Mathematics Winter School worksheet

CALCULUS

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QUESTION 1

1.1 Given: $f(x) = 3x - x^2$

Determine $f'(x)$ from first principles. (5)

1.2 Determine $\frac{dy}{dx}$ if $y = \left(\frac{2}{4x^4} + \frac{4x^4}{2}\right)^2$

Leave your answer with positive exponents. (4)

1.3 Given: $f(x) = \frac{2+3\sqrt{x}}{\sqrt{x}}$

1.3.1 Find the value of $f(4)$. (1)

1.3.2 Find the value of $f'(4)$. (3)

1.3.3 Determine the equation of the tangent to f at $x = 4$ in the form $y = \cdots$ (2)

[15]

QUESTION 2

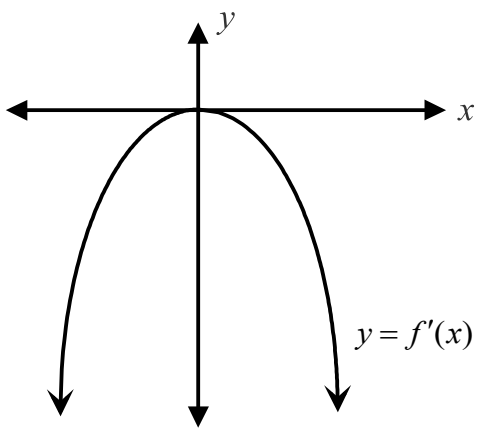
2.1 The graph of $f(x) = x^3 - 4x^2 + 4x$ has a local minimum at $(a; 0)$ and a local maximum at $(b; c)$.

2.1.1 Determine the value of *a*, *b* and *c*. (6)

2.1.2 Draw a neat sketch graph of *f*. (3)

2.1.3 For which values of *k* will $f(x) = x^3 - 4x^2 + 4x + k$ have one real root? (2)

2.2 The sketch below shows the graph of $y = f'(x)$, the derivative of a cubic function $y = f(x)$ It is further given that $f(0) = -5$ and $f(-2) = 0$.



2.2.1 Write down the coordinates of the point of inflection of f. (1)

2.2.2 Write down the coordinates of the point where f cuts the *x*-axis. (1)

2.2.3 For which values of x is the graph of f decreasing? (1)

[14]

QUESTION 3

A rectangular container must have a volume of $1\ \text{m}^2$. The base is a square with each side x- metres in length. The height of the container is h metres. The container has no lid.
The material for the base costs R75 per square metre and the material for the sides cost R50 per square metre.
The volume of a rectangular box is given by the formula: $V = x^2h$.

Show that for a minimum cost, the dimensions of the base must be $1{,}1\ \text{m} \times 1{,}1\ \text{m}$. **[6]**

QUESTION 4

4.1 Given: $f(x) = \frac{x^3-1}{x-1}$

4.1.1 Calculate the average gradient of f between $x = 2$ and $x = 3$. (4)

4.1.2 Show that $f(x)$ can be simplified to the expression $x^2 + x + 1$. (2)

4.1.3 Hence, by using first principles, determine the gradient of the tangent to the graph of f at $x = 2$. (6)

4.1.4 Determine the equation of the tangent to f at $x = 2$. (4)

4.1.5 Determine the values of x for which f is increasing. (3)

4.2 Given: $y = \frac{x^2}{2} + \frac{2\sqrt{x^5}}{x^2} - \frac{\pi}{x^3}$

Determine $\frac{dy}{dx}$ leaving answers with positive exponents. (6)

[25]

QUESTION 5

The graph of $f(x) = x^3 + ax + b$ has a turning point at $(-1\,;0)$.

5.1 Show that $a = -3$ and $b = -2$. (6)

5.2 Show that the point of inflection of f is the same as the y −intercept. (2)

5.3 Determine the values of x for which the graph of f is concave down. (1)

5.4 Sketch the graph of $y = f(x)$ showing the coordinates of the intercepts with the axes, the turning points and the point of inflection. (7)

5.5 For which values of k will $f(x) = k$ have three different roots? (1)

[17]

QUESTION 6

For a certain function *f*, the first derivative is given as $f'(x) = 3x^2 + 8x - 3$.

6.1 Calculate the x-coordinates of the stationary points of f. (3)

6.2 Determine the values of x for which f is strictly increasing. (3)

6.3 If it is further given that $f(x) = ax^3 + bx^2 + cx + d$ and $f(0) = -18$, Determine the equation of f. (5)

[11]

QUESTION 7

A petrol tank at a petrol depot has both the inlet and the outlet pipes which are used to control the amount of petrol it contains. The depth of the tank is given by $\mathrm{D}(t) = 6 + \frac{t^2}{4} - \frac{t^3}{8}$ where D is in metres and t is in hours that are measured from 09h00.

7.1 Determine the rate at which the depth is changing at 12h00, and then state whether there is and increase or decrease in depth. Answer correct to two decimal digits. (3)

7.2 At what time other than 09h00 will the inflow of petrol be the same as the outflow? (3)

[6]

QUESTION 8

A piece of wire 6 metres long is cut into two pieces. One piece, *x* metres long, is bent to form a square ABCD. The other piece is bent into a U-shape so that it forms a rectangle BEFC when placed next to the square, as shown in the diagram below.

8.1 Show that the sum of the areas enclosed by the wire is given by:

$$A(x) = \frac{24x - 3x^2}{32}$$ (7)

8.2 Calculate the value of *x* for which the sum of the areas will be a maximum. (4)

[11]

QUESTION 9

9.1 Given: $f(x) = 3x - x^2$

9.1.1 Determine $f'(x)$ from first principles. (5)

9.1.2 If $y = f(x)$ represents the graph of the derivative of a cubic function *g*, determine:

(a) the x –values of the turning points of g. (3)

(b) the x –value of the point of inflection of g. (2)

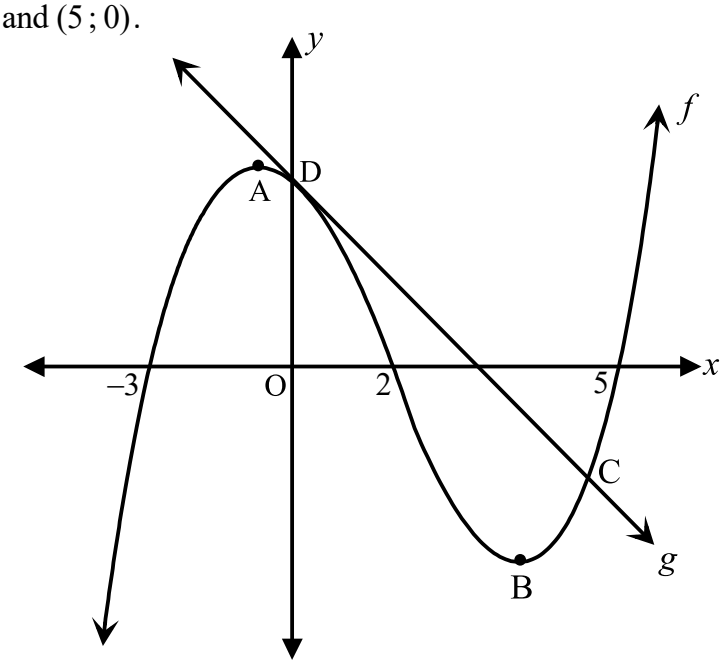
9.2 Determine $\frac{dy}{dx}$ if $y = \left(1 - \sqrt{x}\right)\left(1 - \frac{2}{\sqrt{x}}\right)$

Leave your answer with positive exponents. (4)

[14]

QUESTION 10

The graph defined by $f(x) = x^3 + ax^2 + bx + c$ is shown along with its tangent $g(x) = mx + c$. Both graphs cut the *y*-axis at D. The graph of f cuts the x −axis at $(-3\,;0), (2\,;0)$ and $(5\,;0)$.



10.1 Show that $a = -4, b = -11$ and $c = 30$. (4)

10.2 Determine the coordinates of A, the turning point of f. (4)

10.3 Determine the equation of g. (2)

10.4 Determine the values of x for which:

10.4.1 $f'(x) < 0$ (2)

10.4.2 $f''(x) > 0$ (3)

10.5 For which values of k will $f(x) = g(x) + k$ have non-real roots? (1)

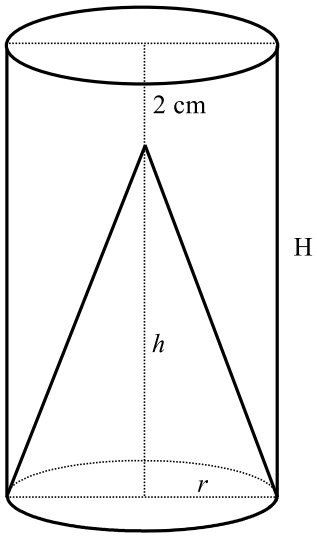
[16]

QUESTION 11

In the diagram below, a cone is cut out of a cylindrical piece of wood. The cone and the cylinder have equal radii and the height of the cylinder is 2 cm more than the height of the cone. The ratio of the radius to the height of the cylinder is 1:3.

Volume of a right circular cone $V = \frac{1}{3}\pi r^2 h$

All measurements are in cm.



11.1 Show that the volume of wood left over, after the cone has been cut out of the cylinder is given by:

$$V = 2\pi r^3 + \frac{2}{3}\pi r^2$$ (5)

11.2 Determine the value of *r* if the rate of change of the volume of the wood with respect to *r* is $498\pi\ cm^3/cm$. (5)

[10]

QUESTION 12

Sketch the graph of the cubic function $f(x)$ which has the following characteristics:

$f(0) = 2$

$f'(x) > 0$ for $x \neq 0$

$f''(x) \leq 0$ for $x \leq 0$

$f(-2) = 0$ (5)