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SOBOLEV TYPE MATHEMATICAL MODELS WITH RELATIVELY POSITIVE OPERATORS IN THE SEQUENCE SPACES

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$$Lu_t = Mu + f, \ \ker L \neq \{0\}$$

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Relatively p-bounded operators

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ (i.e. linear and continuous), $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$ (i.e. linear, closed and densely defined). Sets $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ and $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ are called *L-resolvent set* and *L-spectrum of operator M* respectively. Operator M is (L,σ)-*bounded*, if

$$\exists a \in \mathbb{R}_+ \ \forall \mu \in \mathbb{C} \ (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

If operator is (L,σ)-bounded, then operators

$$P = \frac{1}{2\pi i}\int_\gamma R^L_\mu(M) d\mu \in \mathcal{L}(\mathfrak{U}), \ \ Q = \frac{1}{2\pi i}\int_\gamma L^L_\mu(M) d\mu \in \mathcal{L}(\mathfrak{F}).$$

are the projectors. Here $R^L_\mu(M) = (\mu L - M)^{-1}L$ is called a *right L-resolvent*, and $L^L_\mu(M) = L(\mu L - M)^{-1}$ is called a *left L-resolvent of operator M*; contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$. Here and below, all integrals are understood in the sense of Riemann.

Relatively p-bounded operators

We consider subspaces $\mathfrak{U}^0 = \ker P$, $\mathfrak{U}^1 = \operatorname{im} P$, $\mathfrak{F}^0 = \ker Q$, $\mathfrak{F}^1 = \operatorname{im} Q$; and denote operator of the contraction L (M) on $\mathfrak{U}^k(\mathfrak{U}^k \cap \operatorname{dom} M)$ by L_k (M_k), $k = 0, 1$.

Theorem 1.1.

Let operator M (L, σ)-bounded. Then
(i) operators $L_k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$; and there exist the operator $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$;
(ii) operators $M_k \in \mathcal{C}l(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$; and there exist the operator $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$.

Let operator M be (L, σ)-bounded, construct the operator $H = M_0^{-1} L_0$, $H \in \mathcal{L}(\mathfrak{U}^0)$. Operator M is called (L, p)-*bounded*, $p \in \mathbb{N}$ ($(L, 0)$-*bounded*), if $H^p \neq \mathbb{O}$, a $H^{p+1} = \mathbb{O}$ $(H = \mathbb{O})$.

Phase space

Let operator M be (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$, we consider the equation

$$L\dot{u} = Mu. \tag{1.1}$$

Vector function $u = u(t)$, $t \in \mathbb{R}$, is *solution of equation (1.1)* if it satisfies this equation. Decision $u = u(t)$ is called *solution of the Cauchy problem*

$$u(0) = u_0, \tag{1.2}$$

if it satisfies condition (1.2) at some $u_0 \in \mathfrak{U}$.

The set $\mathfrak{P} \subset \mathfrak{U}$ is *phase space* of equation (1.1) if its any solution $u(t) \in \mathfrak{P}$ at each $t \in \mathbb{R}$; and for any $u_0 \in \mathfrak{P}$ there exists a unique solution $u \in C^1(\mathbb{R}; \mathfrak{U})$ of problem (1.2) for equation (1.1).

Phase space

Finally, we introduce a degenerate (if $\ker L \neq \{0\}$) holomorphic (in the whole plane $\mathbb{C}$) group of operators

$$U^t = \frac{1}{2\pi i} \int_\gamma R^L_\mu(M) e^{\mu t} d\mu, \;\; t \in \mathbb{C}.$$

Notice, that $U^0 = P$, where $\ker P \supset kerL$.

Theorem 1.2.

Let operator M be (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then
(i) any solution $u \in C^1(\mathbb{R}; \mathfrak{U})$ of equation (1.1) has the form $u(t) = U^t u_0$, $t \in \mathbb{R}$, and some $u_0 \in \mathfrak{U}$;
(ii) the phase space of equation (1.1) is subspace $\mathfrak{U}^1$.

Degenerate holomorphic group of operators

Thus, under the conditions of the theorem 1.2 L-resolvent $(\mu L - M)^{-1}$ of operator M in the ring $|\mu| > a$ decomposes into a Laurent series

$$(\mu L - M)^{-1} = \sum_{k=1}^{\infty} \mu^{-k} S^{k-1} L_1^{-1} Q - \sum_{k=0}^{p} \mu^k H^k M_0^{-1}(\mathbb{I} - Q),$$

where operators $S = L_1^{-1} M_1 \in \mathcal{L}(\mathfrak{U}^1)$, $H = M_0^{-1} L_0 \in \mathcal{L}(\mathfrak{U}^0)$. Hence the resolving degenerate group U^t of equation (1.1) is as follows

$$U^t = (\mathbb{I} - Q) + e^{St} Q,$$

where

$$e^{St} = \frac{1}{2\pi i} \int_\gamma (\mu \mathbb{I} - S) e^{\mu t} d\mu = \sum_{k=0}^{\infty} \frac{(St)^k}{k!}$$

is the group of operators of equation (1.1), given on the phase space $\mathfrak{U}^1$.

Banach lattice

Next, we give an order relation $\ll\geq\gg$, compatible with both vector and metric structures, to $\mathfrak{U}^1$. In other words, we assume that $(\mathfrak{U}^1;\geq)$ is a *Banach lattice*.

Spaces $C(\Omega)$, $C(\overline{\Omega})$ and $L_q(\Omega)$, as well as space l_q, where domain $\Omega\subset\mathbb{R}^n$, $q\in[1,+\infty]$ are examples of Banach lattices.

Further, let $\mathfrak{X}$ is vector space. Convex set $\mathfrak{C}\subset\mathfrak{X}$ we call a *cone* if
(ic) $\mathfrak{C}+\mathfrak{C}\subset\mathfrak{C}$;
(iic) $\alpha\mathfrak{C}\subset\mathfrak{C}$ for any $\alpha\in\{0\}\cup\mathbb{R}_+$;
(iiic) $\mathfrak{C}\cap(-\mathfrak{C})=\{0\}$.

The cone $\mathfrak{C}$ is called *generative* if
(ivc) $\mathfrak{C}-\mathfrak{C}=\mathfrak{X}$.

Degenerate positive holomorphic groups of operators

Let $\mathfrak{X}$ be Banach lattice with generative cone $\mathfrak{X}_+$. Linear bounded operator $A \in \mathcal{L}(\mathfrak{X})$ is *positive* if $Au \geq 0$ for all $u \in \mathfrak{X}_+$. Holomorphic group of operators $X^\cdot = \{X^t : X^t \in \mathcal{L}(\mathfrak{X}) \text{ for all } t \in \mathbb{R}\}$ is called *positive* if $X^t u \geq 0$ for all $u \in \mathfrak{X}_+$ and $t \in \mathbb{R}$.

Finally, let us return to the abstract problem

$$L\dot{u} = Mu, \tag{1.1}$$

$$u(0) = u_0. \tag{1.2}$$

We will be interested in its *positive solution* $u = u(t)$ i.e. such that $u(t) \geq 0$ for all $t \in \mathbb{R}$. Therefore, we consider the phase space of equation (1.1) $\mathfrak{U}^1$ Banach lattice, generated by a cone $\mathfrak{U}^1_+$.

Degenerate positive holomorphic groups of operators

(L,p)-bounded operator M is *positive (L,p)-bounded*, $p \in \{0\} \cup \mathbb{N}$, if $Su \in \mathfrak{U}^1_+$ for any $u \in \mathfrak{U}^1_+$.

The degenerate holomorphic group $U^\cdot \in C^\infty(\mathbb{R}; \mathcal{L}(\mathfrak{U}))$, generated by positive (L,p)-bounded operator M is called a *degenerate positive holomorphic group*.

Theorem 1.3.

Let operator M is positive (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then for any $u_0 \in \mathfrak{U}^1_+$ there is the unique positive solution $u = u(t)$, $t \in \mathbb{R}$, of problem (1.1), (1.2), and it has the form $u(t) = U^t u_0$.

Showalter – Sidorov problem

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$, and operator M is (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Consider a linear inhomogeneous equation of Sobolev type

$$L\dot{u} = Mu + f. \tag{2.1}$$

Vector function $u \in C([0,\tau);\mathfrak{U}) \cap C^1((0,\tau);\mathfrak{U})$, $\tau \in \mathbb{R}_+$, is called *solution of equation* (2.1) if it satisfies this equation for some $f = f(t)$. The solution $u = u(t)$ of equation (2.1) is called *solution of the Showalter – Sidorov problem*[a]

$$P(u(0) - u_0) = 0, \tag{2.2}$$

if it also satisfies the initial condition (2.2). Here $P : \mathfrak{U} \to \mathfrak{U}^1$ along $\mathfrak{U}^0$ is projector. Further, let $\mathfrak{U}$ be a Banach lattice generated by the cone $\mathfrak{U}_+$. The solution $u = u(t)$ of problem (2.1), (2.2) is *positive* if $u(t) \in \mathfrak{U}_+$ for any $t \in [0,\tau)$.

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[a] Sviridyuk G.A., Zagrebina S.A. The Showalter–Sidorov problem as a phenomena of the Sobolev-type equations. IIGU. Seriya ≪Matematika≫, Vol. 3, Issue 1, pp. 104–125. (in Russ.).

Strongly positive relatively p-bounded operators

Let $\mathfrak{F}$ be also be a Banach lattice generated by a cone $\mathfrak{F}_+$. If operator M is (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$, then it is not difficult to show that the subspaces $\mathfrak{U}^k$ and $\mathfrak{F}^k$, $k = 0, 1$, are also Banach lattices generated by cones $\mathfrak{U}^k_+ = \mathfrak{U}^k \cap \mathfrak{U}_+$ and $\mathfrak{F}^k_+ = \mathfrak{F}^k \cap \mathfrak{F}_+$, $k = 0, 1$, respectively.

(L,p)-bounded operator M is called *strongly positive* (L,p)*-bounded*, $p \in \{0\} \cup \mathbb{N}$, if
(iP) operator $L_0 : \mathfrak{U}^0_+ \to \mathfrak{F}^0_+$ and operator $L_1 : \mathfrak{U}^1_+ \to \mathfrak{F}^1_+$ is a toplinear isomorphism;
(iiP) operator $M_1 : \mathfrak{U}^1_+ \cap \mathrm{dom} M \to \mathfrak{F}^1_+$ and operator $M_0 : \mathfrak{U}^0_+ \cap \mathrm{dom} M \to \mathfrak{F}^0_+$ and $M_0^{-1}[\mathfrak{F}^0_+] \subset \mathfrak{U}^0_+$.

It is easy to see that strongly positive (L,p)-bounded operator M is positive (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Let be $f = (\mathbb{I} - Q)f + Qf = f^0 + f^1$, where $Q : \mathfrak{F} \to \mathfrak{F}^1$ is projector along $\mathfrak{F}^0$.

Positive solutions

Theorem 2.1.

Let $\mathfrak{U}$ be a Banach lattice and operator M is strongly positive (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then for any vector functions $f : [0, \tau) \to \mathfrak{F}$ such that $f^0 \in C^{p+1}((0, \tau); \mathfrak{F}^0)$, $-f^{0(k)}(t) \in \mathfrak{F}^0_+$, $k = \overline{0, p+1}$, $t \in (0, \tau)$, $f^1 \in C([0, \tau); \mathfrak{F}^1_+)$, and for any vector $u_0 \in \mathfrak{U}$, such that $u_0^1 \in \mathfrak{U}^1_+$ there exists the unique positive solution $u = u(t)$, which also has the form

$$u(t) = -\sum_{k=0}^{p} H^k M_0^{-1} f^{0(k)}(t) + U^t u_0 + \int_0^\tau U^{t-\tau} L_1^{-1} f^1(\tau) d\tau.$$

Here $f^{0(k)}(t) = \dfrac{d^k}{dt^k} f^0(t)$, $k = \overline{0, p+1}$.

Mathematical model in sequence spaces

We consider Sobolev sequence spaces

$$l_q^m = \{u = (u_k) : \sum_{k=0}^{\infty} \lambda_k^{\frac{mq}{2}} |u_k|^q < \infty\}, \;\; m \in \mathbb{R}, \;\; q \in [1, +\infty).$$

First of all, we note that these spaces are Banach spaces with the norm

$$||u||_{m,q} = \left(\sum_{k=1}^{\infty} \lambda_k^{\frac{mq}{2}} |u_k|^q \right)^{1/q}.$$

Then pay attention to dense and continuous embeddings $l_q^m \hookrightarrow l_q^n$ at $m \geq n$. Finally, we set operator $\Lambda u = (\lambda_k u_k)$, where $u = (u_k)$.

$$||\Lambda u||_{m,q} = \left(\sum_{k=1}^{\infty} \lambda_k^{\frac{mq}{2}+q} |u_k|^q \right)^{1/q} = ||u||_{m+2,q} \Rightarrow \Lambda \in \mathcal{L}(l_q^{m+2}; l_q^m).$$

Mathematical model in sequence spaces

Let's construct operators $L = L(\Lambda)$ and $M = M(\Lambda)$, where $L(s)$ and $M(s)$ are polynomials with real (for simplicity) coefficients.

If the condition

$$\deg L \geq \deg M, \tag{2.3}$$

is satisfied, that operators $L,\ M \in \mathcal{L}(l_q^{m+\deg L}; l_q^m)$, $m \in \mathbb{R}$, $q \in [1, +\infty)$.

Lemma 2.1.

Let
(i) the condition (2.3) is satisfied;
(ii) polynomials $L = L(s)$ and $M = M(s)$ have only real roots and have no common roots.
Then operator M is $(L, 0)$-bounded.

Mathematical model in sequence spaces

We introduce in spaces l_q^m, $m \in \mathbb{R}$, $q \in [1, +\infty)$ Banach lattices. In each of them we choose a family of vectors $\{e_k\}$, all components of which are zero except for the component k that is equal to unity. We construct the linear span of these families consisting of linear combinations of these vectors with positive coefficients. The closure of this linear shell in the norm of the space l_q^m we denote by C_q^m, $m \in \mathbb{R}$, $q \in [1, +\infty)$. As is easy to see, C_q^m is generating cone in space l_q^m, $m \in \mathbb{R}$, $q \in [1, +\infty)$.

Lemma 2.2.

Let the conditions of the lemma 2.1 are satisfied, and all the coefficients of the polynomials $L(S)$ and $M(S)$ are positive. Then operator M is strongly positive $(L, 0)$-bounded.

Mathematical model in sequence spaces

Theorem 2.2.

Let the conditions of Lemmas 2.1 and 2.2 are satisfied. Then for any vector function $f = f(t)$ *such that* $f^0 \in C^1((0,\tau); l_{q,0}^{m+\deg L})$ *and* $-f^0(t) \in (C_q^{m+\deg L} \cap l_q^{m+\deg L})$, $t \in (0,\tau)$ *and* $f^1 \in C([0,\tau); C_q^{m+\deg L} \cap l_{q,1}^{m+\deg L})$ *and any vector* $u_0 \in C_q^{m+\deg L}$, *such that* $u_0^1 \in C_q^{m+\deg L} \cap l_{q,1}^{m+\deg L}$, *there exists a unique positive solution of the problem (2.1), (2.2)* $u = u(t)$, *which also has the following form*

$$u(t) = -M_0^{-1} f^0(t) + U^t u_0 + \int_0^t U^{t-s} L_1^{-1} f^1(s) ds, \quad t \in (0,\tau).$$

Here

$$M_0^{-1} f^0(t) = \sum_{\lambda=\lambda_k} \frac{f_k(t) e_k}{M(\lambda_k)},$$

$$U^t u_0 = \sum_{k=1}^{\infty}{}' \exp\left(\frac{M(\lambda_k)}{L(\lambda_k)} t\right) u_{0k} e_k,$$

$$L_1^{-1} f^1(t) = \sum_{k=1}^{\infty}{}' \frac{f_k(t) e_k}{L(\lambda_k)},$$

and the prime at the sum sign means that the summation is over the set $\{k \in \mathbb{N} : \lambda_k \neq \lambda\}$.

Thank you for your attention!