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Abstract

This paper studies general equilibrium theory, for both complete and incomplete markets, under Knightian uncertainty. Noting that the preference represented by Knightian uncertainty induces a set of complete preferences, we set ourselves the task of inquiring the relationship between an equilibrium under Knightian uncertainty and its counterpart under the induced complete preferences. It is shown that they are actually equivalent. The importance of this result is due to its applications, among which the existence of equilibria under Knightian uncertainty and their computation follow at once from the existing knowledge on general equilibrium theory under complete preferences. Moreover, by means of that equivalence, we are in a position to investigate the problem of efficiency and indeterminacy of equilibria under Knightian uncertainty.

Keywords: General equilibrium; Knightian uncertainty; Pareto optimality

1 Introduction

The science of microeconomics starts with the primitive notion of preference which is usually assumed complete. That this assumption is dubious has been felt by several economists, among them von Neumann and Morgenstern (1947), Aumann (1962), Schmeidler (1969). There are at least two reasons for the incompleteness of a preference. From the viewpoint of an individual economic agent, it is doubted "whether a person can always decide which of two alternatives [...] he prefers"(von Neumann and Morgenstern, 1947, pp. 28-29). From the viewpoint of a group of economic agents (as for instance the countries in the European Union), if their individual preferences do not coincide, then the preference of the whole group would necessarily be incomplete (see Shapley and Baucells (1998)). The purpose of this paper however is not to argue whether the preference is complete or not; but, assuming it, to derive its effect on the general equilibrium theory, for both complete and incomplete markets. Although much broader in scope, we shall concern ourselves with four of its many facets: existence, uniqueness, efficiency, and computation.

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To establish general equilibrium theory under incomplete preferences, a most natural approach that suggests itself, is to extend all the technical machinery for complete preferences to the case of incomplete preferences. This approach is taken, for instance, in Gale and Mas-Colell (1975) (which treats also intransitive preferences). Besides being technically complicated it has,for instance, a deficiency about computation of equilibria: The existing computational methods depend in one way or another upon the notion of demand function, which however, as shown in Mas-Colell (1974), is not well-defined under incomplete preferences.

In this paper we shall take a different approach, indirect yet more efficient. To elaborate on it we take as an example the case of complete markets and let there be m consumers. Assume that consumer i's incomplete preference $\succ_i$ can be represented by a set $\mathbb{U}^i$ of utility functions: $x \succ_i y$ if and only if $u(x) > u(y)$ for all u in $\mathbb{U}^i$, where x, y are two bundles of commodities. Let $\bar{\mathbb{U}} = \times_i \mathbb{U}^i$, the Cartesian product of $\mathbb{U}^i$, and abbreviate $(x^1, \ldots, x^m)$ to (x^i), where x^i denotes consumer i's consumption bundle. Then an equilibrium under incomplete preferences (or $\bar{\mathbb{U}}$-equilibrium for short) is a feasible allocation (x^i) and a price vector p such that $x'^i \succ_i x^i$ implies x'^i is outside of consumer i's budget set $B^i(p)$. Recall that, replacing $x'^i \succ_i x^i$ by $u^i(x'^i) > u^i(x^i)$, $u^i \in \mathbb{U}^i$, we get the usual notion of equilibrium under complete preferences, or $\bar{u}$-equilibrium for short, where $\bar{u} = (u^1, \ldots, u^m)$. Now it is proper to ask the question, What is the relationship between a $\bar{\mathbb{U}}$- and a $\bar{u}$-equilibrium; or, more specifically, whether it is true that $((x^i), p)$ is a $\bar{\mathbb{U}}$-equilibrium when and only when it is a $\bar{u}$-equilibrium for some $\bar{u} \in \bar{\mathbb{U}}$.

The point of this question, if its answer turns out in the affirmative, is warranted by its applications. As a trivial one, the existence of $\bar{\mathbb{U}}$-equilibrium follows at once from that of $\bar{u}$-equilibrium, and an algorithm for computing the latter can be applied to compute the former. These two aspects, due to the triviality, will not be discussed in the body of the text; they are mentioned here merely to motivate the question. As a much less trivial application we can discuss the efficiency and indeterminacy of $\bar{\mathbb{U}}$-equilibria.

In view of the complexities of this general question, in this paper we shall limit our efforts, so as to obtain a guide, to a special kind of incomplete preference, namely Knightian uncertainty. This notion has its origin in the work of Knight (1921) and is formalized by Bewley (2002). Roughly speaking, it assumes the existence of a set of probability measures on a state space such that one state-contingent consumption bundle is preferred to another if and only if it has larger expected utility for every measure in that set. The peculiarity of Knightian uncertainty is that it assumes a fixed taste for every consumer, and attributes the incompleteness of her preference entirely to her ambiguity about the likelihood of the states. This makes the determination of commodity prices much easier than in the general case. The main result of the present paper is that, under Knightian uncertainty, the question posed above has a positive answer in various cases: complete markets, incomplete markets with nominal and real assets.

This paper contains six sections. Following this section of introduction, we make in Section 2 a brief review of the notion of Knightian uncertainty. Section 3 studies the case of complete markets. In the case of one commodity in each state the relation between $\bar{\mathbb{U}}$- and $\bar{u}$-equilibria is investigated by Rigotti and Shannon (2005) and Dana and Riedel (2013). In particular, Theorem 7 of the former indicates that any $\bar{u}$-equilibrium is a $\bar{\mathbb{U}}$-equilibrium while Theorem 2 of the latter establishes their equivalence in the setting of a dynamic and infinite-dimensional model. In the static model with any finite number of commodities the equivalence between $\bar{\mathbb{U}}$- and $\bar{u}$-equilibria has been worked out by Carlier and Dana (2013), whose concern consists mainly in the efficiency of $\bar{\mathbb{U}}$-equilibria.

We include their results here, to which we add a new one on indeterminacy of equilibria, on the one hand to ease understanding of the main theme of the present paper, and on the other hand, to set the tone for it.

In Section 4 we study a special kind of incomplete markets, namely those with one commodity in each state. This peculiarity allows us to focus on the central variables of incomplete markets (asset prices and demands) and disregard those only of secondary importance (commodity prices and demands). From the special case we understand how to set asset prices under Knightian uncertainty; which paves the way for the more general studies of the next two sections: Section 5 is concerned with incomplete markets with nominal assets and Section 6 with real assets. Recall that Geanakoplos and Mas-Colell (1989) investigate the dimension of indeterminacy of equilibrium commodity allocations when the assets are nominal, and they conclude that the indeterminacy is independent of the incompleteness of the markets or the number of assets. In contrast, it will be shown in Section 5 that the indeterminacy of equilibria caused by the incomplete preference does depend on the number of assets.

On a personal note I should like to thank the referee of my paper Ma (2015), which treats also general equilibrium theory under Knightian uncertainty. In that paper, one of the concerns is to establish a theorem on the existence of $\bar{\mathbb{U}}$-equilibrium with incomplete markets of real assets, for which I have to, as in Radner (1972), impose an upper bound on forward transactions. The referee asks: since incompleteness of preference leads to indeterminacy of equilibria, is it possible to eliminate that bound? It is in attempt to answer this question that I come across the idea of the present paper. From the equivalence of $\bar{\mathbb{U}}$- and $\bar{u}$-equilibria, the answer is no—incompleteness of preference can not ease the establishment of the equilibrium existence theorem.

2 Knightian Uncertainty

This section makes a brief review of the formalization of Knightian uncertainty. For this purpose, we start with a description of the economic environment with which we shall be concerned. We shall study a pure exchange economy with two dates, denoted 0 and 1, and S possible states of nature at date 1. We index the states by s running from 1 to S, and for notational convenience, call date 0 state 0. Let there be m consumers, J assets, and L commodities in each state. Suppose that every consumer has $X = \mathbb{R}_+^{(S+1)L}$ as her consumption space and that consumer i has $\omega^i \gg 0$[1] as her endowment vector. For $x \in X$, it is often convenient to write it state-wise as $x = (x_0, \cdots, x_S)$ with $x_s \in \mathbb{R}^L$.

Let Δ_S be the set of probability measures on $\{1, \ldots, S\}$, that is,

$$\Delta_S = \{\pi \in \mathbb{R}_+^S | \ \|\pi\|_1 = 1\},$$

where $\|\cdot\|_1$ stands for the 1-norm of a vector. We equip $\mathbb{R}^S$ with the Euclidean topology and Δ_S the relative topology. Let $\succ_i$ be consumer i's preference, $i = 1, \ldots, m$. Then she is said to be a Knightian decision maker, if there exists a closed, convex subset Π^i of Δ_S such that

$$x^i \succ_i x'^i \text{ if, and only if, } U_\pi^i(x^i) > U_\pi^i(x'^i) \text{ for all } \pi \in \Pi^i,$$

where $U_\pi^i(x^i) = u^i(x_0^i) + \sum_{s=1}^S \pi_s u^i(x_s^i)$, u^i being specified in a moment. In case every Π^i is a s-ingleton, we call the environment a risky one, and otherwise, an uncertain one. Historically, the necessity of differentiating between risk and uncertainty is first appreciated by Knight (1921), and

[1] As usual we define a vector $x \gg 0$ as $x_i > 0$ for every i, $x \geq 0$ as $x_i \geq 0$ for every i, and $x > 0$ as $x \geq 0$ but $x \neq 0$.

this notion is formalized recently by Bewley (2002).

For the purpose of obtaining a continuously differentiable demand function, we assume that every $\Pi^i \subset \mathrm{rint}(\Delta_S)$, the relative interior of Δ_S, and that every $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ satisfies these four conditions:

(i) u^i is continuous, strictly concave on $\mathbb{R}^L_+$ and smooth on $\mathbb{R}^L_{++}$;
(ii) The set $\{z \in \mathbb{R}^L_+ | \, u^i(z) \geq u^i(x)\} \subset \mathbb{R}^L_{++}$ for every $x \in \mathbb{R}^L_{++}$;
(iii) The gradient $(\partial u^i(x)/\partial x_1, \ldots, \partial u^i(x)/\partial x_L) \in \mathbb{R}^L_{++}$ for every $x \in \mathbb{R}^L_{++}$.

This set of conditions is similar to that of Magill and Quinzii (2002, pp. 50–51)), except that their fourth condition is strengthened here to strict concavity. We wish to remark that use of these assumptions will be made in the study of indeterminacy, efficiency and computation of equilibria; for existence, they can be relaxed to a great extent.

Before concluding this section we make some notational conventions: Let $\bar{X} = X^m$, the Cartesian product of m copies of X, and $\bar{\Pi} = \Pi^1 \times \cdots \times \Pi^m$; let $\bar{x}$ and $\bar{\pi}$ designate, respectively, generic elements of $\bar{X}$ and $\bar{\Pi}$, and we call $\bar{x}$ an allocation. In the following pages we shall use superscripts to refer to the consumers and subscripts to the components of a vector. Thus, for example, $x^i \in X$ stands for a consumption bundle for consumer i, while x^i_s for her consumption in state s. Finally, given two vectors, v_1, v_2, of the same dimension, by $v_1 v_2$ we mean their inner product.

3 Complete Markets

This section concerns the case of complete markets. Let Δ denote the set of normalized price vectors, that is,

$$\Delta = \{p \in X | \, \|p\|_1 = 1\},$$

and let Δ_{++} be the relative interior of Δ. For each $p \in \Delta$ the budget set of consumer i is given by

$$B^i(p) = \{x \in X | \, px \leq p\omega^i\}.$$

With this we can define the notion of a $\bar{\Pi}$-equilibrium, which is just the usual notion of equilibrium adapted for Knightian uncertainty:

DEFINITION 3.1. (i) An allocation $\bar{x} = (x^1, \ldots, x^m)$ is said to be feasible if $\sum_i x^i = \sum_i \omega^i$. It together with a price vector $p \in \Delta$, is said to be a $\bar{\Pi}$-equilibrium if it is feasible and $x'^i \succ_i x^i$ implies $x'^i \notin B^i(p)$ for every i.
(ii) A feasible allocation $\bar{x} = (x^1, \ldots, x^m)$ is said to be Pareto efficient if there exists no other feasible allocation $\bar{x}' = (x'^1, \ldots, x'^m) \in \bar{X}$ such that $x'^i \succ_i x^i$ for every i.

To study properties of $\bar{\Pi}$-equilibria we introduce another type of equilibrium, namely $\bar{\pi}$-equilibrium. To define it, observe that each $\bar{\pi} = (\pi^1, \ldots, \pi^m) \in \bar{\Pi}$ induces for every consumer a complete preference, denoted $\succcurlyeq_{\pi^i}$:

$$x^i \succcurlyeq_{\pi^i} x'^i \text{ if, and only if, } U^i_{\pi^i}(x^i) \geq U^i_{\pi^i}(x'^i).$$

Let $\succ_{\pi^i}$ be the asymmetric part of $\succcurlyeq_{\pi^i}$. With these complete preferences we can define a $\bar{\pi}$-equilibrium as follows:

DEFINITION 3.2. A pair $(\bar{x}, p)$ with $\bar{x} = (x^1, \ldots, x^m)$ is said to be a $\bar{\pi}$-equilibrium if $\bar{x}$ is feasible and $x'^i \succ_{\pi^i} x^i$ implies $x'^i \notin B^i(p)$ for evey i.

Concerning the relationship of a $\bar{\Pi}$-equilibrium to a $\bar{\pi}$-equilibrium and the Pareto efficiency of the former, we have according to Carlier and Dana (2013, Theorems 4.3, 3.6):

THEOREM 3.1. (i) $(\bar{x}, p)$ is a $\bar{\Pi}$-equilibrium if, and only if, it is a $\bar{\pi}$-equilibrium for some $\bar{\pi} \in \bar{\Pi}$. (ii) Every $\bar{\Pi}$-equilibrium is Pareto efficient.

We turn next to the indeterminacy of $\bar{\Pi}$-equilibria. We shall use the same notion of indeterminacy as in Geanakoplos and Mas-Colell (1989), but define it for a different object. More precisely, what Geanakoplos and Mas-Colell (1989) concern is the indeterminacy of equilibrium commodity allocations; instead, our concern here is the indeterminacy engendered by the variation in $\bar{\pi}$, and so we shall define it for both commodity allocations and prices. This, incidentally, provides us with a great deal of technical convenience.

To define indeterminacy we pick $\bar{\pi} = (\pi^1, \ldots, \pi^m) \in \bar{\Pi}$ and let $x^i(p, \bar{\pi})$ be a solution for some $p \in \Delta$ to the following problem

$$\max U^i_{\pi^i}(x) \text{ subject to } x \in B^i(p). \tag{3.1}$$

From the assumptions on Π^i and u^i made in Section 2, it follows that $x^i(p, \bar{\pi})$ is smooth in p and $\bar{\pi}$. Let x^i_{-1} be the vector formed by deleting from x^i its first component, and similarly for ω^i. By Walras' law, we have as the 'excess demand function':

$$f(p, \bar{\pi}) = \sum_{i=1}^{m} x^i_{-1}(p, \bar{\pi}) - \sum_{i=1}^{m} \omega^i_{-1}. \tag{3.2}$$

From Debreu (1959) we know that function (3.2) has for each $\bar{\pi} \in \bar{\Pi}$ a (not necessarily unique) zero; let $p(\bar{\pi})$ be the set of zeros of f. Then, by reference to statement (i) of Theorem 3.1, the set of $\bar{\Pi}$-equilibria is given by

$$E_c = \{(\bar{x}, p) \in \bar{X} \times \Delta |\, x^i = x^i(p, \bar{\pi}), p \in p(\bar{\pi}), \bar{\pi} \in \bar{\Pi}\}, \tag{3.3}$$

where the subscript, c, stands for 'complete markets.'With a view to later developments we define in the spirit of Geanakoplos and Mas-Colell (1989) the notion of indeterminacy more generally for an arbitrary set E as follows:

DEFINITION 3.3. The degree of indeterminacy of E is said to be τ if it contains the image of a smooth, one-to-one function whose domain contains an open subset of $\mathbb{R}^\tau$ but does not contain any open subset of $\mathbb{R}^{\tau'}$ with $\tau' > \tau$.

It is to be noted that the notion of indeterminacy thus defined is a purely local construction: For instance, in terms of E_c, it concerns a property of E_c only in a neighborhood of $\bar{\pi} \in \bar{\Pi}$. This point will play an important role in our discussion of indeterminacy.

Intuitively, the degree of indeterminacy of E_c should depend on the dimension of $\bar{\Pi}$. To make precise the latter we just assume that $\bar{\Pi}$ is a manifold of dimension K. Then we have

THEOREM 3.2. The degree of indeterminacy of E_c is K.

PROOF. The proof is similar in spirit to that of Geanakoplos and Mas-Colell (1989, Theorem 1), but technically much simpler, due to the completeness of the market structure. It makes use of some tools from differential topology, for which we refer to Mas-Colell (1989) or Guillemin and Pollack (1974).

Step 1. A pair $(\bar{x}, p) \in E_c$ is said to be induced from $\bar{\pi} \in \bar{\Pi}$ if $p \in p(\bar{\pi})$ and each component x^i of $\bar{x}$ is equal to $x^i(p, \bar{\pi})$. We first show that different $\bar{\pi}$ induces different $(\bar{x}, p)$; more specifically, if $\bar{\pi} \neq \bar{\pi}'$ and $(\bar{x}, p)$, $(\bar{x}', p')$ are induced respectively from them, then $(\bar{x}, p) \neq (\bar{x}', p')$. To see this note that if $p \neq p'$ then we are done. Suppose from now on that $p = p'$, and assume to the contrary that $\bar{x} = \bar{x}'$. Let $\bar{\pi}' = (\pi'^1, \ldots, \pi'^m)$; then $\bar{\pi} \neq \bar{\pi}'$ implies $\pi^i \neq \pi'^i$ for some i. Consider Program 3.1; its first-order condition is given by

$$D_x U^i_{\pi^i}(x) = \lambda p, \; \lambda \text{ being the Lagrangian multiplier.}$$

Recalling the functional form of $U^i_{\pi^i}$ from Section 2, this together with $p = p'$ and $x^i = x'^i$, implies that $\pi^i = \pi^{i'}$, a contradiction.

Step 2. Let $\bar{\omega} = (\omega^1, \ldots, \omega^m)$. We take $\bar{\omega}$ as a parameter in function (3.2) and write it as

$$f(p, \bar{\pi}, \bar{\omega}) = \sum_{i=1}^{m} x^i_{-1}(p, \bar{\pi}) - \sum_{i=1}^{m} \omega^i_{-1};$$

let $f_{\bar{\omega}} = f(p, \bar{\pi}, \bar{\omega})$. Recall that Δ_{++} is the relative interior of Δ. From the assumptions on Π^i and u^i made in Section 2, we have $p \in \Delta_{++}$ for any zero $(p, \bar{\pi}, \bar{\omega})$ of f. In view of the local character of indeterminacy, we may take a neighborhood $\bar{\Pi}_0$ of $\bar{\pi} \in \bar{\Pi}$, which is a manifold (without boundary) of dimension K, and consider f as a function on $\Delta_{++} \times \bar{\Pi}_0 \times \mathbb{R}^{(S+1)L}_{++}$. Then, from Mas-Colell et al. (1995, Proposition 17.D.4, p. 596), the Jacobian matrix, $D_{\bar{\omega}} f(p, \bar{\pi}, \bar{\omega})$, has full rank at any zero $(p, \bar{\pi}, \bar{\omega})$ of f. By the transversality theorem, $f_{\bar{\omega}}$ has zero as a regular value for almost every $\bar{\omega}$, which together with the implicit function theorem implies that $f^{-1}_{\bar{\omega}}(0)$ is a smooth manifold of dimension K.

Take $(p, \bar{\pi})$ from $f^{-1}_{\bar{\omega}}(0)$ and let $O(p, \bar{\pi}) \subset f^{-1}_{\bar{\omega}}(0)$ be its neighborhood. Then, by definition of smooth manifold, there exists a neighborhood $\mathcal{N}$ in $\mathbb{R}^K$ such that $O(p, \bar{\pi})$ is diffeomorphic to $\mathcal{N}$. Denote the diffeomorphism by ϕ and its projections on $\Delta_{++}, \bar{\Pi}$ by ϕ_1, ϕ_2 respectively. We then define $g : \mathcal{N} \to E_c$ such that

$$g(t) = (\bar{x}(\phi(t)), \phi_1(t)).$$

It is obvious that g is smooth. For its injectivity we take two distinct $t_1, t_2 \in \mathcal{N}$. If $\phi_1(t_1) \neq \phi_1(t_2)$ then we are done. Otherwise we have $\phi_2(t_1) \neq \phi_2(t_2)$ as ϕ is a diffeomorphism. This together with Step 1 implies that $g(t_1) \neq g(t_2)$, which therefore proves that E_c contains the image of a smooth, one-to-one function whose domain contains an open subset of $\mathbb{R}^K$.

It remains to show that the domain does not contain any open subset of $\mathbb{R}^{K'}$ with $K' > K$. Suppose otherwise; we may assume without loss of generality that $K' = K + 1$ and the open set contained within the domain is $\mathbb{R}^{K+1}$ itself. Consider $\bar{\Pi}_0 \times \mathbb{R}$; it is obviously a manifold of dimension $K + 1$, and so there is a diffeomorphism φ between $\bar{\Pi}_0 \times \mathbb{R}$ and $\mathbb{R}^{K+1}$. Given a $\bar{\pi} \in \bar{\Pi}_0$, we then have $\varphi(\bar{\pi}, r_1) \neq \varphi(\bar{\pi}, r_2)$ for any distinct $r_1, r_2 \in \mathbb{R}$. Since E_c contains the image of a smooth, one-to-one function on $\mathbb{R}^{K+1}$, this means that, to each $\bar{\pi} \in \bar{\Pi}_0$, there correspond infinitely many equilibria $(\bar{x}, p)$. Note that for a fixed $\bar{\pi}$, the economy in question reduces to a usual one with complete preferences, and so by Mas-Colell (1989, Proposition 5.5.2, p. 188), it has only finitely many equilibria, a contradiction. This completes the proof. *Q.E.D*

Before concluding this section, we wish to make a remark on step one: This step shows the advantage of defining the notion of indeterminacy for $(\bar{x}, p)$ rather than for $\bar{x}$ alone, because a variation in $\bar{\pi}$ may not be able to result in a variation in $\bar{x}$, a phenomenon that will recur in somewhat different guises in the following sections.

4 One-commodity Incomplete Markets

This section concerns the situation of incomplete markets of real assets with one commodity in each state. In the case of complete preferences this special economy is studied by Magill and Quinzii (2002, Chapter 2). The reason for studying it here is that, for one thing, it is interesting in its own right, and, for another, it serves as a preliminary to the developments of the ensuing two sections.

We begin with a description of the market structure. Let $L = 1$, i.e. there is one commodity in each state. At date 0 there occur transactions on both assets and the spot commodity; at date 1 there are spot markets for the commodity in each state. Take the commodity in each state as the numéraire, and denominate the asset returns in terms of them. Let $a_{sj} \in \mathbb{R}_+$ be the return of asset j in state s and $A^j = (a_{1j}, \ldots, a_{Sj})^\top$, where the symbol, $\top$, stands for the transpose of a matrix. Let $A = \left[A^1, \ldots, A^J\right]$, and A_s be its s-th row. As we study incomplete markets, we demand $J < S$ and, without loss of generality, that A be of full rank. Let $q = (q_1, \ldots, q_J) \in \mathbb{R}^J_+$ be a vector of asset prices; then the budget set, $B^i(q)$, of consumer i consists of all $(x^i, \theta^i) \in X \times \mathbb{R}^J$ satisfying

$$\begin{aligned} &x^i_0 - \omega^i_0 + q\theta^i \leq 0 \\ &x^i_s - \omega^i_s - A_s\theta^i \leq 0, s = 1, \ldots, S. \end{aligned} \tag{4.1}$$

We now define the notion of general equilibrium for incomplete markets, or, more briefly, GEI equilibrium. Let $\bar{\theta} = (\theta^1, \ldots, \theta^m) \in \mathbb{R}^{mJ}$ and recall $\bar{x} = (x^1, \ldots, x^m)$.

DEFINITION 4.1. A triple $(\bar{x}, \bar{\theta}; q)$ is said to be a $\bar{\Pi}$-GEI equilibrium if

(i) (x'^i, θ'^i) with $x'^i \succ_i x^i$ implies $(x'^i, \theta'^i) \notin B^i(q)$, $i = 1, \ldots, m$;
(ii) $\sum_i x^i = \sum_i \omega^i$, $\sum_i \theta^i = 0$.

Likewise, a $\bar{\pi}$-GEI equilibrium is defined to be a triple $(\bar{x}, \bar{\theta}; q)$ which satisfies the above two conditions but with $\succ_{\pi^i}$ taking the place of $\succ_i$ in condition (i).

We proceed with a study of the relationship between $\bar{\Pi}$- and $\bar{\pi}$-GEI equilibria. For this we first introduce a functional-theoretic notation (see Magill and Quinzii (2002, Section 13)). Define $T : \mathbb{R}^J \to \mathbb{R}^S$ by

$$T(\theta) = A\theta, \quad \forall \theta \in \mathbb{R}^J$$

and let $T^* : \mathbb{R}^S \to \mathbb{R}^J$ be the adjoint of T. This means that $T(\theta)z = T^*(z)\theta$ for all $(z, \theta) \in \mathbb{R}^S \times \mathbb{R}^J$. For later use let $x^i = (x^i_0, x^i_{\mathbf{1}})$, where $x^i_{\mathbf{1}}$ denotes the consumption bundle in period one, and similarly for $\omega^i = (\omega^i_0, \omega^i_{\mathbf{1}})$. We now state that

THEOREM 4.1. $(\bar{x}, \bar{\theta}; q)$ is a $\bar{\Pi}$-GEI equilibrium if, and only if, it is a $\bar{\pi}$-GEI equilibrium for some $\bar{\pi} \in \bar{\Pi}$.

PROOF. Given $v = (v_1, \cdots, v_S)$ with $v_1 \neq 0$ let $N(v) = (\frac{v_2}{v_1}, \cdots, \frac{v_S}{v_1})$. Suppose $\bar{x} = (x^1, \ldots, x^m)$. From the assumptions of Section 2, it follows that $\bar{x} \gg 0$, whether $(\bar{x}, \bar{\theta}; q)$ is a $\bar{\Pi}$- or a $\bar{\pi}$-GEI equilibrium. With this it is reasonable to set

$$\Pi^i(x^i) = \left\{ N(\nabla U^i_\pi(x^i)) \middle| \ \pi \in \Pi^i \right\},$$

where $\nabla U^i_\pi(x^i)$ is the gradient of $U^i_\pi(x^i)$ at x^i. For a generic element $z = (z_1, \ldots, z_S) \in \Pi^i(x^i)$, z_s represents the marginal rate of substitution between states s and 0; or, alternatively, z can be thought of as a vector of normalized marginal utilities. Evidently, $\Pi^i(x^i)$ is compact and convex.

The theorem follows easily from the following lemma, which can be perceived as an incomplete-market analogue of Carlier and Dana (2013, Lemma 4.1).

LEMMA 4.1. Let $(x^i, \theta^i) \in B^i(q)$ for some $q \in \mathbb{R}^J_+$ with $x^i \gg 0$. The following are equivalent:
(a) (x'^i, θ'^i) with $x'^i \succ_i x^i$ implies $(x'^i, \theta'^i) \notin B^i(q)$;
(b) $q \in T^*(\Pi^i(x^i))$;
(c) there exists a $\pi \in \Pi^i$ such that (x^i, θ^i) solves

$$\max_{(y,\varphi)} U^i_\pi(y) \text{ subject to } (y, \varphi) \in B^i(q). \tag{4.2}$$

To prove the lemma, we begin with (a)⇒(b). Note that, in economic terms, each element of $T^*(\Pi^i(x^i))$ represents a vector of normalized marginal utilities of the assets. Assume to the contrary that (b) were not true. Since $\Pi^i(x^i)$ is compact and convex, so also is $T^*(\Pi^i(x^i))$. By the strong separating hyperplane theorem (see, e.g., Magill and Quinzii (2002, Theorem 9.4, p. 73)), there exists a $\theta \in \mathbb{R}^J$ such that

$$T^*(z)\theta > q\theta, \text{ or } (T^*(z) - q)\theta > 0, \text{ for all } z \in \Pi^i(x^i). \tag{4.3}$$

Take $\Delta x^i = (-q\theta, T(\theta))$. Since $x^i \gg 0$, we can choose $r \in \mathbb{R}_{++}$ small enough such that

$$x'^i = x^i + r\Delta x^i \gg 0, \text{ and } \theta'^i = \theta^i + r\theta.$$

It is obvious that $(x'^i, \theta'^i) \in B^i(q)$. Now we shall apply Carlier and Dana (2013, Lemma 3.2) to show $x'^i \succ_i x^i$. For this let $\bar{\Pi}^i(x^i) = \{(1,z) |\, z \in \Pi^i(x^i)\}$, which corresponds to $V_i(x^i)$ of Carlier and Dana (2013) by taking their $\Phi = (1, 0, \ldots, 0) \in \mathbb{R}^{S+1}$. For every $\bar{z} \in \bar{\Pi}^i(x^i)$, we have according to inequality (4.3) and the identity $T^*(z)\theta = zT(\theta)$ that

$$\bar{z}(-q\theta, T(\theta)) = -q\theta + zT(\theta) > 0.$$

From Carlier and Dana's lemma, which says, in our notation, that if $r\Delta x^i \cdot \bar{z} > 0$ for all $\bar{z} \in \bar{\Pi}^i(x^i)$ then $x^i + r\Delta x^i \succ_i x^i$, we conclude that $x'^i \succ_i x^i$, a contradiction with condition (a). This proves (b).

For (b)⇒(c), take $\pi \in \Pi^i$ such that $q = T^*(N(\nabla U^i_\pi(x^i)))$. This means that the price of each asset is equal to its marginal utility in period one. But this is precisely the first-order condition for Problem (4.2), which proves (c), by referring to the strict quasi-concavity of U^i_π. Finally, (c)⇒(a) is obvious. This completes the proof of the lemma, hence of the theorem also. *Q.E.D*

We turn next to the efficiency of $\bar{\Pi}$-GEI equilibrium. For this we first introduce the concept of constrained Pareto efficiency (see, e.g., Magill and Quinzii (2002, Definitions 12.1, 12.2, pp. 103–104)).

DEFINITION 4.2. An allocation $(x^1, \ldots, x^m)$ is A-feasible if it is feasible and there exists a θ^i such that

$$x^i_1 - \omega^i_1 = T(\theta^i) \text{ for every } i.$$

An A-feasible allocation $(x^1, \ldots, x^m)$ is constrained Pareto efficient if there exists no A-feasible allocation $(x'^1, \ldots, x'^m)$ such that $x'^i \succ_i x^i$ for all i.

Informally, an allocation is constrained Pareto efficient (or optimal) if it is not dominated by any other feasible allocation which is achievable through the market system A. When $J = S$, the markets become complete and every feasible allocation is achievable through A, hence the notion reduces to that of (full) Pareto efficiency.

The constrained Pareto efficiency of $\bar{\Pi}$-GEI equilibria has been shown by Ma (2015); but here, using Theorem 4.1, a simpler proof suggests itself.

THEOREM 4.2. Every $\bar{\Pi}$-GEI equilibrium is constrained Pareto efficient.

PROOF. Let $(\bar{x}, \bar{\theta}; q)$ be a $\bar{\Pi}$-GEI equilibrium with $\bar{x} = (x^1, \ldots, x^m)$. Referring to Theorem 4.1, we know that $(\bar{x}, \bar{\theta}; q)$ is also a $\bar{\pi}$-GEI equilibrium for some $\bar{\pi} = (\pi^1, \ldots, \pi^m) \in \bar{\Pi}$. According to Magill and Quinzii (2002, Theorem 12.3, p. 104), every $\bar{\pi}$-GEI equilibrium is constrained Pareto efficient, which by their Proposition 12.4, can be characterized by the equalities:

$$N(\nabla U^1_{\pi^1}(x^1))A = \cdots = N(\nabla U^m_{\pi^m}(x^m))A;$$

or otherwise put, $T^*(z^1) = \cdots = T^*(z^m)$, where $z^i = N(\nabla U^i_{\pi^i}(x^i)$. This means

$$\bigcap_{i=1}^{m} T^*(\Pi^i(x^i)) \neq \emptyset.^2 \tag{4.4}$$

The theorem then follows by referring to Ma (2015, Proposition 2) which says that $\bar{x}$ is constrained Pareto efficient if and only if (4.4) holds valid. *Q.E.D*

We proceed to study the indeterminacy of $\bar{\Pi}$-GEI equilibria. Observe that in (4.2), once the asset demand θ^i is determined (as a function of q of course), so also is the commodity demand x^i, and therefore it is adequate to study the asset demand alone in this case. For any $\bar{\pi} = (\pi^1, \ldots, \pi^m) \in \bar{\Pi}$ and $\bar{\omega} = (\omega^1, \ldots, \omega^m)$, let $\theta^i(q, \bar{\pi}, \bar{\omega})$ be the solution to the following problem

$$\max U^i_{\pi^i}(x) \text{ subject to } (x, \theta) \in B^i(q), \tag{4.5}$$

which is readily seen to be smooth in $(q, \bar{\pi}, \bar{\omega})$. Let

$$f(q, \bar{\pi}, \bar{\omega}) = \sum_{i=1}^{m} \theta^i(q, \bar{\pi}, \bar{\omega}),$$

and $q(\bar{\pi}, \bar{\omega})$ be the set of zeros of f for each $(\bar{\pi}, \bar{\omega})$. Then the set of $\bar{\Pi}$-GEI equilibria is given by

$$E_o = \{(\bar{\theta}, q) |\ \theta^i = \theta^i(q, \bar{\pi}, \bar{\omega}), q \in q(\bar{\pi}, \bar{\omega}), \bar{\pi} \in \bar{\Pi}\},$$

where the subscript, o, stands for 'one-commodity.'

Compared with the case of complete markets, the indeterminacy of E_o is more subtle here. Note that a variation in $\bar{\pi}$ will lead to a variation in the marginal utilities of commodities. With complete markets, this variation will have to be offset, in order to restore the equilibrium, either by a change in the allocation of commodities or by a change in their prices; hence, no matter what, the equilibrium of the economy will change. In the present case, however, what is essential, as explicated above, is the asset demand. The variation in $\bar{\pi}$ may not be able to change the marginal utilities of assets, and so the asset demand, hence the equilibrium of the economy, might remain unaltered.

To be concrete let us take an example. Let there be two states and one asset whose returns are one unit of the commodity in each state, i.e.

$$S = 2,\ J = 1,\ A = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top.$$

Let $U_\pi(x) = \ln x_0 + \pi_1 \ln x_1 + \pi_2 \ln x_2$ and the endowment $\omega = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$. Let q be the asset price;

[2] The defintion of T^* and of $\Pi^i(x^i)$ here are slightly different from those of Ma (2015). But it is not hard to see that this condition is equivalent to the one in Ma (2015, Proposition 2).

then we have at equilibrium

$$\frac{q}{x_0} = \frac{\pi_1}{x_1} + \frac{\pi_2}{x_2},$$

where the left-hand side denotes the marginal disutility of buying one unit of the asset in period zero, and the right-hand side, the corresponding marginal utility in period one. According to the specification of the economy, we always have $x_1 = x_2$, hence

$$\frac{q}{x_0} = \frac{1}{x_1},$$

which means that the asset demand is independent of (π_1, π_2).

To simplify the discussion we make in the rest of the paper the assumption, that every Π^i contains an interior point in the topology of Δ_S.[3] Under this assumption, from Theorem 3.2, the degree of indeterminacy of E_c is $m(S-1)$. Intuitively, this number, in view of the above considerations, is expected to be lowered by the incompleteness of the markets. The following theorem shows that this is indeed the case:

THEOREM 4.3. The degree of indeterminacy of E_o is mJ.

PROOF. The proof proceeds in two steps and resembles that of Theorem 3.2. In the first step, we show that there exists an open subset of $\bar{\Pi}$, of dimension mJ, such that $\bar{\pi} \neq \bar{\pi}'$ implies that for $q \in q(\bar{\pi}, \bar{\omega}), q' \in q(\bar{\pi}', \bar{\omega})$

$$(\bar{\theta}(q, \bar{\pi}, \bar{\omega}), q) \neq (\bar{\theta}(q', \bar{\pi}', \bar{\omega}), q');$$

and any other open subset of higher dimension will not do. To see this, note that $\bar{\pi} \neq \bar{\pi}'$ means $\pi^i \neq \pi'^i$ for some i. In the following we focus on consumer i. Let

$$x = \xi(q, \bar{\pi}, \bar{\omega})$$

be her commodity demand at the equilibrium asset price vector $q \in q(\bar{\pi}, \bar{\omega})$. Without loss of generality we assume that $\bar{\omega}$ is fixed and $\bar{\pi}$ is also fixed except its i-th component π^i. This enable us to write $x = \xi(\pi^i)$, $\pi^i \in \Pi^i$. To simplify the notation we shall use π, π', π^1 etc. to denote a generic element of Π^i. Let $\pi = (\pi_1, \ldots, \pi_S) \in \Pi^i$; then we have at equilibrium

$$\frac{\partial u^i(x)}{\partial x_0} q_j = \sum_s \pi_s \frac{\partial u^i(x)}{\partial x_s} a_{sj}, \; j = 1, \ldots, J. \tag{4.6}$$

Now we show that a variation in π for some $\pi \in \Pi^i$ will lead to a change in the right-hand side member, so that, to restore the equilibrium, there will be an attendant change either in q or in x (hence θ).

For this purpose we take an interior point π of Π^i and let $O(\pi)$ be one if its neighborhoods contained in Π^i. Let $x = \xi(\pi)$, and

$$A_x = \begin{bmatrix} \partial u^i(x)/\partial x_1 & & \\ & \ddots & \\ & & \partial u^i(x)/\partial x_S \end{bmatrix} A.$$

Furthermore, let $\mathrm{sp}(A_x)$ be the column space of A_x and $O^i = O(\pi) \cap \mathrm{sp}(A_x)$, so that $O^i \neq \emptyset$. With these conventions the right-hand side of (4.6) can be written πA_x. For any distinct $\pi^1, \pi^2 \in O^i$ we

[3] This assumption can be weakened to the extent that every Π^i contains an interior point in the topology of aff(Π^i), the affine space of Π^i. It is not very difficult to see that, with this weaker assumption, the following arguments hold still with some small modifications.

have therefore

$$(\pi^1 - \pi^2)A_x \neq 0.$$

For notational convenience let $\tilde{\pi} = (\pi^1, \pi^2)$. Since the expression, $(\pi^1 - \pi^2)A_x$, is continuous in (π^1, π^2, x), there exist a neighborhood $O(\tilde{\pi})$ of $\tilde{\pi}$ and a corresponding neighborhood $O_{\tilde{\pi}}(x)$ of x such that $(\pi' - \pi'')A_{x'} \neq 0$ for all $\pi', \pi'' \in O(\tilde{\pi})$ and $x' \in O_{\tilde{\pi}}(x)$. Let O_c^i be a compact subset of O^i with nonempty interior; then we can cover $O_c^i \times O_c^i$ by finitely many $O(\tilde{\pi})$, say $O(\tilde{\pi}_1), \ldots, O(\tilde{\pi}_n)$. Let $O(x) = \cap_{i=1}^n O_{\tilde{\pi}_i}(x)$ and O_o^i be the interior of O_c^i. We have thus

$$(\pi' - \pi'')A_{x'} \neq 0 \text{ for any } \pi', \pi'' \in O_o^i \text{ and any } x' \in O(x).$$

Recall that $\xi^{-1}(O(x)) = \{\pi \in \Pi^i | \xi(\pi) \in O(x)\}$. Let $\Pi_o^i = O_o^i \cap \xi^{-1}(O(x))$; then $\xi(\pi) \in O(x)$ for any $\pi \in \Pi_o^i$, hence

$$(\pi' - \pi'')A_{x'} \neq 0 \text{ for any } \pi', \pi'' \in \Pi_o^i \text{ and } x' = \xi(\pi'). \tag{4.7}$$

This means that when π' is altered to π'', the right-hand side of (4.6) will alter accordingly, and so there will result a change in the equilibrium (θ, q).

Note that Π_o^i is of dimension J; we now show that any open set, say $\Pi_o'^i$, of higher dimension than J will not do. Namely, (4.7) will not be satisfied if Π_o^i is replaced by $\Pi_o'^i$. To see this note that there exists, as $\Pi_o'^i$ is of higher dimension than J, an interior point π' of $\Pi_o'^i$ satisfying $\pi' \notin \text{sp}(A_x)$. Let $x = \xi(\pi')$ and $\text{sp}(A_x^\perp)$ be the orthogonal complement of $\text{sp}(A_x)$, so that we can write $\pi' = \pi_1' + \pi_2'$, where $\pi_1' \in \text{sp}(A_x)$, $\pi_2' \in \text{sp}(A_x^\perp)$ with $\pi_2' \neq 0$. Pick $\pi'' = \pi_1' + \pi_2''$, that is, π'' differs from π' only in their components in $\text{sp}(A_x^\perp)$; and $\pi'' \in \Pi_o'^i$ when π_2'' is close enough in Euclidean metric to π_2'. Then we have $(\pi' - \pi'')A_x = 0$, which means that the asset demand will remain unaltered when π' is changed to π''. This completes the proof of step 1.

For the second step, let $\bar{\Pi}_o = \Pi_o^1 \times \cdots \times \Pi_o^m$ and we consider f as a function from $\mathbb{R}^J \times \bar{\Pi}_o \times \mathbb{R}^{L(S+1)}$ to $\mathbb{R}^J$. Let us begin by showing that $D_{\omega^1} f(q, \bar{\pi}, \bar{\omega})$ is of full rank for any $(q, \bar{\pi}, \bar{\omega}) \in f^{-1}(0)$. The idea is the same as the one used in the argument of Geanakoplos and Mas-Colell (1989, Lemma 3). Consider consumer 1. For asset j, $j = 1, \ldots, J$, by decreasing ω_0^1 by q_j and increasing ω_s^1 by A_s^j, her commodity demand and demand of other assets than asset j remain unaltered while her demand of asset j reduces by one unit. Let

$$A_q = \begin{bmatrix} -q^\top \\ A \end{bmatrix};$$

then we have $D_{\omega^1} f(q, \bar{\pi}, \bar{\omega})A_q = -\mathbf{I}_J$, where $\mathbf{I}_J$ is the identity matrix of dimension J. Since rank $\mathbf{I}_J = J$, it follows that $\text{rank} D_{\omega^1} f(q, \bar{\pi}, \bar{\omega}) = J$, i.e. $D_{\omega^1} f(q, \bar{\pi}, \bar{\omega})$ is of full rank. Arguing as in the proof of Theorem 3.2, the theorem then follows. *Q.E.D*

5 Incomplete Markets with Nominal Assets

This section concerns the general situation described in Section 2 and assumes that the assets involved are all nominal. By nominal we mean the asset returns are denominated in money, which allows us to specify the asset returns by a matrix $A = (a_{sj}) \in \mathbb{R}_+^S \times \mathbb{R}_+^J$, a_{sj} being the return of asset j in state s. Let A_s be the s-th row of A, i.e. asset returns in state s.

Given a commodity-price vector $p = (p_0, \ldots, p_S) \in \Delta$ and an asset-price vector $q \in \mathbb{R}_+^J$, the budget set of consumer i, $B_n^i(p, q)$, where the subscript, n, stands for 'nominal,'consists of all

$(x^i, \theta^i) \in X \times \mathbb{R}^J$ satisfying

$$\begin{aligned} p_0(x_0^i - \omega_0^i) + q\theta^i \leq 0 \\ p_s(x_s^i - \omega_s^i) - A_s\theta^i \leq 0, s = 1, \dots, S. \end{aligned} \tag{5.1}$$

After this preparation we can define the notions of $\bar{\Pi}$-GEI equilibrium and its counterpart $\bar{\pi}$-GEI equilibrium. Recall $\bar{x} = (x^1, \dots, x^m)$, $\bar{\theta} = (\theta^1, \dots, \theta^m)$.

DEFINITION 5.1. A quadruple $(\bar{x}, \bar{\theta}; p, q)$ is said to be a $\bar{\Pi}$-GEI equilibrium if

(i) (x'^i, θ'^i) with $x'^i \succ_i x^i$ implies $(x'^i, \theta'^i) \notin B_n^i(p, q)$, $i = 1, \dots, m$;
(ii) $\sum_i x^i = \sum_i \omega^i$, $\sum_i \theta^i = 0$.

Likewise, a $\bar{\pi}$-GEI equilibrium is defined to be a quadruple $(\bar{x}, \bar{\theta}; p, q)$ which satisfies the above two conditions but with $\succ_{\pi^i}$ taking the place of $\succ_i$ in condition (i).

Concerning their interrelation we state that

THEOREM 5.1. $(\bar{x}, \bar{\theta}; p, q)$ is a $\bar{\Pi}$-GEI equilibrium if, and only if, it is a $\bar{\pi}$-GEI equilibrium for some $\bar{\pi} \in \bar{\Pi}$.

PROOF. The proof depends on the results of Section 4. For this we take one commodity, say commodity one, from each state and form the vector

$$\hat{x}^i = (x_{01}^i, \dots, x_{S1}^i).$$

Let $\nabla \hat{U}_\pi^i(x^i)$ be the gradient of U_π^i with respect to $\hat{x}^i$, that is,

$$\nabla \hat{U}_\pi^i(x^i) = (\frac{\partial U_\pi^i(x^i)}{\partial x_{01}^i}, \dots, \frac{\partial U_\pi^i(x^i)}{\partial x_{S1}^i}).$$

Recall the definition of $N(\nabla \hat{U}_\pi^i(x^i))$. In the last section where we assume $L = 1$, by normalizing $p_{s1} = 1$ for all s, the quantity, $N(\nabla \hat{U}_\pi^i(x^i))A$, represents the marginal utilities of the assets in period one, so that the equilibrium asset price vector $q = N(\nabla \hat{U}_\pi^i(x^i))A$. Here, with $L > 1$, to find an analogue of that quantity, we have to take into consideration the commodity prices p. For this note that the marginal disutility of buying one unit of asset j in period zero is

$$\frac{\partial U_\pi^i(x^i)}{\partial x_{01}^i} \frac{q_j}{p_{01}},$$

and the marginal utility it yields in period one is

$$\sum_s \frac{\partial U_\pi^i(x^i)}{\partial x_{s1}^i} \frac{a_{sj}}{p_{s1}},$$

so that at equilibrium

$$q_j = \sum_s \frac{\partial U_\pi^i(x^i)}{\partial x_{s1}^i} (\frac{\partial U_\pi^i(x^i)}{\partial x_{01}^i})^{-1} \frac{p_{01}}{p_{s1}} a_{sj}.$$

With this understood it becomes clear how the commodity prices come in. Specifically let $N(\nabla \hat{U}_\pi^i(x^i)) = (v_1, \dots, v_S)$ and define

$$N_p(\nabla \hat{U}_\pi^i(x^i)) = (v_1 \frac{p_{01}}{p_{11}}, \dots, v_S \frac{p_{01}}{p_{S1}}).$$

Let $\hat{\Pi}_p^i(x^i) = \left\{ N_p(\nabla \hat{U}_\pi^i(x^i)) \middle| \ \pi \in \Pi^i \right\}$, which is easily seen to be compact and convex.

Recall that u^i is a function on $\mathbb{R}^L$ and $x^i_s \in \mathbb{R}^L$; let $\nabla u^i(x^i_s)$ be the gradient of u^i at x^i_s, and $P(x^i)$ the set of equilibrium commodity-price vectors at x^i, that is,

$$P(x^i) = \{(p_0, \ldots, p_S) \in \Delta |\ \nabla u^i(x^i_s) = \lambda_s p_s,\ \lambda_s > 0\}.$$

Let T, T^* have the same meaning as in Section 4. The theorem then follows easily from this lemma:
LEMMA 5.1. Let $(x^i, \theta^i) \in B^i_n(p,q)$ with $x^i \gg 0$; then the following are equivalent
(a) (x'^i, θ'^i) with $x'^i \succ_i x^i$ implies $(x'^i, \theta'^i) \notin B^i_n(p,q)$;
(b) $p \in P(x^i)$, $q \in T^*(\hat{\Pi}^i_p(x^i))$;
(c) there exists a $\pi \in \Pi^i$ such that (x^i, θ^i) solves

$$\max_{(y,\varphi)} U^i_\pi(y) \text{ s.t. } (y, \varphi) \in B^i_n(p,q). \tag{5.2}$$

The implication (b)⇒(c) is an immediate consequence of the fact that (b) is the first-order condition for (5.2), and (c)⇒(a) is a matter of definition. There remains (a)⇒(b). We begin by showing $p \in P(x^i)$. Suppose otherwise; then there exists an s such that $\nabla u^i(x^i_s) \neq \lambda_s p_s$ for all $\lambda_s > 0$, hence x^i_s cannot solve the problem:

$$\max u^i(x_s) \text{ s.t. } p_s x_s - p_s \omega^i_s - A_s \theta^i = 0,$$

wherein θ^i is kept fixed. Let x'^i_s be its solution, so that $u^i(x'^i_s) > u^i(x^i_s)$. Let x'^i be the same as x^i but with x^i_s replaced by x'^i_s, so that $(x'^i, \theta^i) \in B^i_n(p,q)$, and, since $\Pi^i \subset \text{rint}(\Delta_S)$, $U^i_\pi(x'^i) > U^i_\pi(x^i)$ for all $\pi \in \Pi^i$. But this means $x'^i \succ_i x^i$, a contradiction. This proves $p \in P(x^i)$.

It remains to show $q \in T^*(\hat{\Pi}^i_p(x^i))$. The idea is essentially the same as in the proof of Lemma 4.1. Again assume to the contrary that $q \notin T^*(\hat{\Pi}^i_p(x^i))$. Then there exists a $\theta \in \mathbb{R}^J$ such that

$$T^*(z)\theta > q\theta, \text{ or } (T^*(z) - q)\theta > 0, \text{ for all } z \in \Pi^i_p(x^i). \tag{5.3}$$

Choose $r \in \mathbb{R}_{++}$ small enough such that $\hat{x}'^i \gg 0$ and $\theta'^i = \theta^i + r\theta$, where

$$x'^i_{01} = x^i_{01} - rq\theta/p_{01}, x'^i_{s1} = x^i_{s1} + rA_s\theta/p_{s1}.$$

Let x'^i be the same as x^i but with $\hat{x}^i$ replaced by $\hat{x}'^i$. Then it is obvious that $(x'^i, \theta'^i) \in B^i(p,q)$.

Now we show $x'^i \succ_i x^i$. For this let $\hat{U}^i_\pi(\hat{x}^i) = U^i_\pi(\hat{x}^i, x^i_{-1})$, where x^i_{-1} denotes the vector formed by deleting $\hat{x}^i$ from x^i. It suffices to show $\hat{U}^i_\pi(\hat{x}'^i) > \hat{U}^i_\pi(\hat{x}^i)$ for all $\pi \in \Pi^i$. Note that $\hat{U}^i_\pi$ behaves just as if there were only one commodity in each state, and so the rest of the argument is the same as in Lemma 4.1. More specifically let

$$\Pi^i(\hat{x}^i) = \left\{ N(\nabla \hat{U}^i_\pi(\hat{x}^i)) \middle|\ \pi \in \Pi^i \right\},$$

and $\bar{\Pi}^i(\hat{x}^i) = \{(1,z) |\ z \in \Pi^i(\hat{x}^i)\}$. Let $\Delta\hat{x}^i = (-q\theta/p_{01}, A_1\theta/p_{11}, \ldots, A_S\theta/p_{S1})$; then we have according to inequality (5.3), after a little algebra,

$$\hat{z} \cdot \Delta\hat{x}^i > 0 \ \text{ for every } \hat{z} \in \bar{\Pi}^i(\hat{x}^i).$$

From this together with Carlier and Dana (2013, Lemma 3.2) it follows that $x'^i \succ_i x^i$, a contradiction with condition (a). This proves (b). *Q.E.D*

We turn now to the problem of efficiency. As opposed to the one-commodity case of Section 4 we have now to take care of the commodity prices.

DEFINITION 5.2. An allocation $(x^1, \cdots, x^m)$ is A-feasible if it is feasible and there exist p, q, θ^i such that $(x^i, \theta^i) \in B^i_n(p,q)$ for every i. An A-feasible allocation $(x^1, \cdots, x^m)$ is constrained Pareto efficient if there exists no A-feasible allocation $(x'^1, \cdots, x'^m)$ such that $x'^i \succ_i x^i$ for all i.

The efficiency of $\bar{\Pi}$-GEI equilibria in this case becomes rather complicated, depending in large measure upon the consumer's ambiguity, or the size of the sets Π^i. And so we have to content ourselves with results about several special examples. At the extreme of each Π^i being a singleton, we know from Geanakoplos and Polemarchakis (1985) that, when $2(L-1) \leq m < S(L-1)$, $\bar{\Pi}$-GEI equilibria are generically constrained Pareto suboptimal.

For the second example, we note that the condition of every Π^i being a singleton can be relaxed a little bit. More precisely, let $\bar{\pi} = (\pi^1, \ldots, \pi^m)$ with $\pi^i \in \mathrm{rint}(\Delta_S)$ and $\bar{x} = (x^1, \ldots, x^m)$ be a $\bar{\pi}$-GEI equilibrium. Appealing to the generic constrained Pareto suboptimality of $\bar{x}$ we may assume the existence of another A-feasible allocation $(x'^1, \cdots, x'^m)$ such that $U^i_{\pi^i}(x'^i) > U^i_{\pi^i}(x^i)$ for every i. Noting that $U^i_{\pi^i}$ is continuous in π^i, there exists a closed set $O(\pi^i)$ containing π^i such that

$$U^i_{\pi'^i}(x'^i) > U^i_{\pi'^i}(x^i) \text{ for every } \pi'^i \in O(\pi^i).$$

Let $\Pi^i = O(\pi^i)$. Then we see that $\bar{x}$ is a $\bar{\Pi}$-GEI equilibrium and it is constrained Pareto suboptimal.

Of course, constrained suboptimality is not always the case, as can be seen from the following trivial example. Let $S = 3, J = 2$, and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

If every consumer believes that the third state is impossible to occur, then the prevailing equilibria will be Pareto efficient, because the markets now are in effect complete.

So much for efficiency. Let us pass to the problem of indeterminacy. As usual, for any $\bar{\pi} = (\pi^1, \ldots, \pi^m) \in \bar{\Pi}$ and $\bar{\omega} = (\omega^1, \ldots, \omega^m)$ let

$$(x^i(p,q,\bar{\pi},\bar{\omega}), \theta^i(p,q,\bar{\pi},\bar{\omega})) = \arg\max\{U^i_{\pi^i}(x) |\, (x,\theta) \in B^i_n(p,q)\}. \tag{5.4}$$

Let x^i_* be the same vector as x^i but with commodity one in every state removed and similarly for ω^i_*. Let

$$f(p,q,\bar{\pi},\bar{\omega}) = (\sum_{i=1}^m x^i_{-1}(p,q,\bar{\pi},\bar{\omega}) - \omega^i, \sum_{i=1}^m \theta^i(q,\bar{\pi},\bar{\omega})).$$

By Walras' law the set of $\bar{\Pi}$-GEI equilibria is given by

$$E_n = \{(\bar{x},\bar{\theta};p,q) |\, (x^i,\theta^i) \text{ solves (5.4)}, (p,q) \text{ is a zero of } f \text{ for } (\bar{\pi},\bar{\omega}) \in \bar{\Pi} \times \bar{X}\}.$$

Our task is to determine the indeterminacy of E_n. Recall the assumption made in Section 4 on Π^i. Note that, in contrast to Geanakoplos and Mas-Colell (1989), which is concerned with the indeterminacy of equilibrium commodity allocations, the set E_n contains the commodity and asset price vectors, and so it is not reasonable to limit p in Δ. For this reason we assume in the rest of this section that $p \in \mathbb{R}^{L(S+1)}_{++}$. Then we have that

THEOREM 5.2. The degree of indeterminacy of E_n is $S + 1 + mJ$.

PROOF. To begin with we shall show the existence of an open subset $\bar{\Pi}_o$ of $\bar{\Pi}$ such that different $\bar{\pi}$ in it will lead to different $\bar{\pi}$-equilibria. For this suppose $(\bar{x},\bar{\theta};p,q)$ is a $\bar{\pi}$-GEI equilibrium. Then we have

$$\frac{\partial u^i(x^i_0)}{\partial x^i_{01}} \frac{q_j}{p_{01}} = \sum_s \pi_s \frac{\partial u^i(x^i_s)}{\partial x^i_{s1}} \frac{a_{sj}}{p_{s1}}.$$

Let

$$A_{x,p} = \begin{bmatrix} \frac{\partial u^i(x^i_s)}{\partial x^i_{s1}} \frac{1}{p_{s1}} & & \\ & \ddots & \\ & & \frac{\partial u^i(x^i_S)}{\partial x^i_{S1}} \frac{1}{p_{S1}} \end{bmatrix} A.$$

Substituting A_x in the proof of Theorem 4.3 for $A_{x,p}$ and following the argument there we will arrive at the desired conclusion.

Now consider f as a mapping from $\mathbb{R}^{L(S+1)+J} \times \bar{\Pi}_o \times \mathbb{R}^{mL(S+1)}$ to $\mathbb{R}^{(L-1)(S+1)+J}$. From Geanakoplos and Polemarchakis (1985) follows that the matrix $\partial_{\omega^1} f$ has full rank at any zero of f. This means, by the transversality theorem, that zero is a regular value of $f_{\bar{\omega}}$ for almost every $\bar{\omega}$, where $f_{\bar{\omega}} = f(\cdot,\cdot,\cdot,\bar{\omega})$. Using the implicit function theorem we have that $f^{-1}_{\bar{\omega}}(0)$ is a smooth manifold of dimension $S+1+mJ$. The rest of the argument is then identical with the corresponding part in the proof of Theorem 3.2. *Q.E.D*

6 Incomplete Markets with Real Assets

This section concerns the same situation as in the last section, but replacing nominal assets there by real ones. By real assets we mean those whose returns are denominated in terms of the commodities in each state. The results about this situation can be derived more or less directly from their counterparts in the preceding sections, and so we shall outline them only without attempting to present the details.

We begin with a description of the asset returns. Since the returns are denominated in commodities, the return of an asset in each state can be formally represented by a vector of dimension L, hence that in period one by a vector of dimension LS. Let $\bar{L} = LS$; then the asset market structure is expressible by a matrix A, of dimension $\bar{L} \times J$, its j-column designating the returns of asset j. For each $p = (p_0, \ldots, p_S) \in \mathbb{R}^{L(S+1)}_{++}$ let

$$\Lambda_p = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_S \end{bmatrix}_{S \times \bar{L}} ;$$

the nominal return matrix at the price vector p of the assets is then given by

$$V(p) = \Lambda_p A \in \mathbb{R}^{S \times J}. \tag{6.1}$$

Let $V_s(p)$ be the s-th row of $V(p)$. The budget set of consumer i, $B^i_r(p,q)$, where the subscript, r, stands for 'real,'then consists of all $(x^i, \theta^i) \in X \times \mathbb{R}^J$ which satisfy (5.1) with $V_s(p)$ taking the place of A_s, that is

$$\begin{aligned} &p_0(x^i_0 - \omega^i_0) + q\theta^i \leq 0 \\ &p_s(x^i_s - \omega^i_s) - V_s(p)\theta^i \leq 0, s = 1, \ldots, S. \end{aligned} \tag{6.2}$$

It is readily seen that these budget constraints are homogenous of degree zero in (p,q), and therefore we may assume in this section that $p \in \mathbb{R}^{L(S+1)}_{++}$ with $p_{s1} = 1$ for every s. This normalization will provide us with some convenience in the proof of Theorem 6.1. Now we can define the notions of $\bar{\Pi}$-GEI equilibrium and its counterpart $\bar{\pi}$-GEI equilibrium. Recall $\bar{x} = (x^1, \ldots, x^m)$,

$\bar{\theta} = (\theta^1, \ldots, \theta^m)$.

DEFINITION 6.1. A quadruple $(\bar{x}, \bar{\theta}; p, q)$ is said to be a $\bar{\Pi}$-GEI equilibrium if

(i) (x'^i, θ'^i) with $x'^i \succ_i x^i$ implies $(x'^i, \theta'^i) \notin B_r^i(p,q)$, $i = 1, \ldots, m$;
(ii) $\sum_i x^i = \sum_i \omega^i$, $\sum_i \theta^i = 0$.

Likewise, a $\bar{\pi}$-GEI equilibrium is defined to be a quadruple $(\bar{x}, \bar{\theta}; p, q)$ satisfying the above two conditions with $\succ_{\pi^i}$ taking the place of $\succ_i$ in condition (i).

Concerning their interrelation we state that

THEOREM 6.1. $(\bar{x}, \bar{\theta}; p, q)$ is a $\bar{\Pi}$-GEI equilibrium if, and only if, it is a $\bar{\pi}$-GEI equilibrium for some $\bar{\pi} \in \bar{\Pi}$.

PROOF. The analysis is similar to that in the proof of Theorem 5.1. More specifically, define $T : \mathbb{R}^J \to \mathbb{R}^{\bar{L}}$ as $T(\theta) = A\theta$, $\forall \theta \in \mathbb{R}^J$, and let $T^* : \mathbb{R}^{\bar{L}} \to \mathbb{R}^J$ be its adjoint. Again let us begin with the equilibrium asset prices. From marginal analysis we have at equilibrium

$$\frac{\partial U_\pi^i(x^i)}{\partial x_{01}^i} q_j = \sum_{s=1}^{S} \sum_{l=1}^{L} \frac{\partial U_\pi^i(x^i)}{\partial x_{sl}} b_{slj},$$

where b_{slj} denotes the units of commodity l that asset j pays off in state s, and so

$$q_j = \sum_{s=1}^{S} \sum_{l=1}^{L} \frac{\partial U_\pi^i(x^i)}{\partial x_{sl}} \left(\frac{\partial U_\pi^i(x^i)}{\partial x_{01}^i}\right)^{-1} b_{slj}, j = 1, \ldots, J. \tag{6.3}$$

Recall from Section 4 the definition of $N(\nabla U_\pi^i(x^i))$ and let $N_{\mathbf{1}}(\nabla U_\pi^i(x^i))$ be the vector formed by deleting from $N(\nabla U_\pi^i(x^i))$ the components in state 0. Let

$$\Pi^i(x^i) = \left\{ N(\nabla U_\pi^i(x^i)) \middle| \; \pi \in \Pi^i \right\}, \Pi_{\mathbf{1}}^i(x^i) = \left\{ N_{\mathbf{1}}(\nabla U_\pi^i(x^i)) \middle| \; \pi \in \Pi^i \right\}.$$

With this notation equation (6.3) can be written compactly as $q = z_{\mathbf{1}} A$ for some $z_{\mathbf{1}} \in \Pi_{\mathbf{1}}^i(x^i)$.

Similarly as in Section 5 we define $P(x^i)$ to be the set of commodity price vectors $(p_0, \ldots, p_S)$ such that $\nabla u^i(x_s^i) = \lambda_s p_s$ for some $\lambda_s > 0$. Then it is sufficient to prove this lemma:
LEMMA 6.1. Let $(x^i, \theta^i) \in B_r^i(p,q)$ with $x^i \gg 0$; then the following are equivalent
(a) (x'^i, θ'^i) with $x'^i \succ_i x^i$ implies $(x'^i, \theta'^i) \notin B_r^i(p,q)$;
(b) $p \in P(x^i)$, $q \in T^*(\Pi_{\mathbf{1}}^i(x^i))$;
(c) there exists a $\pi \in \Pi^i$ such that (x^i, θ^i) solves

$$\max_{(y,\varphi)} U_\pi^i(y) \text{ s.t. } (y, \varphi) \in B_r^i(p,q). \tag{6.4}$$

To prove this lemma note that the proofs of (b)⇒(c), (c)⇒(a), and (a)⇒ $p \in P(x^i)$, are exactly the same as those of Lemma 5.1, and so are omitted here. Therefore it remains (a)⇒ $q \in T^*(\Pi_{\mathbf{1}}^i(x^i))$, which again is similar to (although not identical with) the corresponding part in the proof of Lemma 5.1.

More precisely suppose $q \notin T^*(\Pi_{\mathbf{1}}^i(x^i))$. Then there exists a $\theta \in \mathbb{R}^J$ such that

$$T^*(z_{\mathbf{1}})\theta > q\theta, \text{ or } (T^*(z_{\mathbf{1}}) - q)\theta > 0, \text{ for all } z_{\mathbf{1}} \in \Pi_{\mathbf{1}}^i(x^i). \tag{6.5}$$

We construct (x'^i, θ'^i) such that $x'^i = x^i + r\Delta x$, $\theta'^i = \theta^i + r\theta^i$, where $\Delta x = (-q\theta, 0, \ldots, 0, (A\theta)^\top) \in \mathbb{R}^{L(S+1)}$ and $r \in \mathbb{R}_{++}$ is small enough such that $x'^i \gg 0$. It is obvious that $(x'^i, \theta'^i) \in B_r^i(p,q)$.

To derive a contradiction it suffices to show that $x'^i \succ_i x^i$. For this purpose take $z \in \Pi^i(x^i)$; then according to inequality (6.5)

$$z \cdot \Delta x = -q\theta + z_{\mathbf{1}} A\theta > 0,$$

where $z = (z_{\mathbf{0}}, z_{\mathbf{1}})$ and $z_{\mathbf{1}} \in \Pi^i_{\mathbf{1}}(x^i)$. So, by referring to Carlier and Dana (2013, Lemma 3.2), $x'^i \succ_i x^i$. *Q.E.D*

Finally we turn to the problems of efficiency and indeterminacy. The solving of these two problems, as seen from the previous sections, requires tools from differential topology, the application of which in turn necessitates the smoothness of commodity and asset demand functions. This smoothness however is not at our disposal, because of the rank-dropping behavior of $V(p)$ (see, e.g., Hart (1975)).

One way of avoiding the difficulty is, as in Geanakoplos and Mas-Colell (1989), to study a special kind of real assets, namely those whose returns are paid uniformly in, say, commodity one of each state. In this case, the problem of efficiency has the same nature as in the case of nominal assets; in particular, the equilibria are generically constrained Pareto suboptimal when the preferences involved are complete or the consumers have a 'small'degree of ambiguity. As for indeterminacy, the problem is the same as in the case of Section 4, because the indeterminacy of equilibria depends, not on the number of commodities, but on the Walras' law, the homogeneity of demand, and the consumers' ambiguities in the likelihoods of the states.

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