

DEGENERATE FLOWS OF OPERATORS

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Let \mathfrak{X} and \mathfrak{Y} are Banach spaces, operator $L \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y})$ (i.e. linear and continuous) and operator $M \in \mathcal{C}l(\mathfrak{X}; \mathfrak{Y})$ (i.e. linear and closed with a domain dense in \mathfrak{X}).

On interval $\mathfrak{J} \subset \mathbb{R}$ consider *non-autonomous Sobolev type equation* *

$$L\dot{x}(t) = a(t)Mx(t) + g(t), \quad \ker L \neq \{0\}$$

with a initial conditions

$$x(t_0) = x_0, \quad t_0 \in \mathfrak{J}.$$

Here $a : \mathfrak{J} \rightarrow \mathbb{R}_+$ is a scalar function and $g : \mathfrak{J} \rightarrow \mathfrak{Y}$ is a vector function.

*Sagadeeva M.A. Degenerate Flows of Solving Operators for Nonstationary Sobolev Type Equation. Bulletin of the South Ural State University. Series: Mathematics, Mechanics, Physics, 2017, vol. 9, no. 1, pp 22–30. (in Russian)

Historical review. Sobolev type equations

The autonomous Sobolev type equation

$$L\dot{x} = Mx + g \quad (1)$$

represents a lot of non-classical models of mathematical physics.

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Historical review. Sobolev type equations

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Preliminaries*. Relatively p -Bounded Operators

Let \mathfrak{X} and \mathfrak{Y} are Banach spaces, operators $L \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y})$ (linear and continuous) and $M \in \mathcal{C}l(\mathfrak{X}; \mathfrak{Y})$ (linear, closed and densely defined).

$$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{Y}; \mathfrak{X})\}, \quad \sigma^L(M) = \mathbb{C} \setminus \rho^L(M),$$

Suppose $\rho^L(M) \neq \emptyset$ and consider the operator-valued functions $(\mu L - M)^{-1}$,

$$R_\mu^L(M) = (\mu L - M)^{-1} L, \quad L_\mu^L(M) = L(\mu L - M)^{-1}, \quad \mu \in \rho^L(M),$$

Definition 1. Operator M is called *spectrally bounded with respect to operator L* (shortly (L, σ) -bounded), if $\exists r > 0 \forall \mu \in \mathbb{C} (|\mu| > r) \Rightarrow (\mu \in \rho^L(M))$.

Let operator M be a (L, σ) -bounded. Then the integrals

$$P = \frac{1}{2\pi i} \int_\gamma R_\mu^L(M) d\mu, \quad Q = \frac{1}{2\pi i} \int_\gamma L_\mu^L(M) d\mu, \quad (2)$$

where $\gamma = \{\mu \in \mathbb{C} : |\mu| = h > r\}$ is a closed circuit in complex plane \mathbb{C} .

*G.A. Sviridyuk, V.E. Fedorov, *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*. VSP, Utrecht; Boston; Koln; Tokyo, 2003.

Preliminaries. Relatively p -Bounded Operators

Denote $\mathfrak{X}^0 = \ker P$, $\mathfrak{Y}^0 = \ker Q$, $\mathfrak{X}^1 = \operatorname{im} P$, $\mathfrak{Y}^1 = \operatorname{im} Q$. So we have $\mathfrak{X} = \mathfrak{X}^0 \oplus \mathfrak{X}^1$, $\mathfrak{Y} = \mathfrak{Y}^0 \oplus \mathfrak{Y}^1$. By L_k (M_k) denote restriction of operator L (M) on \mathfrak{X}^k ($\operatorname{dom} M_k = \operatorname{dom} M \cap \mathfrak{X}^k$), $k = 0, 1$.

Theorem 1. (Splitting theorem) *Let operator M be an (L, σ) -bounded. Then*

- (i) $L_k \in \mathcal{L}(\mathfrak{X}^k; \mathfrak{Y}^k)$, $k = 0, 1$;
- (ii) $M_0 \in Cl(\mathfrak{X}^0; \mathfrak{Y}^0)$, $M_1 \in \mathcal{L}(\mathfrak{X}^1; \mathfrak{Y}^1)$;
- (iii) operators $L_1^{-1} \in \mathcal{L}(\mathfrak{Y}^1; \mathfrak{X}^1)$ and $M_0^{-1} \in \mathcal{L}(\mathfrak{Y}^0; \mathfrak{X}^0)$ are exists.

Infinity is called a pole of order $p \in \mathbb{N}_0$ ($\equiv \{0\} \cup \mathbb{N}$) for $(\mu L - M)^{-1}$, if operator $M_0^{-1} L_0 = H = \mathbb{O}$ ($p = 0$) or $H^p \neq \mathbb{O}$ and $H^{p+1} = \mathbb{O}$ with $p \in \mathbb{N}$.

Definition 2. Let ∞ is pole of order $p \in \mathbb{N}_0$ for L -resolvent of operator M . Then (L, σ) -bounded operator M is called (L, p) -bounded.

Preliminaries. Relatively p -Bounded Operators

Definition 3. A one-parameter family $X^\bullet : \mathbb{R} \rightarrow \mathcal{L}(\mathfrak{X})$ is called *degenerate group of operators*, if the following conditions are met

- (i) $X^0 = P$;
- (ii) $X^t X^s = X^{t+s}$ for all $t, s \in \mathbb{R}$.

Degenerate group of operators called of *analytical* if it has an analytic continuation to the whole complex plane \mathbb{C} .

Theorem 2. Let operator M be an (L, σ) -bounded. Then there exists analytical group $\{X^t \in \mathcal{L}(\mathfrak{X}) : t \in \mathbb{R}\}$ ($\{Y^t \in \mathcal{L}(\mathfrak{Y}) : t \in \mathbb{R}\}$) and its operators can be represent by Danford–Taylor type integrals

$$X^t = \frac{1}{2\pi i} \int_{\gamma} R_{\mu}^L(M) e^{\mu t} d\mu \quad \left(Y^t = \frac{1}{2\pi i} \int_{\gamma} L_{\mu}^L(M) e^{\mu t} d\mu \right), \quad (3)$$

where close circuit $\gamma = \{\mu \in \mathbb{C} : |\mu| = h > r\}$.

Preliminaries. Relatively p -Bounded Operators

Definition 4. The vector-function $x \in C^1(\mathbb{R}; \mathfrak{X})$ is called a *solution* of (1) if it satisfies this equation on \mathbb{R} .

Definition 5. The closed set $\mathfrak{P} \subset \mathfrak{X}$ is called a *phase space* of equation

$$L\dot{x} = Mx \quad (4)$$

if

- (i) any solution $x(t)$ of equation lies in \mathfrak{P} (pointwise);
- (ii) there exists a unique solution of the Cauchy problem

$$x(0) = x_0 \quad (5)$$

for equation (4) with any x_0 from \mathfrak{P} .

Theorem 3. Let operator M be an (L, p) -bounded ($p \in \mathbb{N}_0$). Then the set \mathfrak{X}^1 is a phase space of equation (4).

Remark. The solution of (4) has the form $x(t) = X^t x_0$ with $x_0 \in \mathfrak{X}^1$.

Degenerate Flows of Operators

Definition 6. A two-parameter family $X(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{L}(\mathfrak{X})$ is called *degenerate flows of operators*, if the following conditions are met

- (i) $X(t, t) = P$;
- (ii) $X(t, s)X(s, \tau) = X(t, \tau)$.

Degenerate flows of operators called the *analytical* if its operators can be analytically continued to the whole complex plane \mathbb{C} .

Let operator M be an (L, p) -bounded ($p \in \mathbb{N}_0$) and function $a \in C(\mathbb{R}; \mathbb{R})$. Consider

$$X(t, s) = \frac{1}{2\pi i} \int_{\gamma} R_{\mu}^L(M) \exp \left(\mu \int_s^t a(\zeta) d\zeta \right) d\mu, \quad s < t. \quad (6)$$

with close circuit $\gamma = \{\mu \in \mathbb{C} : |\mu| = h > r\}$.

Theorem 4. Let operator M be an (L, p) -bounded ($p \in \mathbb{N}_0$) and function $a \in C(\mathbb{R}; \mathbb{R})$. Then family $\{X(t, s) \in \mathcal{L}(\mathfrak{X}) : t, s \in \mathbb{R}\}$ defined by (6) is an *analytical degenerate flows of operators*.

Solvability of Initial Problems for Non-autonomous Sobolev Type Equations

On the interval $\mathfrak{J} \subset \mathbb{R}$ consider the Cauchy problem

$$x(t_0) = x_0, \quad (t_0 \in \mathfrak{J}) \quad (7)$$

for homogeneous non-autonomous equation

$$L\dot{x}(t) = a(t)Mx(t), \quad (8)$$

where function $a : \mathfrak{J} \rightarrow \mathbb{R}_+$ to be further defined.

Definition 7. Vector-function $x \in C^1(\mathfrak{J}; \mathfrak{X})$ is called *solution of equation (8)*, if it satisfies this equation on \mathfrak{J} . Solution of equation (8) is called *solution of Cauchy problem (7)*, (8), if it also satisfies to (7).

Closed set $\mathfrak{P} \subset \mathfrak{X}$ is called *phase space* of equation (8), if

- (i) any solution $x(t)$ of equation (8) lies in \mathfrak{P} (pointwise);
- (ii) for any x_0 from \mathfrak{P} there exists unique solution of (7), (8).

Theorem 5. Let operator M be an (L, p) -bounded ($p \in \mathbb{N}_0$) and function $a \in C(\mathbb{R}, \mathbb{R}_+)$. Then the set \mathfrak{X}^1 is a phase space of equation (8).

Solvability of Initial Problems for Non-autonomous Sobolev Type Equations

Definition 8. The flows of operators $X(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{L}(\mathfrak{X})$ is called *flows of solving operators* for equation (8), if for any $x_0 \in \mathfrak{X}$ the vector-function $x(t) = X(t, t_0)x_0$ is a solution of equation (8) by the definition 7.

Consider the Showalter–Sidorov problem

$$P(x(0) - x_0) = 0 \tag{9}$$

for non-homogeneous equation

$$L\dot{x}(t) = a(t)Mx(t) + g(t) \tag{10}$$

with function $g : \mathfrak{J} \rightarrow \mathfrak{Y}$. Denote $(\mathbb{I}_{\mathfrak{Y}} - Q)g(t) = g^0(t)$.

Definition 9. Solution of equation (10) is called *solution of Showalter–Sidorov problem* (9), (10), if it satisfies (9).

Solvability of Initial Problems for Non-autonomous Sobolev Type Equations

Theorem 6. *Let $0, T \in \mathfrak{J}$, operator M be an (L, p) -bounded ($p \in \mathbb{N}_0$) and function $a \in C^{p+1}([0, T]; \mathbb{R}_+)$. Then for all $x_0 \in \mathfrak{X}$ and vector-function $g : [0, T] \rightarrow \mathfrak{Y}$, such that $Qg \in C^1([0, T]; \mathfrak{Y}^1)$ and $g^0 \in C^{p+1}([0, T]; \mathfrak{Y}^0)$ there exists a unique solution $x \in C^1([0, T]; \mathfrak{X})$ of Showalter–Sidorov problem (9) for equation (10), given by the next formula*

$$x(t) = X(t, 0)Px_0 + \int_0^t X(t, s)L_1^{-1}Qg(s)ds - \sum_{k=0}^p H^k M_0^{-1} \left(\frac{1}{a(t)} \frac{d}{dt} \right)^k \frac{g^0(t)}{a(t)}. \quad (11)$$

If in addition initial data x_0 satisfies

$$(\mathbb{I}_{\mathfrak{X}} - P)x_0 = - \sum_{k=0}^p H^k M_0^{-1} \left(\frac{1}{a(0)} \frac{d}{dt} \right)^k \frac{g^0(0)}{a(0)},$$

then solution (11) is a unique solution of the Cauchy problem (7), (10).

THANK YOU FOR YOUR ATTENTION!