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Rationalizable Information Equilibria

Alexander Zimper
University of Pretoria
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Department of Economics
University of Pretoria
0002, Pretoria
South Africa
Tel: +27 12 420 2413

Rationalizable information equilibria*

Alexander Zimper[†]

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Abstract

Rational expectations equilibria (REE) assume that the ex post equilibrium price function is able to reveal ex ante information. This paper drops the assumption of *information revealing prices* and instead constructs an internal reasoning process through which highly rational price-takers can infer information from other market participants under the assumption that their utility maximization problems are common knowledge. Based on this reasoning process, we introduce the novel competitive equilibrium concept of *rationalizable information equilibria* (RIE). Our formal analysis establishes that (i) the RIE concept amounts to a refinement of the (generalized) REE concept whereby (ii) REE with interior net-trades are generically RIE.

Keywords: General Equilibrium, Asset Exchange Economies, Asymmetric Information, Rational Expectations, Generalized Rational Expectations, Rationalizability

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[†]Department of Economics, University of Pretoria, and Kiel Institute for the World Economy. E-mail: alexander.zimper@up.ac.za

1 Introduction

The concept of *rational expectations equilibria* (REE) is the standard general equilibrium concept for asset exchange economies with asymmetric information.[1] Denote by $P : \Omega \rightarrow \mathbb{R}_{+}^{M}$ a price function which assigns in every state of the world $\omega \in \Omega$ prices to the economy's M assets. The REE concept is based on two assumptions:

1. The price-taking agents ex ante anticipate a price function P which ex post turns out to be the equilibrium price function;

2. In addition to his private information, every agent uses ex ante the information that is ex post revealed through the equilibrium price function formally defined as the inverse of P, i.e.,
$$[P]^{-1}(\omega) \subseteq \Omega \text{ for } \omega \in \Omega. \tag{1}$$

Because of the commonly observed information (1), REE with a sufficiently information-sensitive price function reveal to each agent the other agents' private information (cf. Radner 1979; Allen 1981; Grossman 1981). Irrespective of any asymmetric private information REE thus reduce generically to equilibria under uncertainty whose properties are well-studied in the literature (see, e.g., Radner 1982 and references therein). In addition, such fully revealing REE offer a theoretical justification for Fama's (1970) notion of information efficient asset markets.

Despite these attractive features of REE it is an unresolved issue through which mechanism ex post equilibrium prices might be able to reveal ex ante information.[2] Consider, e.g., a risky asset with random payoff

$$X(\omega) = \begin{cases} x & \text{if } \omega \in \Omega_1 \\ y & \text{if } \omega \in \Omega_2 \end{cases}$$

such that $x \neq y$ where $\{\Omega_1, \Omega_2\}$ is a partition of the state space. Under the assumption of *information revealing prices* an ex-post market clearing price function

$$P(\omega) = X(\omega) \text{ for } \omega \in \Omega \tag{2}$$

[1] Early examples of "fulfilled expectations" equilibria (=REE) appear in Green (1975), Grossman (1977), and Kreps (1977). Radner (1979) provides a general definition of REE as well as a generic existence result for revealing REE.

[2] Compare, e.g., Kreps (1977, p. 36): "At this point, the reader is entitled to conclude that (1) [=the REE concept, the author] is a highly idealized, and highly suspect, way of modeling the "prices transmit information" phenomenon." For a more recent criticism of the REE concept see, e.g., O'Hara (1997), Brunnermeier (2001), and references therein.

informs every agent ex ante about the asset's true value irrespective of the agents' private information in the ex ante situation. That is, even if all agents are completely uncertain about the asset's true value before they enter the market, the price function (2) is supposedly able to tell them the asset's true value. In other words, without further qualifications the assumption of *information revealing prices* amounts to the magic power of foretelling the asset's future value. Foretelling the future, however, does not capture the notion of highly rational market participants who try to infer private information of other price-taking market participants from their net-trade choices at given prices.

To address the open issue of how highly rational agents may infer information in a competitive equilibrium environment, this paper introduces the novel equilibrium concept of *rationalizable information equilibria* (RIE). The RIE concept drops the REE assumption of *information revealing prices* and instead assumes that information might be inferred in a market situation through an internal reasoning process. In analogy to game-theoretic *rationalizability* concepts[3] (Bernheim 1984; Pearce 1984; Moulin 1984; Zimper 2006), we assume that every agent understands the utility maximization problem of any other agent, who in turn also understands the other agents' utility maximization problems, and so forth. However, whereas the players of a strategic game determine through the rationalization process which strategies might possibly be chosen, our price-taking agents determine which information cells of the other agents are consistent with utility maximizing behavior and the common knowledge thereof. That is, in an RIE the equilibrium net-trade choices of all agents at equilibrium prices must be consistent with the assumption that it is common knowledge between all agents that they are utility maximizing price-takers. In contrast to the REE concept, in an RIE the equilibrium prices themselves do not reveal any information but rather the agents' net-trade choices at given prices.

We start out in Section 2 with a motivating example that further illustrates why the assumption of *information revealing prices* is not a good characterization of the information that can be inferred by highly rational market participants. Asset exchange economies are formally introduced in Section 3. Section 4 defines the central notion of "market information equilibria" which subsumes all equilibria discussed in the present paper. In addition to the standard general equilibrium conditions of utility maximization and market-clearing, any market information equilibrium must satisfy epistemic and measurability conditions. In particular, we require that in any equilibrium each agent

[3] As a general equilibrium concept with price-taking agents, the RIE concept does not amount to any strategic reformulation of exchange economies under asymmetric information in the sense of, e.g., Guesnerie (1992) or Heinemann (1997) who both use rationalizable strategy profiles as a strategic foundation for specific competitive equilibria.

(i) knows his equilibrium information, (ii) cannot learn more than the aggregate private (=full communication) information that exists in the economy, and (iii) will not forget his private information in the equilibrium. Moreover, the equilibrium price function and the net-trade function of each agent must be measurable with respect to his equilibrium information. Section 5 then constructs the internal reasoning process which is used in the formal definition of our RIE concept.

The mathematical analysis of Section 6 first investigates the formal relationship between RIE, on the one hand, and equilibria with rational expectations, on the other hand. In addition to the REE concept, we also consider *generalized rational expectations equilibria* (GREE) discussed in Allen and Jordan (1998). The equilibrium information of an GREE in state ω consists of each agent i's private information augmented with the information revealed by the *market-signal*

$$[P]^{-1}(\omega) \cap [\Theta_{j \neq i}]^{-1}(\omega) \subseteq \Omega$$

which combines (1) with the inverse of the other agents' net-trade functions $[\Theta_{j \neq i}]^{-1}$ evaluated at ω. The following findings emerge.

- Any RIE is also an GREE but the converse statement is not true.
- Any RIE with one-one[4] price function is also an REE but not every REE with one-one price function is also an RIE.
- RIE without one-one price functions are not necessarily REE.

In other words, the RIE concept stands for a refinement of the GREE concept that subjects GREE price- and net-trade functions to a consistency test in the form of the RIE internal reasoning process.

Next, Section 6 investigates the relationship between RIE, on the one hand, and *full communication equilibria* (FCE), on the other hand. FCE are market information equilibria whose equilibrium information coincides with the full communication information. Our main analytical insight can be roughly stated as follows.

- For asset-exchange economies whose net-trade correspondences are characterized through first-order conditions, FCE with interior equilibrium net-trades are generically RIE.

[4]A price (or market-signal) function is one-one if it takes on different values for different full communication information cells.

Since REE with one-one price functions, as well as GREE with one-one market-signal functions, are also FCE, such equilibria with rational expectations are–under suitable circumstances–also RIE. In other words, the REE assumption of *information revealing prices* works often, but not always, as a shortcut for the RIE internal reasoning process.

In developing a notion of highly rational market participants which is an alternative to the REE assumption of *information revealing prices*, this paper shares some motivation with the *market-microstructure literature* (cf. O'Hara 1997). This literature has been challenging the realistic appeal of the REE concept by arguing that REE ignore, for example, the possible existence of liquidity/noise traders, the role of market makers, the strategic dimension of trading under asymmetric information and so forth. However, in contrast to strategic bidding models (e.g., Kyle 1989; Reny and Perry 2006) described in the market-microstructure literature, the RIE concept is not about strategic foundations of asset trade but rather it combines general equilibrium analysis with an explicit description of the agents' reasoning processes. As a competitive equilibrium concept with Walrasian price-takers the RIE concept thus remains prone to some of the realistic-appeal criticism originally directed against the REE concept.

2 Motivating example

We analyze an example which illustrates that the REE assumption of *information revealing prices* allows an uninformed investors to correctly predict a risky asset's true value. We further argue that such foretelling of the asset's value is not equivalent to the information that a highly rational but uninformed investor may infer from his understanding of an insider's utility maximization problem.

Example 1: Insider information. Consider the state space

$$\Omega = \{\omega_H, \omega_L\}$$

and suppose that there exist two real assets–one risky, the other risk-free–with the following payoffs measured in the single consumption good:

	X_1	X_2
ω_H	2	1
ω_L	$\frac{1}{2}$	1

The two agents, A and B, have identical asset endowments $(e_{i,1}, e_{i,2}) = (1, 2)$ whereby their physically possible net-trades lie in the intervals $\Gamma_{i,1} = [-1, 1]$,

$\Gamma_{i,2} = [-2, 2]$. The agents' private information cells are given as

$$\begin{aligned} I_A^{PI} &\in \{\{\omega_H\}, \{\omega_L\}\}, \\ I_B^{PI} &= \{\Omega\}, \end{aligned}$$

respectively. Agent A is thus the "insider" who has perfect knowledge about the risky asset's true value whereas agent B is an "uninformed investor" who cannot directly observe the asset's true value.

Next suppose that in an equilibrium agent B somehow learns the true state of the world. Then every agent i maximizes for each $\omega \in \Omega$ the utility

$$u_i(\theta_i, \omega) = X_1(\omega)(\theta_{i,1} + e_{i,1}) + X_2(\omega)(\theta_{i,2} + e_{i,2})$$

over possible net-trades $\theta_i \in \Gamma_i$ subject to the budget constraint at equilibrium prices $P : \Omega \to \mathbb{R}^2_+$

$$P_1(\omega)\theta_{i,1} + P_2(\omega)\theta_{i,2} = 0.$$

By choosing the single consumption good as numeraire, we obtain the unique equilibrium price function $P_m(\omega) = X_m(\omega)$ for all m and ω. At equilibrium prices agent i's utility in state $\omega \in \Omega$ becomes

$$u_i(\theta_i, \omega) = X_1(\omega) e_{i,1} + e_{i,2}$$

so that he is indifferent between all possible net-trades. The following equilibrium net-trade functions $\Theta_A : \Omega \to \Gamma_A$ of the insider A are thus supported by this equilibrium price function: for any $x, y \in [-1, 1]$

	$\Theta_{A,1}$	$\Theta_{A,2}$
ω_H	x	$-2x$
ω_L	y	$-\frac{1}{2}y$

(3)

whereby agent B's equilibrium net-trades satisfy $\Theta_B = -\Theta_A$.$\square$

The above equilibrium price function is one-one because of

$$P_1(\omega_H) = 2 \text{ and } P_1(\omega_L) = \frac{1}{2}, \tag{4}$$

implying

$$[P]^{-1}(\omega) = \{\omega\} \text{ for all } \omega.$$

Consequently, the Θ_A in (3) constitute for all values $x, y \in [-1, 1]$ REE net-trades. Suppose that we are in state ω_H and focus on the REE where the insider offers to sell all units of the risky asset irrespective of the state of the world, i.e.,

$$\Theta_{A,1}(\omega_H) = \Theta_{A,1}(\omega_L) = -1 \tag{5}$$

In this situation, the uninformed agent B would observe exactly two things on the market: (i) agent A wants to sell all units of the risky asset and (ii) the price is high, i.e., 2. Through the REE assumption of *information revealing prices* the REE price function (4) guarantees that the high price of 2 can only be observed in the good state ω_H whereas the low price of $\frac{1}{2}$ can only be observed in the bad state ω_L. In other words, the one-one REE price function allows the uninformed investor to correctly predict the asset's true value. As a consequence, there cannot be any market exchange in an REE through which the uninformed investor B might be ripped off by the insider A. But since the REE concept does not further explain where this magic power of foretelling is coming from, the Kreps quote (footnote 2) about "a highly idealized, and highly suspect, way of modeling the "prices transmit information" phenomenon" comes to mind.

To better understand why the REE net-trades (5) are problematic, let us abandon the REE assumption of *information revealing prices* and sketch instead an internal reasoning process through which agent B might (or might not) infer A's private information. The price-taking agent A's net-trade offer $\Theta_A(\omega)$ in state ω is, by assumption, utility maximizing at the observed prices $P(\omega)$ conditional on A's private information $I_A^{PI}(\omega)$ in state ω; that is,

$$\Theta_A(\omega) \in \varphi_A\left(P(\omega), I_A^{PI}(\omega)\right)$$

where $\varphi_A(P(\omega), I_A(\omega))$ denotes agent A's net-trade correspondence evaluated at state ω and $I_A^{PI}(\omega) = \{\omega\}$. Fix for the risk-free asset $P_2(\omega) = 1$ and rewrite agent A's utility maximization problem over net-trades in the risky asset 1 as follows

$$\varphi_{A,1}\left(P_1(\omega), I_A^{PI}(\omega)\right) = \arg \max_{\theta_{A,1} \in [-1,1]} (X_1(\omega) - P_1(\omega))\,\theta_{A,1} + X_1(\omega)\,e_{A,1} + e_{A,2}$$

for all ω. At the high price $P_1(\omega) = 2$, agent A's utility maximizing net-trades become

$$\begin{aligned} \varphi_{A,1}(2, \{\omega_H\}) &= [-1, 1] \\ \varphi_{A,1}(2, \{\omega_L\}) &= \{-1\}. \end{aligned} \tag{6}$$

If B understands A's maximization problem (6), he knows that the price-taking utility maximizer A would choose any net-trade $\Theta_{A,1}(\omega) \in (-1, 1]$ only in the good state ω_H. Agent B can then infer from a net-trade offer $\Theta_{A,1}(\omega) \in (-1, 1]$ at price $P_1(\omega) = 2$ that the true state of the world must be ω_H. However, if agent B observes the net-trade offer $\Theta_{A,1}(\omega) = -1$ at price $P_1(\omega) = 2$, he cannot infer the true state of the world because $\Theta_{A,1}(\omega) = -1$ is utility maximizing in both states. Similarly, at the low price $P_1(\omega) = \frac{1}{2}$ we have that

$$\begin{aligned}\varphi_{A,1}\left(\frac{1}{2}, \{\omega_H\}\right) &= \{1\} \\ \varphi_{A,1}\left(\frac{1}{2}, \{\omega_L\}\right) &= [-1, 1]\end{aligned}$$

so that agent B infers at the low price the true state of the world from any net-trade offer $\Theta_{A,1}(\omega) \in [-1, 1)$ but not from $\Theta_{A,1}(\omega) = 1$. To sum up: Under the assumptions that (i) A is a utility-maximizing price-taker and that (ii) B understands A's utility-maximization problem, A's insider information becomes revealed to B except for A's REE net-trade offers where

$$\Theta_{A,1}(\omega_H) = -1 \text{ or } \Theta_{A,1}(\omega_L) = 1. \tag{7}$$

The above idea that a highly rational agent may infer private information of other price-taking agents from their utility maximizing behavior will be at the heart of our novel RIE concept. In Section 6 we come back to this example and establish that only the REE (7) fail to be also RIE. Our Theorem 3 will also show that it is not particular to this example that REE with non-interior trades may fail to be RIE.

3 Asset exchange economies

We consider a static asset exchange economy with $n \geq 2$ expected utility maximizing agents who trade $M \geq 2$ assets on an ex ante market at given prices. Agent $i \in \{1, ..., n\}$ has initial endowment $e_{i,m} \geq 0$ of asset $m \in \{1, ..., M\}$. The ex post payoffs of all assets are units of a single consumption good. There are $K \geq 2$ different payoff-relevant states of the world $\omega \in \Omega$. Denote by Σ the powerset of Ω, which we use as our default sigma-algebra. The ex post payoff of asset $m \in \{1, ..., M\}$ is given by the Σ-measurable function $X_m : \Omega \to [0, \infty)$. In addition to the payoffs of his asset-portfolio each agent i may receive a random income given by the Σ-measurable function $Y_i : \Omega \to [0, \infty)$. For each agent i define the probability space (Ω, Σ, π_i) such that agent i's subjective beliefs are described by the additive probability measure π_i with full support on Ω.

Central to our approach are *information mappings* $I_i : \Omega \to \Sigma$ satisfying, for all $\omega \in \Omega$, $\omega \in I_i(\omega)$. We call $I_i(\omega)$ the *information-cell* of agent i in state ω with the interpretation that if ω is the true state of the world, i regards all states $\omega' \in I_i(\omega)$ as possible whereas he regards all states $\omega' \notin I_i(\omega)$ as impossible. Note that, by assumption, the agent always regards the true state as possible. Every information mapping I_i generates the collection of sets

$$\{I_i(\omega) \mid \omega \in \Omega\} \tag{8}$$

which is a covering of Ω but not necessarily a partition. If (8) constitutes a partition, we also denote this collection by Π and write $\Sigma(\Pi)$ for the sigma-algebra generated by Π.

Let $u_i : [0,\infty) \to \mathbb{R}$ denote a strictly increasing utility function defined over ex post consumption. Conditional on the information $I_i(\omega)$ agent i's expected utility (EU) from net-trade $\theta_i \in \mathbb{R}^M$ is given as

$$\begin{aligned} & E\left[u_i(\cdot), \pi_i(\cdot \mid I_i(\omega))\right] \\ = & \sum_{\omega' \in I_i(\omega)} u_i\left(\sum_{m=1}^{M} X_m(\omega')(\theta_{i,m} + e_{i,m}) + Y_i(\omega')\right) \pi_i(\omega' \mid I_i(\omega)). \end{aligned} \tag{9}$$

Next define the function $P : \Omega \to \mathbb{R}^M_+$ with the interpretation that $P(\omega)$ is the asset price vector that agent i observes when he enters the market in state ω. For a fixed price function P the set of *budget-feasible net-trades* of agent i in state $\omega \in \Omega$ is

$$B_i(\omega) = \{\theta_i \in \Gamma_i \mid P(\omega)\,\theta_i = 0\} \tag{10}$$

where the non-empty set $\Gamma_i \subseteq \mathbb{R}^M$ contains all units of assets that can physically be traded by agent i. Agent i's net-trade correspondence at prices $P(\omega)$ and information cell $I_i(\omega)$ is then defined as

$$\varphi_i(P(\omega), I_i(\omega)) \equiv \arg\max_{\theta_i \in B_i(\omega)} \sum_{\omega' \in I_i(\omega)} u_i\left(\sum_{m=1}^{M} X_m(\omega')(\theta_{i,m} + e_{i,m}) + Y_i(\omega')\right) \pi_i(\omega' \mid I_i(\omega)). \tag{11}$$

The above notion of asset exchange economies with a single consumption good covers several classes of economies discussed in the literature. For example, consider a *real-asset* exchange economy such that, for all i and all m, $e_{i,m} \geq 0$ and $\Gamma_{i,m} = \left[-e_{i,m}, \sum_{j \neq i} e_{j,m}\right]$. Regardless of the price vector, any agent i cannot sell more units of the real asset m than he initially owns; nor can he buy more units of the asset than owned by all other agents. Under symmetric information, a real-asset exchange economy would reduce to a static version of Lucas's (1978) 'fruit-tree' economy in which different assets correspond to different apple orchards whose apple (=the numeraire) crop is uncertain from an ex ante perspective. Real-asset exchange economies (with $Y_i = 0$ and without an upper bound on Γ_i) are, e.g., considered in Radner (1979).

In contrast to real-assets, which stand for (random) single-good production units, *financial assets* stand for state-contingent claims (='payoff promises') to the single consumption good.[5] Making such 'promises' does not require any physical endowments of the good nor is there necessarily any limit to the physical amount that might be promised, e.g., we can have that, for all i and all m, $e_{i,m} = 0$ and $\Gamma_{i,m} = \mathbb{R}$. Financial asset economies with a single consumption good are the workhorse models of the asset pricing literature because they allow to investigate the relationship between consumption-based general equilibrium and arbitrage-free asset pricing models (cf., e.g., Chapter 1 in Duffie 2001 or Dybvig and Ross 2003). In particular, the theoretically important benchmark model of *complete* asset markets is characterized through $\Gamma_i = \mathbb{R}^m$ for all i combined with a payoff matrix of rank K so that the possible portfolio payoffs span the whole space $\mathbb{R}^K$.

Finally, recall that *pure speculation* economies are characterized through agents who share a common prior and who only care about gains from trade (Tirole 1982). Pure speculation economies are special cases of asset exchange economies under the following specifications: (i) all agents share a common prior $\pi = \pi_i$, (ii) there exists a risk-free numeraire good M such that $P_M(\omega) = X_M(\omega) = 1$ for all ω whereby $\Gamma_{i,M} = \mathbb{R}$ is unbounded, and (iii) there is neither any initial endowment, i.e., $e_i = 0$, nor any random income, i.e., $Y_i = 0$. Substituting the budget-condition (10) in the expected utility function (9) results in the following net-trade correspondence on $\Gamma_i \subseteq \mathbb{R}^{M-1}$ for a pure speculation economy:

$$\varphi_i(P(\omega), I_i(\omega)) \equiv \arg\max_{\theta_i \in \Gamma_i} \sum_{\omega' \in I_i(\omega)} u_i\left(\sum_{m=1}^{M-1}(X_m(\omega') - P_m(\omega))\,\theta_{i,m}\right)\pi(\omega' \mid I_i(\omega)). \tag{12}$$

4 Market information equilibria

4.1 Formal definition

Prior to meeting on the market the information of any agent $i \in \{1, ..., n\}$ is described by his *private information mapping* $I_i^{PI} : \Omega \to \Sigma$ such that

$$\left\{I_i^{PI}(\omega) \mid \omega \in \Omega\right\}$$

constitutes a partition of Ω. The standard epistemic interpretation is that agent i *knows* in state ω every event $A \subseteq \Sigma$ such that $I_i^{PI}(\omega) \subseteq A$. Private information mappings

[5] By defining in a financial asset exchange economy the expected utility (9) over 'payoff promises' rather than physical consumption, we implicitly assume that all agents are convinced that any such promises will be fulfilled regardless of who promises how much.

are exogenous to the model. We speak of an economy under *asymmetric information* whenever there are agents $i, j \in \{1, ..., n\}$ such that $I_i^{PI}(\omega) \neq I_j^{PI}(\omega)$ for some ω.

For a given I^{PI} define the *full communication information mapping* $I^{FC} : \Omega \to \Sigma$ such that, for all $\omega \in \Omega$,

$$I^{FC}(\omega) = \bigcap_{i=1}^{n} I_i^{PI}(\omega). \tag{13}$$

Note that

$$\left\{ I^{FC}(\omega) \mid \omega \in \Omega \right\}$$

constitutes a partition Π^{FC} given as the *join* (=coarsest common refinement) of all private information partitions.[6] The full communication information partition Π^{FC} would arise if all agents truthfully shared their private information. As a compact notation for fixed individual information mappings and priors we write $\langle I, \pi \rangle$ where $I = (I_1, ..., I_n)$ and $\pi = (\pi_1, ..., \pi_n)$.

Definition. Market Information Equilibria. *Fix some private information-belief structure* $\left\langle I^{PI}, \pi \right\rangle$ *and consider a mapping*

$$(P, \Theta) \equiv (P_1, ..., P_M; \Theta_1, ..., \Theta_n) : \Omega \to \mathbb{R}_+^M \times \mathbb{R}^{nM}.$$

A market information equilibrium is given as $(P, \Theta)\langle I, \pi \rangle$ *such that* P, Θ, *and the equilibrium information* $I = (I_1, ..., I_n)$ *satisfy the following conditions:*

1. *for all* $i \in \{1, ..., n\}$, *the equilibrium information mapping* I_i *constitutes a partition*

$$\Pi_i = \{I_i(\omega) \mid \omega \in \Omega\};$$

2. *for all* $i \in \{1, ..., n\}$ *and all* $\omega \in \Omega$,

$$I^{FC}(\omega) \subseteq I_i(\omega) \subseteq I_i^{PI}(\omega).$$

3. *for all* $i \in \{1, ..., n\}$, P *and* Θ_i *are* $\Sigma(\Pi_i)$*-measurable;*

4. *for all* $i \in \{1, ..., n\}$ *and all* $\omega \in \Omega$,

$$\Theta_i(\omega) \in \varphi_i(P(\omega), I_i(\omega));$$

5. *for all* $\omega \in \Omega$,

$$\sum_{i=1}^{n} \Theta_i(\omega) = 0.$$

[6] Cf., Aumann (1976).

The Partition Condition 1 is a rationality requirement by which each agent's equilibrium information conforms to the standard set-theoretic knowledge axioms (see, e.g., Battigalli and Bonanno 1998). In other words, the rational agent is required to know his equilibrium information. The Boundary Condition 2 imposes a lower and an upper bound for any possible equilibrium information cells. By this condition, an agent can neither forget his private information nor can he learn more about the assets' true values than might be revealed by the full communication information. The Measurability Condition 3 ensures that the commonly observable equilibrium prices and any agent's net-trade decision cannot possibly reveal more information than the agent actually knows in the equilibrium. Conditions 4 and 5 are the familiar general equilibrium utility-maximization and market-clearing conditions, respectively, which must hold in every state of the world.

4.2 Full communication equilibria

A relevant special case of market information equilibria are equilibria in which the equilibrium information is given as the full communication information.

Definition. Full Communication Equilibria (FCE). *An FCE is a market information equilibrium* $(P, \Theta) \langle I, \pi \rangle$ *such that, for all* $i \in \{1, ..., n\}$, $I_i = I^{FC}$.

FCE are analytically important because they are equilibria under uncertainty whose properties have been extensively studied in the financial markets literature. For example, for complete financial-asset markets FCE (i) exist under standard conditions (guaranteeing, e.g., that the expected utility functions are continuous and quasiconcave in net-trades), (ii) are Pareto optimal, and (iii) equilibrium asset prices are characterized through equilibrium prices for Arrow-Debreu securities.

Since our notion of asset exchange economies falls under the class of single-consumption (=numeraire) good economies, FCE also exist for incomplete asset markets under standard conditions (cf. Geanakoplos and Polemarchakis 1986; for surveys on the related literature see Polemarchakis 1990; Geanakoplos 1990).

For pure speculation economies (12) Tirole (1982, Proposition 1) shows that there always exist zero-trade FCE when the utility functions are concave; (for strictly concave utility functions any FCE must be a zero-trade FCE). Moreover, it can be shown that the equilibrium prices of these FCE are given as the assets' I^{FC}-conditional expected values.

4.3 Equilibria with rational expectations

The standard approach towards determining the equilibrium information in market information equilibria is Radner's (1979) concept of *rational expectations equilibria* (REE). According to the REE concept, every agent's equilibrium information results from his private information augmented with the information revealed through the equilibrium price function.[7] Formally, define the *price signal* in state ω as the inverse of the price function P evaluated in state ω:

$$[P]^{-1}(\omega) \equiv \{\omega' \in \Omega \mid P(\omega') = P(\omega)\}.$$

Definition. Rational Expectations Equilibria (REE). *An REE is a market information equilibrium*

$$(P, \Theta) \langle I^* [P], \pi \rangle$$

such that, for all i and all ω,

$$I_i^* [P](\omega) \equiv I_i^{PI}(\omega) \cap [P]^{-1}(\omega).$$

An alternative rational expectations concept is the concept of *generalized rational expectations equilibria* (GREE) (Allen and Jordan 1998). In an GREE agents augment their private information with the information revealed through the *market signal* which consists of the price signal plus the information that can be learnt from the other agents' net-trades. The information that agent i learns from observing agent j's net-trade offer in state ω is defined as the inverse of j's net-trade function:

$$[\Theta_j]^{-1}(\omega) \equiv \{\omega' \in \Omega \mid \Theta_j(\omega') = \Theta_j(\omega)\}.$$

The information that i learns from observing all other agents' net-trade offers in state ω is thus given as

$$[\Theta_{-i}]^{-1}(\omega) \equiv \bigcap_{j \neq i} [\Theta_j]^{-1}(\omega).$$

[7] Note that Radner's definition of *rational expectations* in terms of a correctly understood equilibrium price function under asymmetric information is very different from Muth's (1961) *rational expectations hypothesis* according to which producers should base their supply decisions on an unbiased estimator for the mean of the economy's objective distribution of future prices.

Definition. Generalized Rational Expectations Equilibria (GREE). *An GREE is a market information equilibrium*

$$(P, \Theta) \langle I^* [P, \Theta], \pi \rangle$$

such that, for all i and all ω,

$$I_i^* [P, \Theta_{-i}] (\omega) \equiv I_i^{PI} (\omega) \cap [P]^{-1} (\omega) \cap [\Theta_{-i}]^{-1} (\omega) . \tag{14}$$

An equilibrium $(P, \Theta) \langle I, \pi \rangle$ has an one-one price function if, and only if, for all ω,

$$[P]^{-1} (\omega) = I^{FC} (\omega) .$$

Similarly, an equilibrium $(P, \Theta) \langle I, \pi \rangle$ has an one-one market-signal function if, and only if, for all i and all ω,

$$[P]^{-1} (\omega) \cap [\Theta_{-i}]^{-1} (\omega) = I^{FC} (\omega) .$$

Clearly, if the price function is one-one so is the market-signal function whereas the converse statement is not necessarily true. Except for the last fact the following relationships are obvious and follow readily from the definitions.

Facts.

(i) *Any REE with one-one price function is also an FCE. Any FCE with one-one price function is also an REE.*

(ii) *Any GREE with one-one market-signal function is also an FCE. Any FCE with one-one market-signal function is also an GREE.*

(iii) *Any REE with one-one price function is also an GREE. Any GREE with one-one price function is also an REE.*

(iv) *An GREE with one-one market signal function is not necessarily an REE.*

For real-asset exchange economies with single-valued net-trade correspondences, Radner (1979) argues that REE with one-one price functions exist generically whereby the formal proof establishes generic existence of FCE with one-one price functions.[8] Kreps

[8] The details of this genericity argument are somewhat complicated, (in particular, see Radner's (1979) discussion of his Assumptions A3(a) and A3(b) on page 676.)

(1977) constructs a parametrized family of economies such that an REE fails to exist whenever the parameter value implies a price function that is not one-one. It can be shown that for these (non-generic) parameter values an GREE with one-one market-signal function exists despite the non-existence of REE (Kreps's example is further discussed below). For a (quite complicated) discussion on how existence of GREE may hold in situations where existence of REE fails, see Allen and Jordan (1998) and references therein.

5 Rationalizable information equilibria

The RIE concept is based on an internal reasoning process of every agent, which we refer to as *rationalization process*. To focus thoughts, we first describe an economy with two agents $i \in \{1,2\}$ only. Fix an arbitrary price function P and arbitrary net-trade functions $\Theta = (\Theta_1, \Theta_2)$ with the interpretation that both agents consider the net-trade functions Θ_1 and Θ_2 as possible candidates for utility maximizing net-trade functions at the price function P.

Under the hypothesis that the net-trade function Θ_j is utility maximizing at the price function P with respect to j's information, agent $i \neq j$ might be able to infer something about j's private information. More specifically, i knows in every given state ω all possible information cells of j at which $\Theta_j(\omega)$ is utility maximizing at $P(\omega)$. If there is no such information cell, then the function Θ_j can be rejected as not utility maximizing at P.

Suppose now that both agents are highly rational in the sense that they both understand what the other agent has just inferred. Then agent $i \in \{1,2\}$ might be able to infer something new about j's information under the maintained hypothesis that Θ_j is utility maximizing at the price function P with respect to j's information that must now include whatever j had just inferred from i's supposed utility maximizing choice Θ_i at P.

Again suppose that both agents understand what the other agent has just inferred at the previous rationalization stage and proceed with this rationalization process until either one of the following two events occurs:

1. Either the hypothesis of utility maximizing net-trade functions is rejected for at least one agent. Then the rationalization process is prematurely terminated.

2. Or both agents stop inferring anything new from this internal reasoning process to the effect that this process has converged and leaves each agent with his *rationalizable information at* P, Θ.

Let us now construct a formal model for this iterative reasoning process of both agents at arbitrary P, Θ. For all $\omega \in \Omega$, we initialize the rationalizable information at "rationalization stage" $k = 0$ as the agents' private information, i.e.,

$$I_i^{R,0}(\omega) \equiv I_i^{PI}(\omega).$$

At stage $k = 1$, we assume that agent i understands that agent j's net-trade $\Theta_j(\omega)$ maximizes j's expected utility at $P(\omega)$ with respect to some information cell I_j that agent j may hold. First, let us characterize the "possibility" set, denoted $\mathcal{P}_{i,j}^k(\omega)$, which contains all information cells I_j that j may possibly hold from the perspective of agent i given his rationalizable information $I_i^{R,k-1}(\omega)$ at stage $k-1$.

On the one hand, agent i knows from his information $I_i^{R,k-1}(\omega)$ that agent j's private information could be any information cell $I_j^{R,k-1}(\omega')$ such that $\omega' \in I_i^{R,k-1}(\omega)$ are the states that agent i perceives as possible with respect to his rationalizable information in state ω at stage $k-1$. We take $I_j^{R,k-1}(\omega')$ as j's least informative information possible in any state. On the other hand, agent i also knows that agent j's maximally possible information in any state $\omega' \in I_i^{R,k-1}(\omega)$ is given as $I^{FC}(\omega')$.

Having identified for each state an upper and a lower bound for j's possible information from the perspective of i's information cell $I_i^{R,k-1}(\omega)$, we define the possibility set as all information cells between these two bounds as follows

$$\mathcal{P}_{i,j}^k\left(I_i^{R,k-1}(\omega)\right) \equiv \bigcup_{\omega' \in I_i^{R,k-1}(\omega)} \left\{I_j \in \Sigma \mid I^{FC}(\omega') \subseteq I_j \subseteq I_j^{R,k-1}(\omega')\right\}. \tag{15}$$

Given the possibility set (15), agent i can now rationalize $\Theta_j(\omega)$ as j's expected utility maximizing choice at $P(\omega)$ with respect to some information cell $I_j \in \mathcal{P}_{i,j}^k(\omega)$. Formally, agent i infers from the observation of $P(\omega), \Theta_j(\omega)$ the following information at stage $k = 1$

$$\begin{aligned} I_{i,j}^k(\omega) &\equiv \bigcup_{I_j \in \mathcal{P}_{i,j}^k\left(I_i^{R,k-1}(\omega)\right)} \left\{\omega' \in I_j \mid \Theta_j(\omega) \in \varphi_j(P(\omega), I_j)\right\} \\ &= \bigcup_{\left\{I_j \in \mathcal{P}_{i,j}^k\left(I_i^{R,k-1}(\omega)\right) \mid \Theta_j(\omega) \in \varphi_j(P(\omega), I_j)\right\}} I_j. \end{aligned}$$

In words: $I_i^k(\omega)$ is the union of all information cells I_j of agent j such that (i) agent i deems I_j possible with respect to his information $I_i^{R,k-1}(\omega)$ and (ii) $\Theta_j(\omega)$ maximizes at I_j j's utility given the price vector $P(\omega)$.

Finally, we augment this information $I_{i,j}^k(\omega)$ with i's information at stage $k-1$ to obtain the information cell

$$I_i^{R,1}(\omega) \equiv I_{i,j}^1(\omega) \cap I_i^{R,0}(\omega) \tag{16}$$

which we interpret as the rationalizable information of agent i in state ω at stage $k=1$ and which is (weakly) more informative than agent i's rationalizable information in state ω at stage $k-1$. An analogous argument results in agent j's rationalizable information in state ω at stage $k=1$

$$I_j^{R,1}(\omega) \equiv I_{j,i}^1(\omega) \cap I_j^{R,0}(\omega). \tag{17}$$

At $k=2$, we assume that, for all $\omega \in \Omega$, (16) and (17) is common knowledge between both agents. Consequently, we can apply the same reasoning process as under $k=1$ whereby we have to substitute $I_i^{R,1}$ and $I_j^{R,1}$ for $I_i^{R,0}$ and $I_j^{R,0}$, respectively. For example, the set of j's information cells which i deems possible with respect to his newly rationalized information (16) becomes, for $\omega \in \Omega$,

$$\mathcal{P}_{i,j}^2\left(I_i^{R,1}(\omega)\right) \equiv \bigcup_{\omega' \in I_i^{R,1}(\omega)} \left\{ I_j \in \Sigma \mid I^{FC}(\omega') \subseteq I_j \subseteq I_j^{R,1}(\omega') \right\}.$$

The following definition summarizes the above argument and generalizes it to an arbitrary number of agents and rationalization stages.

Definition. Rationalizable information given P, Θ

1. *Fix some private information-belief structure* $\langle I^{PI}, \pi \rangle$ *and some* $\Sigma\left(\Pi^{FC}\right)$*-measurable* $(P, \Theta): \Omega \to \mathbb{R}_+^M \times \mathbb{R}^{nM}$.
2. *Initialize, for all* i *and all* $\omega \in \Omega$,

$$I_i^{R,0}(\omega) \equiv I_i^{PI}(\omega).$$

For all $k \geq 1$ *and all* $\omega \in \Omega$, *define recursively the rationalizable information of agent* i *at stage* k *as*

$$I_i^{R,k}(\omega) \equiv \bigcap_{j \neq i} I_{i,j}^k(\omega) \cap I_i^{R,k-1}(\omega)$$

such that

$$I_{i,j}^k(\omega) \equiv \bigcup_{\left\{ I_j \in \mathcal{P}_{i,j}^k\left(I_i^{R,k-1}(\omega)\right) \mid \Theta_j(\omega) \in \varphi_j(P(\omega), I_j) \right\}} I_j$$

where

$$\mathcal{P}_{i,j}^k\left(I_i^{R,k-1}(\omega)\right) \equiv \bigcup_{\omega' \in I_i^{R,k-1}(\omega)} \left\{ I_j \in \Sigma \mid I^{FC}(\omega') \subseteq I_j \subseteq I_j^{R,k-1}(\omega') \right\}.$$

3. *If there is some $I^k_{i,j}(\omega) = \emptyset$, we prematurely terminate the rationalization process and conclude that Θ_j cannot be utility maximizing at P. If there are no $I^k_{i,j}(\omega) = \emptyset$, we define agent i's rationalizable information at P, Θ as the information mapping $I^R_i[P,\Theta] : \Omega \to \Sigma$ such that*

$$I^R_i[P,\Theta](\omega) \equiv \bigcap_{k=0}^{\infty} I^{R,k}_i(\omega). \tag{18}$$

If there is no premature[9] termination, the $\left\{I^{R,k}_i(\omega)\right\}_{k\geq 0}$ constitute a nested sequence

$$I^{R,0}_i(\omega) \supseteq I^{R,1}_i(\omega) \supseteq ....$$

By finiteness of Ω and n, the rationalization process will thus converge after finitely many steps for any functions P, Θ whenever there is no premature termination. Our preferred interpretation is that the highly rational agents of our model "mentally test" the functions P, Θ whereby they go through the above reasoning process to determine the rationalizable information at P, Θ in every state of the world $\omega \in \Omega$. A *rationalizable information equilibrium* (RIE) is then characterized through the consistency condition that the equilibrium price function clears markets in every state whereby the agents' net-trade functions are utility maximizing with respect to the rationalizable information at the equilibrium price and net-trade functions.

Definition. Rationalizable information equilibria (RIE). *Fix some private information-belief structure $\langle I^{PI}, \pi \rangle$. An RIE is a market information equilibrium*

$$(P,\Theta)\left\langle I^R[P,\Theta], \pi\right\rangle \tag{19}$$

whenever $I^R_i[P,\Theta]$, given by (18), constitutes a partition for all i.

Note that the above definition emphasizes that the rationalizable information at P and Θ has to be a partition. If $I^R_i[P,\Theta]$ exists (i.e., $I^R_i[P,\Theta](\omega)$ is non-empty for all ω) but does not constitute a partition for every i, then $(P,\Theta)\left\langle I^R[P,\Theta], \pi\right\rangle$ fails to be a market information equilibrium and is therefore not an RIE. We illustrate in some detail the above rationalization process in Appendix B.

[9] Premature termination happens whenever some market variables P, Θ_j violate the assumption that a price-taking agent j maximizes his utility conditional on the information that he has learnt from the above rationalization process.

6 Formal analysis

This section investigates the formal relationships between RIE and other market information equilibria.

6.1 RIE versus GREE

We start out with the relationship between RIE and GREE.

Theorem 1. RIE versus GREE.

(i) *Any RIE is also an GREE.*

(ii) *An GREE is not necessarily an RIE.*

The proof of part (i), relegated to the Appendix, uses the following lemma (also proved in the Appendix).

Lemma 1. *In an RIE* $(P, \Theta)\left\langle I^R[P, \Theta], \pi\right\rangle$ *an agent cannot learn more information than his private information augmented with the market signal, i.e., for all* i *and all* $\omega \in \Omega$,

$$I_i^*[P, \Theta_{-i}](\omega) \subseteq I_i^R[P, \Theta](\omega) \tag{20}$$

with $I_i^*[P, \Theta_{-i}]$ *given by (14).*

By Lemma 1 there exists a lower bound for the rationalizable information $I^R[P, \Theta]$ in any RIE which is given by the GREE equilibrium information $I^*[P, \Theta]$. The formal proof of part (i) of Theorem 1 then simply uses the Measurability Condition of market information equilibria to show that

$$I_i^*[P, \Theta_{-i}](\omega) \subset I_i^R[P, \Theta](\omega)$$

is impossible. Consequently, (20) holds with equality whenever the RIE $(P, \Theta)\left\langle I^R[P, \Theta], \pi\right\rangle$ exists, which gives the desired result.

To prove part (ii), revisit Example 1 'Insider Information' from Section 2. Recall that all REE $(P, \Theta)\langle I^*[P], \pi\rangle$ are characterized as follows: for all $x, y \in [-1, 1]$,

	P_1	P_2	$\Theta_{A,1}$	$\Theta_{A,2}$
ω_H	2	1	x	$-2x$
ω_L	$\frac{1}{2}$	1	y	$-\frac{1}{2}y$

with $\Theta_B = -\Theta_A$. Since the REE price function is one-one, all these REE are also GREE. For $\Theta_{A,1}(\omega_H) = x \in (-1, 1]$, we have

$$\begin{aligned} I_B^R[P,\Theta](\omega_H) &= I_{B,A}^1(\omega_H) = \bigcup_{\left\{I_A \in \{\{\omega_H\},\{\omega_L\}\} \mid x \in \varphi_{A,1}(P_1(\omega_H)=2, I_A)\right\}} I_A \\ &= \{\omega_H\} \end{aligned}$$

while we have for $\Theta_{A,1}(\omega_L) = y \in [-1, 1)$ that

$$\begin{aligned} I_B^R[P,\Theta](\omega_L) &= I_{B,A}^1(\omega_L) = \bigcup_{\left\{I_A \in \{\{\omega_H\},\{\omega_L\}\} \mid y \in \varphi_{A,1}\left(P_1(\omega_L)=\frac{1}{2}, I_A\right)\right\}} I_A \\ &= \{\omega_L\}. \end{aligned}$$

For the values $\Theta_{A,1}(\omega_H) \in (-1, 1]$ and $\Theta_{A,1}(\omega_L) \in [-1, 1)$, agent B thus learns A's private information through the rationalization process so that the corresponding REE/GREE are also RIE.

However, for $\Theta_{A,1}(\omega_H) = x = -1$ we have that

$$\begin{aligned} I_B^R[P,\Theta](\omega_H) &= I_{B,A}^1(\omega_H) = \bigcup_{\left\{I_A \in \{\{\omega_H\},\{\omega_L\}\} \mid -1 \in \varphi_{A,1}(P_1(\omega_H)=2, I_A)\right\}} I_A \\ &= \{\omega_H, \omega_L\} \end{aligned}$$

while we have for $\Theta_{A,1}(\omega_L) = y = 1$

$$\begin{aligned} I_B^R[P,\Theta](\omega_L) &= I_{B,A}^1(\omega_L) = \bigcup_{\left\{I_A \in \{\{\omega_H\},\{\omega_L\}\} \mid 1 \in \varphi_{A,1}\left(P_1(\omega_L)=\frac{1}{2}, I_A\right)\right\}} I_A \\ &= \{\omega_H, \omega_L\}. \end{aligned}$$

Consequently, the following three different cases of B's rationalizable information can arise for which A's private information is not revealed through the rationalization process at REE/GREE prices and net-trades:

	$I_B^R[P,\Theta]$ for $x = -1$, $y \in [-1, 1)$	$x \in (-1, 1]$, $y = 1$	$x = -1$, $y = 1$
ω_H	$\{\omega_H, \omega_L\}$	$\{\omega_H\}$	$\{\omega_H, \omega_L\}$
ω_L	$\{\omega_L\}$	$\{\omega_H, \omega_L\}$	$\{\omega_H, \omega_L\}$

In all three cases the rationalizable information $I_B^R[P,\Theta]$ violates the conditions of market information equilibria. The REE/GREE corresponding to the first two cases violate the Partition Condition whereas the REE/GREE corresponding to the last case violates the Measurability Condition for the one-one equilibrium price function. Consequently, these REE/GREE are not RIE. □□

6.2 RIE versus REE

Turn now to the formal relationship between RIE and REE.

Theorem 2. RIE versus REE.

(i) *Any RIE with one-one price function is also an REE.*

(ii) *An REE with one-one price function is not necessarily an RIE.*

(iii) *An RIE without an one-one price function is not necessarily an REE.*

Part (i) follows from Theorem 1 since an RIE with one-one price function is also an GREE with one-one price function which is also an REE (see the facts from Section 3). We did already prove part (ii) when we revisited Example 1 to show that not all REE/GREE with one-one price function are RIE. To prove part (iii), let us recall Kreps's (1977) example about the possible non-existence of REE.

Example 2. Kreps (1977) derives the net-trade functions

$$\begin{aligned} \Theta_A(\omega) &= E[X, \pi(\cdot \mid I(\omega))] - P(\omega) \\ \Theta_B(\omega) &= E[X, \pi(\cdot \mid I(\omega))] - P(\omega) - Y_B(\omega) \end{aligned}$$

under the assumption that both agents, A and B, have information $I(\omega) \in \{\Omega_1, \Omega_2\}$. Suppose now that $\Pi_A^{PI} = \{\Omega_1, \Omega_2\}$ and $\Pi_B^{PI} = \{\Omega\}$; that is, whereas agent A has private information about Ω_1 versus Ω_2, agent B cannot distinguish between both events. For the parametrization

$$\begin{aligned} E[X, \pi(\cdot \mid I(\omega))] &= \begin{cases} 4 & \text{if } \omega \in \Omega_1 \\ 5 & \text{if } \omega \in \Omega_2 \end{cases} \\ Y_B(\omega) &= \begin{cases} 2 & \text{if } \omega \in \Omega_1 \\ 4 & \text{if } \omega \in \Omega_2 \end{cases} \end{aligned}$$

there exists an GREE $(P, \Theta)\langle I^*[P, \Theta], \pi\rangle$ with one-one market signal function such that

	$P(\omega)$	$\Theta_A(\omega)$
$\omega \in \Omega_1$	3	1
$\omega \in \Omega_2$	3	2

and $\Theta_B = -\Theta_A(\omega)$ since the different equilibrium net-trade offers by A reveal Ω_1 versus Ω_2 to agent B. However, this GREE does not come with an one-one price function so that agent B cannot learn from $P(\omega)$ whether event Ω_1 or Ω_2 has occurred. Since B's net-trade must thus, by the Measurability Condition, be constant across Ω_1 and Ω_2 in any REE, markets cannot clear in both events. Consequently, there does not exist any REE for this example.$\square$

To establish that an RIE without an one-one price function is not necessarily an REE, it remains to be shown that the GREE in Kreps's example is also an RIE. Observe that

$$\begin{aligned}\mathcal{P}^1_{B,A}\left(I^{R,0}_B(\omega)=\Omega\right) &= \bigcup_{\omega'\in\Omega}\left\{I_A\in\Sigma \mid I^{FC}(\omega')\subseteq I_A\subseteq I^{R,0}_A(\omega')\right\}\\ &= \{\Omega_1,\Omega_2\}\end{aligned}$$

so that, for all ω with $P(\omega)=3$ and $\Theta_A(\omega)=1$,

$$\begin{aligned}I^1_{B,A}(\omega) &= \bigcup_{\{I_A\in\{\Omega_1,\Omega_2\}\mid 1\in\varphi_A(3,I_A)\}} I_j\\ &= \Omega_1\end{aligned}$$

whereas, for all ω with $P(\omega)=3$ and $\Theta_A(\omega)=2$,

$$\begin{aligned}I^1_{B,A}(\omega) &= \bigcup_{\{I_A\in\{\Omega_1,\Omega_2\}\mid 2\in\varphi_A(3,I_A)\}} I_j\\ &= \Omega_2.\end{aligned}$$

That is, B infers at rationalization stage $k=1$ the full communication information. Consequently, the rationalizable information, $I^R[P,\Theta]$, coincides with the GREE information, $I^*[P,\Theta]$, so that $(P,\Theta)\langle I^*[P,\Theta],\pi\rangle$ is an RIE.$\square\square$

6.3 RIE versus FCE

We address the question under which conditions FCE are (generically) also RIE. Recall that Radner (1979) presents conditions which ensure generic existence of FCE with one-one price functions. Since FCE with one-one price functions are REE, Radner's (1979) analysis implies a generic existence result for REE.

Our approach is different. We start out with an existing FCE–with or without one-one price or/and one-one market signal function–whereby such existence might or might not be implied by standard conditions. In a next step, we ask under which circumstances this FCE might fail to be an RIE. Finally, we identify properties of FCE which ensure that such circumstances can only occur in non-generic situations.

Technical assumptions.

A1. *For all i, Γ_i is a convex subset of $\mathbb{R}^M$.*

A2. *For all i, the utility functions u_i are continuously differentiable and concave.*

Assumptions A1 and A2 are standard in the literature as they allow us to describe optimal net-trades in terms of first-order conditions. We now list possible properties which may or may not be satisfied by any existing FCE.

Equilibrium properties. *Consider an FCE $(P, \Theta)\left\langle I^{FC}, \pi\right\rangle$ with the following properties.*

P1. Strictly positive prices. *For all ω and m, $P_m(\omega) > 0$.*

P2. Interior solution. *For all i, the equilibrium net-trade Θ_i lies in the interior of Γ_i.*

P3. Information sensitivity. *For all i,*

$$\Theta_i(\omega) \in \varphi_i\left(P(\omega), I^{FC}(\omega)\right) \text{ implies } \Theta_i(\omega) \notin \varphi_i\left(P(\omega), I^{FC}(\omega')\right) \tag{21}$$

for any $I^{FC}(\omega') \neq I^{FC}(\omega)$.

Properties P1 and P2 are straightforward. By standard arbitrage arguments, P1 would be implied by strictly positive I^{FC}-conditional expected payoffs, i.e., $E\left[X_m, \pi\left(\cdot \mid I^{FC}(\omega)\right)\right] > 0$ for all m and ω. P2 must, e.g., hold for any existing FCE whenever the Γ_i are open sets (e.g., $\Gamma_i = \mathbb{R}^M$).

The interesting property is P3 "Information Sensitivity", which is stronger than an one-one market signal function because it additionally considers "out-of-equilibrium" maximization behavior. By P3, a net-trade which is optimal at information cell $I^{FC}(\omega)$ for the equilibrium prices $P(\omega)$ at this information cell shall not be optimal at any different full communication information cell $I^{FC}(\omega')$ at these fixed prices $P(\omega)$. In other words, if the information changes in the economy from one full communication information cell to another while prices remain fixed, P3 requires the agents to react with different optimal net-trade decisions (in at least one asset) to this change in information at fixed prices.

It is not difficult to see (without making a full-blown formal argument) that property P3 is generic on the payoff-space whenever A1-A2 and P2 hold: in case (21) is violated, one can keep $X(\omega)$ (and therefore $P(\omega)$) for all $\omega \in I^{FC}(\omega)$ while perturbing $X(\omega'')$ for all $\omega'' \in I^{FC}(\omega') \neq I^{FC}(\omega)$ so that $\Theta_i(\omega) \in \varphi_i\left(P(\omega), I^{FC}(\omega')\right)$ will generically break down for these new payoffs at an interior solution.[10]

Theorem 3. *Suppose that the asset exchange economy satisfies Assumptions A1-A2. An FCE* $(P, \Theta)\left\langle I^{FC}, \pi\right\rangle$ *that satisfies properties P1-P3 is, on the space of beliefs, generically an RIE.*

The proof of Theorem 3 uses the following lemma, which applies to all asset exchange economies and all FCE (that is, the lemma does neither assume A1-A2 nor P1-P3.)

Lemma 2. *An FCE* $(P, \Theta)\left\langle I^{FC}, \pi\right\rangle$ *can only fail to be an RIE if there is some* $\hat{I} \neq I^{FC}(\omega)$ *such that*

$$\Theta_i(\omega) \in \varphi_i\left(P(\omega), \hat{I}\right) \tag{22}$$

whereby $I^{FC}(\cdot) \subseteq \hat{I}$ *for some* $I^{FC}(\cdot) \in \Pi^{FC}$.

The proof of Lemma 2 (relegated to the Appendix) is based on the observation that an FCE can only fail to be an RIE whenever the rationalization process does not resolve the ambiguous situation in which there is some state ω such that the FCE net-trade $\Theta_i(\omega)$ of some agent i at FCE prices $P(\omega)$ is also optimal at some other information cell $\hat{I} \neq I^{FC}(\omega)$ at the fixed price $P(\omega)$.

Intuitively, one would think that such ambiguous situation could only occur non-generically on the space of beliefs whenever perturbations of these beliefs have sufficient impact on the agent's net-trade decision. The proof of Theorem 3 formalizes exactly this intuition. More specifically, let h denote the number of full communication information cells $I^{FC}(\cdot)$ in Π^{FC}. Our formal proof constructs a set of perturbed beliefs π^{ε} with $h-1$

[10] This can be most easily seen for (interior) zero-trade FCE of pure speculation economies where

$$P(\omega) = E\left[X, \pi\left(\cdot \mid I^{FC}(\omega)\right)\right] \text{ for all } \omega.$$

Any difference between the two I^{FC}-conditional expected values $E\left[X_m(\cdot), \pi\left(\cdot \mid I^{FC}(\omega')\right)\right]$ and $E\left[X_m(\cdot)\,\pi\left(\cdot \mid I^{FC}(\omega)\right)\right]$ for some asset m would result in

$$\mathbf{0} \notin \varphi_i\left(P(\omega), I^{FC}(\omega')\right) \text{ whenever } \mathbf{0} \in \varphi_i\left(P(\omega), I^{FC}(\omega)\right).$$

dimensional Lebesgue measure strictly greater zero such that $(P,\Theta)\left\langle I^{FC},\pi^{\varepsilon}\right\rangle$ remains an FCE for all perturbed beliefs π^{ε} in this set whenever $(P,\Theta)\left\langle I^{FC},\pi\right\rangle$ is an FCE. On the other hand, for almost all perturbed beliefs π^{ε} the relationship (22) breaks down. As a consequence, for almost all π^{ε} the FCE $(P,\Theta)\left\langle I^{FC},\pi^{\varepsilon}\right\rangle$ is also an RIE. The property P3 "Information Sensitivity" is central to our proof because it ensures that net-trades react sufficiently strongly to changes in beliefs across different information cells; (the formal role of P3 for the proof of Theorem 3 is further explained in the Remark following the proof in the Appendix).

If one ore more of the properties P1-P3 are violated, an FCE might generically fail to be an RIE even for asset exchange economies that satisfy assumptions A1-A2. This is, for instance, the case in Example 1 where the FCE/REE that fail to be RIE violate P2 "Interior Solution" as well as P3 "Information Sensitivity". To see this, consider some FCE/REE in Example 1 where the insider sells all units of the risky asset at the high price $P_1(\omega_H)=2$, i.e., $\Theta_{A,1}(\omega_H)=-1$. Obviously, P2 "Interior Solution" is violated since $\Gamma_{A,1}=[-1,1]$. Next note that selling all units of the risky asset at this high price is optimal for the insider A at both full communication information cells $I^{FC}(\omega_H)=\{\omega_H\}$ and $I^{FC}(\omega_L)=\{\omega_L\}$, respectively. That is,

$$-1\in\varphi_A\left(P_1(\omega_H)=2,\{\omega_H\}\right) \text{ as well as } -1\in\varphi_A\left(P_1(\omega_H)=2,\{\omega_L\}\right),$$

which is a violation of P3 "Information Sensitivity". Any perturbation $\varepsilon\in\left[-\frac{1}{2},\frac{3}{2}\right)$ of the risky asset's payoff in state ω_L such that $X_1^{\varepsilon}(\omega_L)=\frac{1}{2}+\varepsilon$ does not change this situation as the net-trade -1 remains optimal in state ω_L at $P_1(\omega_H)=2$. Consequently, P2 as well as P3 are generically violated. The genericity argument of Theorem 3, which is based on first-order conditions (and therefore always applicable to interior solutions), does thus not apply to non-interior net-trades that are not critical points as they are not sensitive enough to changes in beliefs and/or payoffs.

Appendix A: Mathematical proofs

Proof of Lemma 1

Suppose to the contrary that

$$I_i^R [P, \Theta] (\omega) \subset I_i^* [P, \Theta_{-i}] (\omega),$$

which is, by the rationalization process, only possible if

$$\Theta_i (\omega) \notin \varphi_i (P (\omega), I_i^* [P, \Theta_{-i}] (\omega)). \tag{23}$$

Because of

$$I_i^* [P, \Theta_{-i}] (\omega) \equiv I_i^{PI} (\omega) \cap [P]^{-1} (\omega) \cap [\Theta_{-i}]^{-1} (\omega)$$

the market signal consists in an RIE exclusively of states in which i chooses an utility maximizing net-trade such that markets clear at the RIE price function. That is, for all $\omega' \in I_i^* [P, \Theta_{-i}] (\omega)$,

$$-\sum_{j \neq i} \Theta_j (\omega') = \Theta_i (\omega') \in \varphi_i (P (\omega'), I_i^* [P, \Theta_{-i}] (\omega')). \tag{24}$$

But $\omega' \in I_i^* [P, \Theta_{-i}] (\omega)$ implies, by the Partition Condition, $I_i^* [P, \Theta_{-i}] (\omega') = I_i^* [P, \Theta_{-i}] (\omega)$ so that, by the Measurability Condition, $P (\omega') = P (\omega)$ as well as $-\sum_{j \neq i} \Theta_j (\omega') = -\sum_{j \neq i} \Theta_j (\omega)$. Substituting in (24) gives

$$-\sum_{j \neq i} \Theta_j (\omega) \in \varphi_i (P (\omega), I_i^* [P, \Theta_{-i}] (\omega)).$$

Consequently, by market clearing,

$$\Theta_i (\omega) \in \varphi_i (P (\omega), I_i^* [P, \Theta_{-i}] (\omega)),$$

a contradiction to (23).□□

Proof of Theorem 1

We formally prove part (i). To prove that any given RIE $(P, \Theta) \left\langle I^R [P, \Theta], \pi \right\rangle$ is also an GREE, we must show that $(P, \Theta) \langle I^* [P, \Theta], \pi \rangle$ with

$$I_i^* [P, \Theta_{-i}] \equiv I_i^{PI} \cap [P]^{-1} \cap [\Theta_{-i}]^{-1}$$

is a market information equilibrium. That is, we have to show that, for all ω,

$$\Theta_i (\omega) \in \varphi_i (P (\omega), I_i^* [P, \Theta_{-i}] (\omega)).$$

Suppose to the contrary that

$$\Theta_i(\omega) \notin \varphi_i\left(P(\omega), I_i^*[P, \Theta_{-i}](\omega)\right). \tag{25}$$

By definition of an RIE,

$$\Theta_i(\omega) \in \varphi_i\left(P(\omega), I_i^R[P, \Theta](\omega)\right)$$

so that (25) implies, by the lower bound result of Lemma 1,

$$I_i^*[P, \Theta](\omega) \subset I_i^R[P, \Theta](\omega). \tag{26}$$

But by the Measurability Condition, Θ and P must be constant on $I_i^R[P, \Theta](\omega)$ implying

$$\begin{aligned} I_i^R[P,\Theta](\omega) &\subseteq [P]^{-1}(\omega) \cap [\Theta_{-i}]^{-1}(\omega) \\ &\Leftrightarrow \\ I_i^R[P,\Theta](\omega) &\subseteq I_i^*[P,\Theta_{-i}](\omega) \end{aligned}$$

whereby the last step follows from $I_i^R(\omega) \subseteq I_i^{PI}(\omega)$. But this is a contradiction to (26).□□

Remark. To see why the proof of Theorem 1 does work for all GREE but not for REE without one-one price function (cf. Theorem 2), observe that an RIE $(P, \Theta)\left\langle I^R, \pi\right\rangle$ is an REE if, and only if, for all i and ω,

$$\Theta_i(\omega) \in \varphi_i\left(P(\omega), I_i^*[P](\omega)\right)$$

where

$$I_i^*[P] \equiv I_i^{PI} \cap [P]^{-1}.$$

If the REE price function is not one-one, we may encounter the case

$$I_i^*[P, \Theta](\omega) \subset I_i^*[P](\omega).$$

But then a situation where

$$\theta_i \equiv \Theta_i(\omega) \in \varphi_i\left(P(\omega), I_i^R[P,\Theta](\omega)\right) \text{ whereas } \theta_i \notin \varphi_i\left(P(\omega), I_i^*[P](\omega)\right)$$

for some $I_i^R[P, \Theta](\omega)$ such that

$$I_i^*[P,\Theta](\omega) \subseteq I_i^R[P,\Theta](\omega) \subset I_i^*[P](\omega)$$

does not violate the Measurability Condition according to which $\Theta_i(\omega)$ must only be identically θ_i for all $\omega \in I_i^R[P, \Theta]$.

Exactly such situation is exploited by the example in Kreps (1977).

Proof of Lemma 2

Step 1. An FCE can only fail to be an RIE if there is some agent whose rationalizable information does not coincide with the full communication information. That is, for any FCE $(P, \Theta)\left\langle I^{FC}, \pi\right\rangle$ that fails to be an RIE there must exist some i and some ω such that

$$I^{FC}(\omega) \subset I_i^R\left[P, \Theta_{-i}\right](\omega). \tag{27}$$

Step 2. By the rationalization process, condition (27) is equivalent to

$$I^{FC}(\omega) \subset I_i^{R,k}(\omega) \equiv \bigcap_{j \neq i} I_{i,j}^k(\omega) \cap I_i^{R,k-1}(\omega) \text{ for all } k \geq 0$$

where

$$I_{i,j}^k(\omega) \equiv \bigcup_{\left\{I_j \in \mathcal{P}_{i,j}^k\left(I_i^{R,k-1}(\omega)\right) \mid \Theta_j(\omega) \in \varphi_j\left(P(\omega), I_j\right)\right\}} I_j.$$

Condition (27) thus holds only if the following ambiguous situation occurs throughout the whole rationalization process. Agent i observes some choice $\Theta_j(\omega)$ of which he knows that it maximizes j's utility at the information cell $I^{FC}(\omega)$; (else, $\Theta_j(\omega)$ would not be an utility maximizing choice in an FCE). However, agent i cannot rule out the possibility that $\Theta_j(\omega)$ has actually been chosen as an utility maximizing choice at some different information cell I_j in the possibility set evaluated at state ω.

Consequently, an FCE $(P, \Theta)\left\langle I^{FC}, \pi\right\rangle$ can only fail to be an RIE if there is some $I_j \neq I^{FC}(\omega)$ with

$$I_j \in \mathcal{P}_{i,j}^k\left(I_i^{R,k-1}(\omega)\right) \text{ for all } k \geq 0,$$

such that

$$\Theta_j(\omega) \in \varphi_j\left(P(\omega), I_j\right).$$

Step 3. Next observe that

$$\mathcal{P}_{i,j}^k\left(I_i^{R,k-1}(\omega)\right) \subseteq \mathcal{P}_{i,j}^1\left(I_i^{R,0}(\omega)\right) \text{ for all } k \geq 0$$

whereby

$$\begin{aligned}
\mathcal{P}_{i,j}^1\left(I_i^{R,0}(\omega)\right) & \equiv \bigcup_{\omega' \in I_i^{PI}(\omega)}\left\{I_j \in \Sigma \mid I^{FC}\left(\omega'\right) \subseteq I_j \subseteq I_j^{PI}\left(\omega'\right)\right\} \\
& \subseteq \bigcup_{\omega' \in \Omega}\left\{I_j \in \Sigma \mid I^{FC}\left(\omega'\right) \subseteq I_j \subseteq I_j^{PI}\left(\omega'\right)\right\}.
\end{aligned} \tag{28}$$

Using the upper bound (28) for any possibility set $\mathcal{P}_{i,j}^k\left(I_i^{R,k-1}(\omega)\right)$, $i \neq j$, gives us the desired result.□□

Proof of Theorem 3

Step 1. Consider the FCE $(P,\Theta)\left\langle I^{FC},\pi\right\rangle$. As the equilibrium net-trade Θ_i is, by P2, an interior point of Γ_i, it must satisfy, by Assumptions A1-A2, the following first-order conditions for all $I^{FC}(\omega)$ and all m

$$\begin{aligned}\frac{\partial}{\partial\theta_{i,m}}E\left[u_i,\pi_i\left(\cdot\mid I^{FC}(\omega)\right)\right]\Big|_{\Theta_i(\omega)}-\mu P_m(\omega) &= 0\\ &\Leftrightarrow\\ \sum_{\omega'\in I^{FC}(\omega)}u_i'\left(\sum_{k=1}^{M}X_k(\omega')\left(\Theta_{i,k}(\omega)+e_{i,k}\right)+Y_i(\omega')\right)X_m(\omega')\pi_i\left(\omega'\mid I^{FC}(\omega)\right)-\mu P_m(\omega) &= 0\end{aligned}$$

for some Lagrange multiplier $\mu>0$. By P1, the following equation is well-defined for arbitrary assets $1,2$

$$\begin{aligned}\frac{\sum_{\omega'\in I^{FC}(\omega)}u_i'(\cdot)X_1(\omega')\pi_i\left(\omega'\mid I^{FC}(\omega)\right)}{\sum_{\omega'\in I^{FC}(\omega)}u_i'(\cdot)X_2(\omega')\pi_i\left(\omega'\mid I^{FC}(\omega)\right)} &= \frac{P_1(\omega)}{P_2(\omega)}\\ &\Leftrightarrow\\ \sum_{\omega'\in I^{FC}(\omega)}z\left(\Theta_i(\omega),\omega'\right)\pi_i\left(\omega'\mid I^{FC}(\omega)\right) &= 0.\end{aligned}\tag{29}$$

where

$$z\left(\Theta_i(\omega),\omega'\right)\equiv u_i'(\cdot)\left[X_1(\omega')-\frac{P_1(\omega)}{P_2(\omega)}X_2(\omega')\right].$$

Step 2. By Lemma 2, this FCE can only fail to be an RIE if there is some $\hat{I}\neq I^{FC}(\omega)$ such that for all m

$$\Theta_{i,m}(\omega)\in\varphi_i\left(P(\omega),\hat{I}\right),$$

which is, by A1-A2 and P2, equivalent to

$$\begin{aligned}\frac{\partial}{\partial\theta_{i,m}}E\left[u_i,\pi_i\left(\cdot\mid\hat{I}\right)\right]\Big|_{\Theta_i(\omega)}-\mu' P_m(\omega) &= 0\\ &\Leftrightarrow\\ \sum_{\omega'\in\hat{I}}u_i'\left(\sum_{k=1}^{M}X_k(\omega')\left(\Theta_{i,k}(\omega)+e_{i,k}\right)+Y_i(\omega')\right)X_m(\omega')\pi_i\left(\omega'\mid\hat{I}\right)-\mu' P_m(\omega) &= 0\end{aligned}$$

for some $\mu'>0$. By P1, this FCE can thus only fail to be an RIE if there is some $\hat{I}\neq I^{FC}(\omega)$, satisfying the conditions of Lemma 2, such that for all assets 1 and 2

$$\sum_{\omega'\in\hat{I}}z\left(\Theta_i(\omega),\omega'\right)\pi_i\left(\omega'\mid\hat{I}\right)=0.\tag{30}$$

Step 3. We construct a set of perturbed beliefs for which (29) will hold for all $I^{FC}(\cdot)\in\Pi^{FC}$ whereas (30) will generically break down. Suppose that there are h

different information cells $I^{FC}(\cdot)$ in Π^{FC} and label them as $I_1^{FC}, ..., I_h^{FC}$. (Note that $h \geq 2$ by Lemma 2.) Fix π_i and write, for notational convenience, the probabilities of the full communication information cells as

$$\lambda^j \equiv \pi_i\left(I_j^{FC}\right).$$

Note that the set of h-tupels $\left(\lambda^1, ..., \lambda^h\right)$ for all possible π_i corresponds to the $h-1$ dimensional open simplex

$$\triangle \equiv \left\{\lambda \in \mathbb{R}_{++}^h \mid \sum_{j=1}^h \lambda^j = 1\right\}. \tag{31}$$

Since $h \geq 2$, this simplex has strictly positive $h-1$ dimensional Lebesgue measure $\frac{1}{(h-1)!}$.

Now pick an arbitrary $I_j^{FC} \in \Pi^{FC}$ and introduce the following open interval of possible perturbations of probability λ^j

$$\Lambda^j \equiv \left(-\lambda^j, 1-\lambda^j\right). \tag{32}$$

Fix a j-perturbation parameter $\varepsilon^j \in \Lambda^j$ and define δ_{ε^j} as follows

$$\delta_{\varepsilon^j} = 1 + \frac{\varepsilon^j}{\lambda^j}.$$

Given π_i introduce a perturbed belief π_i^ε such that (i) the perturbation parameter $\varepsilon = \left(\varepsilon^1, ..., \varepsilon^h\right)$ lies in the set

$$\mathcal{E} \equiv \left\{\left(\varepsilon^1, ..., \varepsilon^h\right) \mid \varepsilon^j \in \left(-\lambda^j, 1-\lambda^j\right) \text{ and } \left(\lambda^1, ..., \lambda^h\right) \in \triangle\right\}$$

and (ii), for all $j = 1, ..., h$,

$$\pi_i^\varepsilon(\omega') = \delta_{\varepsilon^j}\pi_i(\omega') \text{ for all } \omega' \in I_j^{FC}. \tag{33}$$

Step 4. Note that we have, by construction, for all $\varepsilon \in \mathcal{E}$ and all $I_j^{FC} \in \Pi^{FC}$ that

$$\begin{aligned}
\pi_i^\varepsilon\left(\omega' \mid I_j^{FC}\right) &= \frac{\delta_{\varepsilon^j}\pi_i(\omega')}{\sum_{\omega'' \in I_j^{FC}} \delta_{\varepsilon^j}\pi_i(\omega'')} = \frac{\left(1+\frac{\varepsilon^j}{\lambda^j}\right)\pi_i(\omega')}{\sum_{\omega'' \in I_j^{FC}}\left(1+\frac{\varepsilon^j}{\lambda^j}\right)\pi_i(\omega'')} \\
&= \frac{\pi_i(\omega')}{\lambda^j} = \pi_i\left(\omega' \mid I_j^{FC}\right) \text{ for all } \omega' \in I_j^{FC}.
\end{aligned} \tag{34}$$

Consequently, the set of all these perturbed beliefs

$$\triangle^{per} \equiv \{\pi_i^\varepsilon \mid \varepsilon \in \mathcal{E}\}$$

corresponds one-one through the mapping (33) to the $h-1$ dimensional open simplex (31) since only the probabilities λ^j of the $I_1^{FC}, ..., I_h^{FC}$ are changed while the I_j^{FC}-conditional probabilities $\frac{\pi_i(\omega')}{\lambda^j}$, $\omega' \in I_j^{FC}$, remain unchanged. The set of all these perturbed beliefs $\triangle^{per}$ has therefore strictly positive $h-1$ dimensional Lebesgue measure $\frac{1}{(h-1)!}$.

By (34), we also have for every $\pi_i^\varepsilon \in \triangle^{per}$ that for all $I^{FC}(\omega) \in \Pi^{FC}$

$$\begin{aligned} \sum_{\omega' \in I^{FC}(\omega)} z(\Theta_i(\omega), \omega') \pi_i \left(\omega' \mid I^{FC}(\omega)\right) &= 0 \\ &\Leftrightarrow \\ \sum_{\omega' \in I^{FC}(\omega)} z(\Theta_i(\omega), \omega') \pi_i^\varepsilon \left(\omega' \mid I^{FC}(\omega)\right) &= 0. \end{aligned} \tag{35}$$

This establishes the following result: If Θ_i is an FCE net-trade function at equilibrium prices P for belief π_i, then Θ_i is also an FCE net-trade function at equilibrium prices P for all perturbed beliefs π_i^ε that live in the set $\triangle^{per}$ with strictly positive $h-1$ dimensional Lebesgue measure $\frac{1}{(h-1)!}$.

Step 5. Consider now some $\hat{I} \neq I^{FC}(\omega)$, satisfying the conditions of Lemma 2, such that (30) holds for all assets 1 and 2. Our ultimate aim is to show that

$$\sum_{\omega' \in \hat{I}} z(\Theta_i(\omega), \omega') \pi_i^\varepsilon \left(\omega' \mid \hat{I}\right) = 0 \text{ for all assets 1 and 2} \tag{36}$$

breaks down for almost all perturbed beliefs $\pi_i^\varepsilon \in \triangle^{per}$ despite the fact that it holds, by assumption, for $\varepsilon = (0, ..., 0)$. To make this argument, we have to transform (36) in several stages.

Rewrite (36) as

$$\sum_{I^{FC}(\cdot) \neq I^{FC}(\omega)} \pi_i^\varepsilon \left(I^{FC}(\cdot) \mid \hat{I}\right) \sum_{\omega' \in I^{FC}(\cdot)} z(\Theta_i(\omega), \omega') \pi_i^\varepsilon \left(\omega' \mid I^{FC}(\cdot)\right) + b^\varepsilon = 0$$

where

$$b^\varepsilon = \pi_i^\varepsilon \left(I^{FC}(\omega) \mid \hat{I}\right) \sum_{\omega' \in I^{FC}(\omega)} z(\Theta_i(\omega), \omega') \pi_i^\varepsilon \left(\omega' \mid I^{FC}(\omega)\right).$$

Observe that, by (35), $b^\varepsilon = 0$ for all ε. By (34), equation (36) thus becomes

$$\sum_{I^{FC}(\cdot) \neq I^{FC}(\omega)} \pi_i^\varepsilon \left(I^{FC}(\cdot) \mid \hat{I}\right) \sum_{\omega' \in I^{FC}(\cdot)} z(\Theta_i(\omega), \omega') \pi_i \left(\omega' \mid I^{FC}(\cdot)\right) = 0.$$

Step 6. Consider the information cells $I^{FC}(\cdot) \neq I^{FC}(\omega)$ in Π^{FC} such that $\pi_i^\varepsilon \left(I^{FC}(\cdot) \mid \hat{I}\right) > 0$ and relabel them as $I_1^{FC}, .., I_k^{FC}$. Obviously, $k \leq h-1$ but also $k \geq 1$ as there must be at least one $\omega^* \in \hat{I}$ such that $\omega^* \notin I^{FC}(\omega)$ by Lemma 2. For notational simplicity let

$$\begin{aligned} y_j &\equiv \sum_{\omega' \in I_j^{FC}} z(\Theta_i(\omega), \omega') \pi_i \left(\omega' \mid I_j^{FC}\right), \\ \lambda_j^{\varepsilon j} &\equiv \pi_i^\varepsilon \left(I_j^{FC} \mid \hat{I}\right) \end{aligned}$$

so that equation (36) becomes

$$\sum_{j=1}^{k} y_j \lambda_j^{\varepsilon_j} = 0. \tag{37}$$

Step 7. Note that the

$$y_j = u_i' \left(\Theta_i(\omega), \omega'\right) \left[X_1(\omega') - \frac{P_1(\omega)}{P_2(\omega)} X_2(\omega')\right]$$

in (37) actually depend on the two assets 1 and 2 and write, for the moment, $y_j \equiv y_j[1,2]$ to emphasize this dependence. For (37) being not trivially true for all assets irrespective of perturbed beliefs $\lambda_j^{\varepsilon_j}$, we need to ensure that there are at least two assets 1 and 2 for some j such that $y_j[1,2]$ is not zero. Recall from Step 2 that, by A1-A2 and P1-P2, $y_j[1,2] = 0$ for all assets 1 and 2 implies

$$\Theta_i(\omega) \in \varphi_i\left(P(\omega), I_j^{FC}\right).$$

But because $\Theta_i(\omega) \in \varphi_i\left(P(\omega), I_j^{FC}\right)$ with $I_j^{FC} \neq I^{FC}(\omega)$ would violate P3 Information Sensitivity, there must exist for every $j = 1, .., k$ some assets 1 and 2 (possibly depending on j) such that $y_j[1,2] \neq 0$. From now on we pick for any $j = 1, .., k$ such assets 1 and 2 implying $y_j \equiv y_j[1,2] \neq 0$ for all $j = 1, .., k$ in (37). (Actually, considering only one $y_j \neq 0$ for an arbitrary j would be sufficient for our argument.)

Step 8. Next we need to ensure that there are $\lambda_j^{\varepsilon_j}$ in (37) which take on varying values in $(0,1)$ for different perturbations $\varepsilon \in \mathcal{E}$. More specifically, there are two different cases why $\lambda_j^{\varepsilon_j}$ might remain constant (either zero or one) for all $\varepsilon \in \mathcal{E}$. We consider each case in turn.

Case i.) Observe that $\lambda_j^{\varepsilon_j} = 0$ if, and only if,

$$I_j^{FC} \cap \hat{I} = \emptyset.$$

Denote by $k^* \leq k$ the number of $\lambda_j^{\varepsilon_j}$ such that $\lambda_j^{\varepsilon_j} > 0$ and write (without loss of generality) $I_1^{FC}, .., I_{k^*}^{FC}$ for all the I_j^{FC} with $\lambda_j^{\varepsilon_j} > 0$. Let us rewrite (37) as

$$\sum_{j=1}^{k^*} y_j \lambda_j^{\varepsilon_j} = 0 \tag{38}$$

and note that Lemma 2 implies $k^* \geq 1$ (i.e., there must be some $\omega^* \notin I^{FC}(\omega)$).

Case ii.) By Lemma 2, we have either $I^{FC}(\omega) \subset \hat{I}$ or $I_j^{FC} \subseteq \hat{I}$. That is,

$$\lambda_j^{\varepsilon_j} = \pi_i^{\varepsilon}\left(I_j^{FC} \mid \hat{I}\right) = 1$$

if, and only if, for some j, $I_j^{FC} = \hat{I}$. But this would imply

$$\begin{array}{rcl}\Theta_i(\omega) & \in & \varphi_i\left(P(\omega), \hat{I}\right) \\ & \Leftrightarrow & \\ \Theta_i(\omega) & \in & \varphi_i\left(P(\omega), I_j^{FC}\right),\end{array}$$

a violation of P3 Information Sensitivity. Consequently, $\lambda_j^{\varepsilon_j} \neq 1$ by P3.

To summarize: For all $j = 1, ..., k^*$, with $k^* \geq 1$, the probabilities $\lambda_j^{\varepsilon_j}$ in (38) take on varying values in $(0,1)$ for different perturbations $\varepsilon \in \mathcal{E}$. Collect the $\lambda^\varepsilon = (\lambda_1^{\varepsilon_1}, ..., \lambda_{k^*}^{\varepsilon_{k^*}})$ for all possible perturbations in the set

$$\Lambda^\varepsilon \equiv \left\{ \lambda^\varepsilon = (\lambda_1^{\varepsilon_1}, ..., \lambda_{k^*}^{\varepsilon_{k^*}}) \mid \lambda_j^\varepsilon = \frac{\pi_i^\varepsilon\left(I_j^{FC} \cap \hat{I}\right)}{\pi_i^\varepsilon\left(\hat{I}\right)} \text{ and } \pi_i^\varepsilon \in \triangle^{per} \right\}$$

and observe that Λ^ε is an open, non-empty subset of $\mathbb{R}^{k^*}$.

Step 9. By Step 7 together with Step 8, we have $y_j \lambda_j^{\varepsilon_j} \neq 0$ for all $j = 1, .., k^*$ in (38) for any perturbations. Observe that the set

$$\left\{ (\lambda_1^{\varepsilon_1}, ..., \lambda_{k^*}^{\varepsilon_{k^*}}) \in \Lambda^\varepsilon \mid \sum_{j=1}^{k^*} y_j \lambda_j^{\varepsilon_j} = 0 \right\}$$

has k^*-dimensional Lebesgue measure zero because it is a subset of the $k^* - 1$ dimensional hyperplane

$$\left\{ (\tau_1, .., \tau_{k^*}) \in \mathbb{R}^{k^*} \mid \sum_{j=1}^{k^*} y_j \tau_j = 0 \right\}$$

(cf. Billingsley 1995, p. 172). Because of $k^* \leq k \leq h - 1$, the subset of the perturbed beliefs in $\triangle^{per}$ that satisfy equation equation (38), and thereby equation (36), can thus only have an $h - 1$-dimensional Lebesgue measure of zero.

However, under Step 4 we had already established that the set of perturbed beliefs $\triangle^{per}$ has a strictly positive $h - 1$-dimensional Lebesgue measure. Consequently, equation (36) will be violated for almost all perturbations $\left(\varepsilon^1, ..., \varepsilon^h\right) \in \mathcal{E}$, i.e., for almost all $\pi_i^\varepsilon \in \triangle^{per}$. This proves the Theorem.□□

Remark. Note that P3 Information Sensitivity was used twice in the above proof. First, it was used as a sufficient condition for ensuring that the y_j, $j = 1, ..., k$, are non-zero (cf. Step 7). Else, (38) might reduce to $0 = 0$ so that (36) would trivially hold for all $\pi_i^\varepsilon \in \triangle^{per}$.

Second, it was used as a sufficient condition for ensuring that not all $\lambda_j^{\varepsilon j}$, $j = 1, ..., k^*$, become one (cf. case (ii) of Step 8). Else, (38) might reduce to

$$\sum_{j=1}^{k^*} y_j = 0$$

so that, again, (36) holds for all $\pi_i^{\varepsilon} \in \triangle^{per}$ regardless of the value of the perturbation parameter ε. That is, P3 Information Sensitivity ensures that the perturbation of beliefs may have any impact on optimal net-trades that are described through first-order conditions.

Appendix B: Illustration of the rationalization process

We illustrate the rationalization process for the following example.

Example 3. Consider a pure speculation economy with one risky asset and state space

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}.$$

The private information partitions of the risk-neutral agents, A and B, are given as

$$\begin{aligned} \Pi_A^{PI} &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \\ \Pi_B^{PI} &= \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}. \end{aligned}$$

Both agents have a common prior π such that

ω	$X(\omega)$	$\pi(\omega)$
ω_1	1	$\frac{1}{4}$
ω_2	5	$\frac{1}{4}$
ω_3	3	$\frac{1}{4}$
ω_4	2	$\frac{1}{4}$

We assume that each agent can choose any net-trade in $\mathbb{R}$. Agent i's net-trade correspondence is thus given as

$$\varphi_i(P(\omega), I_i(\omega)) = \arg\max_{\theta_i \in \mathbb{R}} \sum_{\omega' \in I_i(\omega)} ((X(\omega') - P(\omega))\theta_i)\pi(\omega' \mid I_i(\omega)).$$

□

Restrict attention to the zero-trade FCE $(P,\Theta)\left\langle I^{FC},\pi\right\rangle$ such that, for all ω,

$$\begin{aligned} P(\omega) &= X(\omega), \\ \Theta_A(\omega) &= \Theta_B(\omega)=0, \\ I^{FC}(\omega) &= \{\omega\}. \end{aligned}$$

Since the FCE price function is one-one, this FCE is also an REE/GREE. The rationalization process at (P,Θ) terminates at $k=3$ revealing the full communication information to both agents. Consequently, the FCE is also an RIE.[11] In what follows we describe in detail the rationalization process at (P,Θ) whereby the rationalizable information at different stages will be given as follows:

ω	$I_A^{R,0}(\omega)$	$I_A^{R,1}(\omega)$	$I_A^{R,2}(\omega)$	$I_A^{R,3}(\omega)=I_A^{R,\infty}(\omega)$
ω_1	$\{\omega_1,\omega_2\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
ω_2	$\{\omega_1,\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
ω_3	$\{\omega_3,\omega_4\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
ω_4	$\{\omega_3,\omega_4\}$	$\{\omega_3,\omega_4\}$	$\{\omega_3,\omega_4\}$	$\{\omega_4\}$

ω	$I_B^{R,0}(\omega)$	$I_B^{R,1}(\omega)$	$I_B^{R,2}(\omega)$	$I_B^{R,3}(\omega)=I_B^{R,\infty}(\omega)$
ω_1	$\{\omega_1,\omega_3\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
ω_2	$\{\omega_2,\omega_4\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
ω_3	$\{\omega_1,\omega_3\}$	$\{\omega_1,\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
ω_4	$\{\omega_2,\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$

Let $k=0$. Initialize

$$\begin{aligned} I_A^{R,0} &= I_A^{PI}, \\ I_B^{R,0} &= I_B^{PI}. \end{aligned}$$

Let $k=1$. Start with agent A and identify for each state the possibility sets:

$$\begin{aligned} & \mathcal{P}_{A,B}^{1}\left(I_A^{R,0}(\omega_1)\right)=\mathcal{P}_{A,B}^{1}\left(I_A^{R,0}(\omega_2)\right) \\ = & \left\{I_B\in\Sigma \mid I^{FC}(\omega_1)\subseteq I_B\subseteq I_B^{R,0}(\omega_1)\right\}\cup\left\{I_B\in\Sigma \mid I^{FC}(\omega_2)\subseteq I_B\subseteq I_B^{R,0}(\omega_2)\right\} \\ = & \left\{\{\omega_1\},\{\omega_1,\omega_3\},\{\omega_2\},\{\omega_2,\omega_4\}\right\} \end{aligned}$$

[11] At the FCE one-one price function all net-trades $\Theta_A(\omega),\Theta_B(\omega)\in\mathbb{R}$ such that $\Theta_A(\omega)=-\Theta_B(\omega)$ constitute an FCE which is also an RIE. This situation is an example for the generic case described in Theorem 3 since we only have interior FCE net-trades.

as well as

$$
\begin{aligned}
& \mathcal{P}_{A,B}^{1}\left(I_{A}^{R,0}\left(\omega_{3}\right)\right)=\mathcal{P}_{A,B}^{1}\left(I_{A}^{R,0}\left(\omega_{4}\right)\right) \\
= & \left\{I_{B}\in\Sigma\mid I^{FC}\left(\omega_{3}\right)\subseteq I_{B}\subseteq I_{B}^{R,0}\left(\omega_{3}\right)\right\}\cup\left\{I_{B}\in\Sigma\mid I^{FC}\left(\omega_{4}\right)\subseteq I_{B}\subseteq I_{B}^{R,0}\left(\omega_{4}\right)\right\} \\
= & \left\{\left\{\omega_{3}\right\},\left\{\omega_{1},\omega_{3}\right\},\left\{\omega_{4}\right\},\left\{\omega_{2},\omega_{4}\right\}\right\}.
\end{aligned}
$$

With respect to all information cells in the possibility set $\mathcal{P}_{A,B}^{1}\left(I_{A}^{R,0}\left(\omega_{1}\right)\right)$, the zero trade is optimal for agent B at price $P\left(\omega_{1}\right)=1$ only at information cell $\left\{\omega_{1}\right\}$ as B would love to buy infinitely many units of the asset at price 1 at all other information cells in $\mathcal{P}_{A,B}^{1}\left(I_{A}^{R,0}\left(\omega_{1}\right)\right)$, i.e.,

$$
\begin{aligned}
I_{A,B}^{1}\left(\omega_{1}\right) &= \bigcup_{\left\{I_{B}\in\left\{\left\{\omega_{1}\right\},\left\{\omega_{1},\omega_{3}\right\},\left\{\omega_{2}\right\},\left\{\omega_{2},\omega_{4}\right\}\right\}\mid 0\in\varphi_{B}\left(1,I_{B}\right)\right\}} I_{B} \\
&= \left\{\omega_{1}\right\}.
\end{aligned}
$$

For the remaining states we obtain

$$
\begin{aligned}
I_{A,B}^{1}\left(\omega_{2}\right) &= \bigcup_{\left\{I_{B}\in\left\{\left\{\omega_{1}\right\},\left\{\omega_{1},\omega_{3}\right\},\left\{\omega_{2}\right\},\left\{\omega_{2},\omega_{4}\right\}\right\}\mid 0\in\varphi_{B}\left(5,I_{B}\right)\right\}} I_{B} \\
&= \left\{\omega_{2}\right\},
\end{aligned}
$$

$$
\begin{aligned}
I_{A,B}^{1}\left(\omega_{3}\right) &= \bigcup_{\left\{I_{B}\in\left\{\left\{\omega_{3}\right\},\left\{\omega_{1},\omega_{3}\right\},\left\{\omega_{4}\right\},\left\{\omega_{2},\omega_{4}\right\}\right\}\mid 0\in\varphi_{B}\left(3,I_{B}\right)\right\}} I_{B} \\
&= \left\{\omega_{3}\right\},
\end{aligned}
$$

$$
\begin{aligned}
I_{A,B}^{1}\left(\omega_{4}\right) &= \bigcup_{\left\{I_{B}\in\left\{\left\{\omega_{3}\right\},\left\{\omega_{1},\omega_{3}\right\},\left\{\omega_{4}\right\},\left\{\omega_{2},\omega_{4}\right\}\right\}\mid 0\in\varphi_{B}\left(2,I_{B}\right)\right\}} I_{B} \\
&= \left\{\omega_{1},\omega_{3}\right\}\cup\left\{\omega_{4}\right\} \\
&= \left\{\omega_{1},\omega_{3},\omega_{4}\right\}
\end{aligned}
$$

whereby $I_{A,B}^{1}\left(\omega_{4}\right)$ is not a singleton because of the ambiguity arising from

$$
0\in\varphi_{B}\left(2,\left\{\omega_{1},\omega_{3}\right\}\right)=\varphi_{B}\left(2,\left\{\omega_{4}\right\}\right).
$$

As A's rationalizable information at stage $k=1$, we thus obtain

$$
\begin{aligned}
I_{A}^{R,1}\left(\omega_{1}\right) &= I_{A}^{PI}\left(\omega_{1}\right)\cap I_{A,B}^{1}\left(\omega_{1}\right)=\left\{\omega_{1}\right\}, \\
I_{A}^{R,1}\left(\omega_{2}\right) &= I_{A}^{PI}\left(\omega_{2}\right)\cap I_{A,B}^{1}\left(\omega_{2}\right)=\left\{\omega_{2}\right\}, \\
I_{A}^{R,1}\left(\omega_{3}\right) &= I_{A}^{PI}\left(\omega_{3}\right)\cap I_{A,B}^{1}\left(\omega_{3}\right)=\left\{\omega_{3}\right\}, \\
I_{A}^{R,1}\left(\omega_{4}\right) &= I_{A}^{PI}\left(\omega_{4}\right)\cap I_{A,B}^{1}\left(\omega_{4}\right)=\left\{\omega_{3},\omega_{4}\right\}.
\end{aligned}
$$

Analogously, we obtain for agent B the following rationalizable information at stage $k = 1$

$$\begin{array}{rcl} I_B^{R,1}(\omega_1) & = & \{\omega_1\}, \\ I_B^{R,1}(\omega_2) & = & \{\omega_2\}, \\ I_B^{R,1}(\omega_3) & = & \{\omega_1, \omega_3\}, \\ I_B^{R,1}(\omega_4) & = & \{\omega_4\}, \end{array}$$

whereby $I_B^{R,1}(\omega_3)$ is not a singleton because of

$$0 \in \varphi_A(3, \{\omega_3\}) = \varphi_A(3, \{\omega_1, \omega_2\}).$$

Let $k = 2$**.** Start with A and note that it is sufficient to focus on ω_4 since the rationalizable information in all other states has already converged to the full communication information. As possibility set we obtain

$$\begin{array}{rl} & \mathcal{P}_{A,B}^2\left(I_A^{R,1}(\omega_4)\right) \\ = & \left\{I_B \in \Sigma \mid I^{FC}(\omega_3) \subseteq I_B \subseteq I_B^{R,1}(\omega_3)\right\} \cup \left\{I_B \in \Sigma \mid I^{FC}(\omega_4) \subseteq I_B \subseteq I_B^{R,1}(\omega_4)\right\} \\ = & \{\{\omega_3\}, \{\omega_1, \omega_3\}, \{\omega_4\}\} \end{array}$$

so that

$$\begin{array}{rcl} I_{A,B}^2(\omega_4) & = & \bigcup_{\{I_B \in \{\{\omega_3\}, \{\omega_1, \omega_3\}, \{\omega_4\}\} \mid 0 \in \varphi_B(2, I_B)\}} I_B \\ & = & \{\omega_1, \omega_3, \omega_4\}. \end{array}$$

Consequently, the rationalizable information of agent A in state ω_4 remains the same at $k = 2$ as at $k = 1$, i.e.,

$$I_A^{R,2}(\omega_4) = I_A^{R,1}(\omega_4) \cap I_{A,B}^2(\omega_4) = \{\omega_3, \omega_4\}.$$

Turn now to agent B and focus on ω_3. Observe that

$$\begin{array}{rl} & \mathcal{P}_{B,A}^2\left(I_B^{R,1}(\omega_3)\right) \\ = & \left\{I_A \in \Sigma \mid I^{FC}(\omega_1) \subseteq I_A \subseteq I_A^{R,1}(\omega_1)\right\} \cup \left\{I_A \in \Sigma \mid I^{FC}(\omega_3) \subseteq I_A \subseteq I_A^{R,1}(\omega_3)\right\} \\ = & \{\{\omega_1\}\} \cup \{\{\omega_3\}\} \\ = & \{\{\omega_1\}, \{\omega_3\}\} \end{array}$$

implying

$$\begin{array}{rcl} I_{B,A}^2(\omega_3) & = & \bigcup_{\{I_A \in \{\{\omega_1\}, \{\omega_3\}\} \mid 0 \in \varphi_A(3, I_A)\}} I_A \\ & = & \{\omega_3\}. \end{array}$$

Consequently, the rationalizable information of agent B at $k=2$ coincides with the full communication information, i.e., for all ω,

$$I_B^{R,2}(\omega) = \{\omega\}. \tag{39}$$

Let $k=3$. It remains to consider A's rationalizable information in state ω_4. Because of (39), $I^{FC}(\omega) = I_B^{R,2}(\omega)$ so that A's possibility set becomes

$$\begin{aligned} & \mathcal{P}_{A,B}^3\left(I_A^{R,2}(\omega_4)\right) \\ = & \left\{I_B \in \Sigma \mid I^{FC}(\omega_3) \subseteq I_B \subseteq I_B^{R,2}(\omega_3)\right\} \cup \left\{I_B \in \Sigma \mid I^{FC}(\omega_4) \subseteq I_B \subseteq I_B^{R,2}(\omega_4)\right\} \\ = & \{\{\omega_3\}, \{\omega_4\}\}. \end{aligned}$$

Consequently,

$$\begin{aligned} I_{A,B}^2(\omega_4) &= \bigcup_{\{I_B \in \{\{\omega_3\},\{\omega_4\}\} \mid 0 \in \varphi_B(2, I_B)\}} I_B \\ &= \{\omega_4\} \end{aligned}$$

so that

$$I_A^{R,3}(\omega_4) = \{\omega_4\}.$$

That is, the rationalization process terminates at $k=3$ whereby both agents have inferred the full communication information.□□

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