RMSE π_t	67.42%	72.88%	91.79%	94.14%	92.31%	98.81%	96.35%	83.8%
	RMSE y_t	0.06%	0%	0%	0%	0%	0%	0%	0.01%
	LPS r_t	99.82%	99.31%	97.82%	97.02%	99.17%	97.34%	99.31%	99%
	LPS π_t	99.98%	99.97%	100%	99.99%	100%	100%	99.99%	100%
	LPS y_t	8.71%	27.12%	8.21%	0.22%	0%	0.05%	0.08%	0%
	LPSG	95.19%	81.98%	86%	89.45%	92.31%	90.02%	98.32%	90.61%
	LPSGM	97.56%	72.88%	77.96%	82.56%	86.59%	83.16%	87.21%	95.06%
POL [1st]	RMSE r_t	91.29%	96.7%	95.58%	73.36%	86.59%	94.59%	96.35%	99%
	RMSE π_t	85.44%	93.69%	56.12%	73.36%	99.17%	83.16%	43.57%	50%
	RMSE y_t	98.87%	72.88%	86%	94.14%	98.08%	94.59%	96.35%	95.06%
	LPS r_t	99.52%	99.73%	99.86%	99.78%	99.89%	99.95%	99.92%	99.62%
	LPS π_t	67.42%	99.73%	99.6%	99.98%	99.97%	99.95%	100%	100%
	LPS y_t	95.19%	99.99%	100%	100%	100%	100%	99.99%	99.99%
	LPSG	99.94%	99.97%	99.6%	99.99%	100%	100%	99.92%	99.62%
	LPSGM	100%	100%	99.95%	99.93%	99.89%	99.95%	98.32%	99.62%
NOS [2nd]	RMSE r_t	67.42%	61.96%	91.79%	82.56%	99.17%	98.81%	99.92%	99.87%
	RMSE π_t	97.56%	72.88%	32.2%	50%	86.59%	62.54%	56.43%	50%
	RMSE y_t	0.06%	0%	0.01%	0%	0.11%	0%	0.25%	0.13%
	LPS r_t	95.19%	96.7%	99.02%	99.78%	99.97%	99.99%	99.99%	99.99%
	LPS π_t	97.56%	99.73%	99.95%	99.99%	100%	100%	99.99%	99.99%
	LPS y_t	0%	0%	0%	0%	0%	0%	0%	0%
	LPSG	2.44%	6.31%	91.79%	98.62%	99.97%	99.83%	99.75%	99.96%
	LPSGM	2.44%	6.31%	91.79%	98.62%	99.97%	99.83%	99.75%	99.96%
VOL [2nd]	RMSE r_t	85.44%	72.88%	86%	89.45%	95.97%	99.53%	99.92%	99.96%
	RMSE π_t	44.02%	6.31%	8.21%	10.55%	31.79%	26.12%	31.36%	37.14%
	RMSE y_t	0.02%	0%	0.14%	0.58%	0.83%	0.17%	0.69%	0.13%
	LPS r_t	99.82%	98.42%	95.58%	94.14%	99.17%	97.34%	98.32%	99%
	LPS π_t	98.87%	88.9%	91.79%	89.45%	92.31%	97.34%	87.21%	97.65%
	LPS y_t	8.71%	11.1%	2.18%	0.22%	0.01%	0.01%	0.02%	0%
	LPSG	95.19%	61.96%	56.12%	73.36%	78.52%	50%	79.12%	83.8%
	LPSGM	98.87%	88.9%	67.8%	62.24%	92.31%	83.16%	92.83%	90.61%
POL [2nd]	RMSE r_t	98.87%	98.42%	99.6%	97.02%	99.97%	99.99%	99.98%	99.96%
	RMSE π_t	67.42%	50%	14%	17.44%	7.69%	5.41%	3.65%	2.35%
	RMSE y_t	77.43%	38.04%	77.96%	62.24%	78.52%	50%	43.57%	50%
	LPS r_t	0%	0%	0.14%	2.98%	4.03%	37.46%	68.64%	74.43%
	LPS π_t	0.06%	0%	0%	0%	0%	0%	0%	0%
	LPS y_t	1.13%	50%	99.02%	97.02%	86.59%	83.16%	79.12%	90.61%
	LPSG	0%	0%	0%	0.22%	0.32%	0.47%	0.02%	0%
	LPSGM	100%	99.97%	99.86%	99.78%	99.68%	99.53%	98.32%	97.65%

Table 7: Significance test of equal forecasting quality of models and AR

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
NOS [1st]	RMSE r_t	0.18%	3.3%	8.21%	17.44%	43.73%	73.88%	87.21%	90.61%
	RMSE π_t	85.44%	50%	8.21%	5.86%	7.69%	0.17%	0.02%	0%
	RMSE y_t	1.13%	0.1%	0%	0.01%	0.03%	0.05%	0.08%	0.13%
	LPS r_t	91.29%	96.7%	95.58%	97.02%	86.59%	97.34%	99.31%	97.65%
	LPS π_t	98.87%	99.73%	99.95%	98.62%	98.08%	94.59%	87.21%	83.8%
	LPS y_t	0%	0%	0%	0%	0%	0%	0%	0%
	LPSG	22.57%	50%	86%	89.45%	95.97%	98.81%	92.83%	95.06%
	LPSGM	22.57%	50%	86%	89.45%	95.97%	98.81%	92.83%	95.06%
VOL [1st]	RMSE r_t	44.02%	81.98%	97.82%	99.78%	99.89%	100%	100%	100%
	RMSE π_t	95.19%	93.69%	95.58%	89.45%	99.68%	90.02%	99.98%	100%
	RMSE y_t	0.48%	0%	0%	0%	0%	0%	0.01%	0.01%
	LPS r_t	99.94%	99.9%	95.58%	99.98%	99.89%	99.83%	99.92%	99.87%
	LPS π_t	99.94%	100%	100%	100%	100%	100%	100%	100%
	LPS y_t	8.71%	18.02%	4.42%	0.22%	0%	0%	0%	0%
	LPSG	91.29%	72.88%	95.58%	98.62%	98.08%	99.53%	99.75%	99.62%
	LPSGM	95.19%	50%	86%	97.02%	98.08%	90.02%	87.21%	90.61%
POL [1st]	RMSE r_t	22.57%	61.96%	91.79%	98.62%	99.68%	99.95%	99.99%	100%
	RMSE π_t	91.29%	88.9%	77.96%	89.45%	98.08%	90.02%	98.32%	95.06%
	RMSE y_t	97.56%	61.96%	77.96%	94.14%	78.52%	83.16%	79.12%	74.43%
	LPS r_t	99.98%	99.97%	99.99%	100%	99.99%	100%	99.99%	100%
	LPS π_t	67.42%	99.31%	100%	100%	100%	100%	100%	100%
	LPS y_t	95.19%	99.97%	100%	100%	100%	100%	99.98%	100%
	LPSG	100%	100%	99.99%	100%	100%	100%	100%	100%
	LPSGM	100%	100%	99.99%	99.98%	99.99%	99.99%	99.98%	99.87%
NOS [2nd]	RMSE r_t	2.44%	27.12%	56.12%	82.56%	92.31%	99.53%	99.31%	99.87%
	RMSE π_t	95.19%	72.88%	32.2%	82.56%	68.21%	73.88%	43.57%	83.8%
	RMSE y_t	0.18%	0%	0%	0.01%	0.03%	0.01%	0.02%	0.04%
	LPS r_t	99.82%	98.42%	95.58%	99.78%	99.89%	99.83%	99.98%	99.87%
	LPS π_t	95.19%	99.97%	99.02%	100%	100%	100%	99.99%	99.96%
	LPS y_t	0%	0%	0%	0%	0%	0%	0%	0%
	LPSG	22.57%	27.12%	99.6%	98.62%	99.89%	99.53%	99.75%	99.87%
	LPSGM	22.57%	27.12%	99.6%	98.62%	99.89%	99.53%	99.75%	99.87%
VOL [2nd]	RMSE r_t	22.57%	61.96%	67.8%	89.45%	95.97%	99.53%	99.92%	100%
	RMSE π_t	55.98%	18.02%	4.42%	73.36%	31.79%	16.84%	12.79%	9.39%
	RMSE y_t	0.06%	0%	0%	0.01%	0.01%	0.01%	0.02%	0.04%
	LPS r_t	99.99%	96.7%	97.82%	98.62%	98.08%	97.34%	92.83%	97.65%
	LPS π_t	95.19%	98.42%	95.58%	98.62%	99.17%	90.02%	92.83%	99.62%
	LPS y_t	8.71%	3.3%	0.14%	0%	0%	0%	0%	0%
	LPSG	99.94%	81.98%	77.96%	82.56%	78.52%	73.88%	56.43%	83.8%
	LPSGM	99.82%	96.7%	77.96%	89.45%	92.31%	90.02%	92.83%	95.06%
POL [2nd]	RMSE r_t	55.98%	81.98%	86%	94.14%	98.08%	99.53%	99.31%	99.87%
	RMSE π_t	67.42%	27.12%	14%	10.55%	13.41%	2.66%	12.79%	2.35%
	RMSE y_t	77.43%	27.12%	67.8%	37.76%	31.79%	37.46%	20.88%	16.2%
	LPS r_t	0%	0%	0%	0.01%	0.11%	0.47%	0.69%	0.38%
	LPS π_t	0.18%	0%	0%	0%	0%	0%	0%	0%
	LPS y_t	2.44%	72.88%	99.86%	99.42%	95.97%	97.34%	92.83%	95.06%
	LPSG	0.18%	0.1%	0.01%	0.58%	0.83%	0.05%	0.02%	0%
	LPSGM	100%	99.97%	99.6%	99.78%	92.31%	62.54%	79.12%	97.65%