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THE CAUCHY PROBLEM FOR THE SEMILINEAR SOBOLEV TYPE EQUATION OF HIGHER ORDER

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Formulation of the Problems

Let $\Omega = (0,a) \times (0,b) \times (0,c) \subset \mathbb{R}^3$.

$$(\Delta - \lambda)u_{tttt} + (\Delta - \lambda')u_{tt} + \alpha \frac{\partial^2 u}{\partial x_3^2} = \Delta(u^3), (x,t) \in \Omega \times \mathbb{R} \tag{1}$$

with the Cauchy – Dirichlet conditions

$$\begin{gathered} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\ u_{tt}(x,0) = u_2(x), \quad u_{ttt}(x,0) = u_3(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \end{gathered} \tag{2}$$

In suitable Banach spaces $\mathfrak{U}$ and $\mathfrak{F}$ mathematical model (1), (2) can be reduced to the Cauchy problem

$$u^{(k)}(0) = u_k, \quad k = 0, 1, \ldots, n-1, \tag{3}$$

$$Au^{(n)} = B_{n-1}u^{(n-1)} + B_{n-2}u^{(n-2)} + \ldots + B_0 u + N(u), \tag{4}$$

where the operators $A, B_{n-1}, B_{n-2}, \ldots, B_0 \in \mathcal{L}(\mathfrak{U}; \mathfrak{F}), N \in C^{\infty}(\mathfrak{U}; \mathfrak{F})$.

Theory Relatively Polynomially Bounded Operator Pencils I

$\mathfrak{U}, \mathfrak{F}$ be Banach spaces, $A, B_0, B_1, \ldots, B_{n-1} \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$.

$\rho^A(\overrightarrow{B}) = \{\mu \in \mathbb{C} : (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and $\sigma^A(\overrightarrow{B}) = \overline{\mathbb{C}} \setminus \rho^A(\vec{B})$ are called an A-*resolvent set* and an A-*spectrum* of the pencil $\overrightarrow{B}$ respectively. The operator-function $R^A_\mu(\overrightarrow{B}) = (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1}$ is called an A-*resolvent* of the pencil $\overrightarrow{B}$.

Definition

The operator pencil $\vec{B}$ is called *polynomially bounded with respect to an operator* A (*or polynomially* A*-bounded*) if $\exists a \in \mathbb{R}_+ \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (R^A_\mu(\vec{B}) \in \mathcal{L}(\mathfrak{F}; \mathfrak{U}))$.

$$\int_\gamma \mu^k R^A_\mu(\vec{B}) d\mu \equiv \mathbb{O}, \quad k = 0, 1, \ldots, n-2, \text{[1]} \tag{5}$$

where $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$

1. Zamyshlyaeva A.A. Linear Sobolev Type Equations of Higher Order. Chelyabinsk, Izd. Center of SUSU, 2012.

Theory Relatively Polynomially Bounded Operator Pencils II

Lemma 1[1]

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R^A_\mu(\vec{B})\mu^{n-1}A d\mu, \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} \mu^{n-1}AR^A_\mu(\vec{B})d\mu$$

$\mathfrak{U}^0 = \ker P$, $\mathfrak{F}^0 = \ker Q$, $\mathfrak{U}^1 = \text{im } P$, $\mathfrak{F}^1 = \text{im} Q$
$\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1, \quad \mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$
A^k (B^k_l) – restriction of operators A (B_l) on $\mathfrak{U}^k$, $k = 0, 1;$ $l = 0, 1, \dots, n-1.$

Theorem 1[1]

Let the operator pencil $\vec{B}$ be polynomially A-bounded and condition (5) be fulfilled. Then
(i) $A^k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k),\ k = 0, 1;$
(ii) $B^k_l \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k),\ k = 0, 1,\ l = 0, 1, \dots, n-1;$
(iii) operator $(A^1)^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$ exists;
(iv) operator $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ exists.

Using last theorem construct operators $H_0 = (B^0_0)^{-1}A^0 \in \mathcal{L}(\mathfrak{U}^0)$, $H_1 = (B^0_0)^{-1}B^0_1 \in \mathcal{L}(\mathfrak{U}^0)$,. . ., $H_{n-1} = (B^0_0)^{-1}B^0_{n-1} \in \mathcal{L}(\mathfrak{U}^0)$ and $S_0 = (A^1)^{-1}B^1_0 \in \mathcal{L}(\mathfrak{U}^1)$, $S_1 = (A^1)^{-1}B^1_1 \in \mathcal{L}(\mathfrak{U}^1)$, . . ., $S_{n-1} = (A^1)^{-1}B^1_{n-1} \in \mathcal{L}(\mathfrak{U}^1)$.

1. Zamyshliaeva A.A. Linear Sobolev Type Equations of Higher Order. Chelyabinsk, Izd. Center of SUSU, 2012.

Theory Relatively Polynomially Bounded Operator Pencils III

Definition

Define the *family of operators* $\{K_q^1, K_q^2, \ldots, K_q^n\}$ as follows:

$$K_0^s = \mathbb{O},\ s \neq n,\ K_0^n = \mathbb{I},$$
$$K_1^1 = H_0,\ K_1^2 = -H_1, \ldots, K_1^s = -H_{s-1}, \ldots, K_1^n = H_{n-1},$$
$$K_q^1 = K_{q-1}^n H_0,\ K_q^2 = K_{q-1}^1 - K_{q-1}^n H_1, \ldots, K_q^s = K_{q-1}^{s-1} - K_{q-1}^n H_{s-1}, \ldots,$$
$$K_q^s = K_{q-1}^{n-1} - K_{q-1}^n H_{n-1},\ q = 1, 2, \ldots.$$

The A-resolvent can be represented by a Laurent series[1]

$$R_\mu^A(\overrightarrow{B}) = -\sum_{q=0}^{\infty} \mu^q K_q^n (B_0^0)^{-1}(\mathbb{I} - Q) + \sum_{q=1}^{\infty} \mu^{-q}(\mu^{n-1}S_{n-1} + \cdots + \mu S_1 + S_0)^q L_1^{-1} Q.$$

Definition

The point ∞ is called

- *a pole of order* $p \in \{0\} \cup \mathbb{N}$ *of an* A*-resolvent of the pencil* $\vec{B}$, if $\exists\, p$ such that $K_p^s \not\equiv \mathbb{O}, s = 1, 2, \ldots, n$, but $K_{p+1}^s \equiv \mathbb{O}, s = 1, 2, \ldots, n$;
- *an essential singularity of an* A*-resolvent of the pencil* $\vec{B}$, if $K_q^n \not\equiv \mathbb{O}$ for all $q \in \mathbb{N}$.

1. Zamyshlyaeva A.A. The Phase Space of a High Order Sobolev Type Equation. *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2011, vol. 4, no. 4, pp. 45–57.

Banach Manifolds I

Let $\mathfrak{M}$ be a C^k-manifold modelled by a Banach space $\mathfrak{U}$. By $T\mathfrak{M}$ denote a tangent bundle of the manifold $\mathfrak{M}$ and by $T^n\mathfrak{M}$ denote a tangent bundle of order n. The set $T\mathfrak{M}$ has the structure of a smooth C^{k-1}-manifold, modelled by Banach space $\mathfrak{U}$ by construction, and tangent bundle $T^n\mathfrak{M}$ is a manifold of class C^{k-n}.

By π^l denote a canonical projection from a tangent bundle of order l to a tangent bundle of order $l-1$ where $l = 1, 2, \ldots, n$ and by π^l_* denote projection from tangent bundle of order l to a manifold $\mathfrak{M}$, i.e. $\pi^l_* = \pi^1\pi^2\ldots\pi^l$.

Consider a curve $\alpha : J \to \mathfrak{M}$ of class $C^s, (s \leq k)$ where J is some interval containing zero. By canonical lifting of the curve α we call a curve α^1 in $T\mathfrak{M}$ $\alpha^1 : J \to T\mathfrak{M}$ such that $\pi^1\alpha^1 = \alpha$. Similarly, by the lifting of order l of curve α in $T^l\mathfrak{M}$ we call a curve $\alpha^l : J \to T^l\mathfrak{M}$ such that $\pi^l_*\alpha^l = \alpha$. Therefore lifting of order l of the curve is a mapping of class $s - l \geq 1$.

On the basis of the definition of a second-order differential equation[1] introduce

Definition

A differential equation of order n on a manifold $\mathfrak{M}$ is a vector field ξ of class C^{k-n} on the tangent bundle $T^{n-1}\mathfrak{M}$ such that for all $v \in T^{n-1}\mathfrak{M}$ the equality

$$\pi^n\xi(v) = v$$

holds.

1. Leng S. *Introduction to Differentiable Manifolds*. N. Y., Springer-Verlag, 2002.

Banach Manifolds II

Let $\mathfrak{M}$ be an open set in the Banach space $\mathfrak{U}$. In this case, for any vector field on $T^{n-1}\mathfrak{M}$, the main part of differential equation

$$f : T^{n-1}\mathfrak{M} \to \mathfrak{U}^n$$

has n components $f = (f_1, f_2, \dots, f_n)$ each of which maps $T^{n-1}\mathfrak{M}$ into $\mathfrak{U}$.

Lemma 2[1]

The mapping f of class C^{k-n} is the main part of a differential equation of order n iff

$$f(g_1, g_2, \dots, g_n) = (g_2, g_3, \dots, g_n, f_n(g_1, g_2, \dots, g_n)).$$

Theorem 3

Let $\mathfrak{M}$ be a Banach C^k-manifold, ξ be a differential equation of order n of class C^{k-n}. Then for any point $(u_0, u_1, \dots, u_{n-1}) \in T^{n-1}\mathfrak{M}$ there exists a unique curve $u \in C^l\big((-\tau, \tau); \mathfrak{M}\big)$, $\tau = \tau(u_0, u_1, \dots, u_{n-1}) > 0$, $l \geq n$, lying in $\mathfrak{M}$, passing through the point $(u_0, u_1, \dots, u_{n-1})$ such that

$$\begin{gathered} u^{(n)} = f_n(u, \dot{u}, \ddot{u}, \dots, u^{(n-1)}) \\ u^{(k)}(0) = u_k, \quad k = 0, 1, \dots, n-1. \end{gathered} \tag{6}$$

1. Leng S. *Introduction to Differentiable Manifolds*. N. Y., Springer-Verlag, 2002.

Abstract Problem I

$$u^{(k)}(0) = u_k, \quad k = 0, 1, \ldots, n-1, \tag{3}$$

$$Au^{(n)} = B_{n-1}u^{(n-1)} + B_{n-2}u^{(n-2)} + \ldots + B_0 u + N(u), \tag{4}$$

Definition

If a vector-function $u \in C^\infty((-\tau, \tau); \mathfrak{U})$, $\tau \in \mathbb{R}_+$ satisfies equation (4) then it is called *a solution of this equation*. If the vector-function satisfies in addition condition (3) then it is called *a solution of* (3), (4).

Definition

The set $\mathfrak{P}$ is called a *phase space of* (4), if
(i) for all $(u_0, u_1, \ldots, u_{n-1}) \in T^{n-1}\mathfrak{P}$ there exists a unique solution of (3), (4);
(ii) a solution $u = u(t)$ of (4) lies in $\mathfrak{P}$ as a trajectory, i.e. $u(t) \in \mathfrak{P}$ for all $t \in (-\tau, \tau)$.

If $\ker A = \{0\}$ then equation (4) can be reduced to an equivalent equation

$$u^{(n)} = F(u, \dot{u}, \ldots, u^{(n-1)}),$$

where $F(u, \dot{u}, \ldots, u^{(n-1)}) = A^{-1}(B_{n-1}u^{(n-1)} + B_{n-2}u^{(n-2)} + \ldots + B_0 u + N(u))$ is a mapping of class C^∞ by construction. The existence of a unique solution u of (3), (4) for all $(u_0, u_1, \ldots, u_{n-1})$ follows from theorem 3.

Abstract Problem II

Let $\ker A \neq \{0\}$ and operator pencil $\vec{B}$ be $(A, 0)$-bounded.

$$\begin{cases} 0 = (\mathbb{I} - Q)(B_0 + N)(u^0 + u^1), \\ \dfrac{d^n}{dt^n} u^1 = A_1^{-1} Q\left(B_{n-1} \dfrac{d^{n-1}}{dt^{n-1}} + B_{n-2} \dfrac{d^{n-2}}{dt^{n-2}} + \ldots + B_0 + N\right)(u^0 + u^1), \end{cases} \tag{7}$$

where $u^1 = Pu, u^0 = (I - P)u$.
Now consider a set $\mathfrak{M} = \{u \in \mathfrak{U} : (I - Q)(B_0 u + N(u)) = 0\}$. Let the set $\mathfrak{M}$ be not empty, i.e. there is a point $u_0 \in \mathfrak{M}$. Denote ${u_0}^1 = Pu \in \mathfrak{U}^1$.

$$(\mathbb{I} - Q)(B_0 + N'_{u_0}) : \mathfrak{U}^0 \to \mathfrak{F}^0 \quad \text{is a toplinear isomorfism.} \tag{8}$$

Lemma

The set $\mathfrak{M} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(B_0 u + N(u)) = 0\}$ under condition (8) is a C^∞-manifold at point u_0.

Abstract Problem III

Lets act with the Frechet derivative $\delta^{(n)}_{(u^1_0,u^1_1,\dots,u^1_{n-1})}$ of order n on the second equation of system (7). Since $\delta(u^1)=u$ and

$$\delta^{(n)}_{(u^1_0,u^1_1,\dots,u^1_{n-1})}u^{1\,(n)}=\frac{d^n}{dt^n}\left(\delta(u^1)\right)$$

we obtain equation $u^{(n)}=F(u,\dot{u},\dots,u^{(n-1)})$, where

$$F(u,\dot{u},\dots,u^{(n-1)})=\delta^{(n)}_{(u^1_0,u^1_1,\dots,u^1_{n-1})}A^{-1}Q(B_{n-1}u^{(n-1)}+B_{n-2}u^{(n-2)}+\dots$$

$$+B_0u+N(u))\in C^{\infty}(\mathfrak{U}).$$

By virtue of theorem 3, we get

Theorem 4

Let the operator pencil $\vec{B}$ be $(A,0)$-bounded, $N\in C^{\infty}(\mathfrak{U};\mathfrak{F})$ and condition (8) be fulfilled. Then for any $(u_0,u_1,\dots,u_{n-1})\in T^{n-1}\mathfrak{M}$ there exists a unique solution of (3), (4) lying in $\mathfrak{M}$ as trajectory.

Mathematical Model I

$$(\Delta - \lambda)u_{tttt} + (\Delta - \lambda')u_{tt} + \alpha \frac{\partial^2 u}{\partial x_3^2} = \Delta(u^3), (x,t) \in \Omega \times \mathbb{R} \quad (9)$$

$$\begin{gathered} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\ u_{tt}(x,0) = u_2(x), \quad u_{ttt}(x,0) = u_3(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \end{gathered} \quad (10)$$

$$\mathfrak{U} = \{u \in W_2^{l+2}(\Omega) : u(x) = 0, x \in \partial\Omega\}, \quad \mathfrak{F} = W_2^l(\Omega).$$

$A = \Delta - \lambda$, $B_2 = (\lambda' - \Delta)$, $B_0 = -\alpha \frac{\partial^2}{\partial x_3^2}$, $B_3 = B_1 = \mathbb{O}$.

$\varphi_{kmn} = \left\{ \sin \frac{\pi k x_1}{a} \sin \frac{\pi m x_2}{b} \sin \frac{\pi n x_3}{c} \right\}$, where k, m, $n \in \mathbb{N}$

$\lambda_{kmn} = -\sqrt{\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2 + \left(\frac{\pi n}{c}\right)^2}$.

Mathematical Model II

Since $\{\varphi_{kmn}\} \subset C^\infty(\Omega)$ we obtain

$$\mu^4 A - \mu^3 B_3 - \mu^2 B_2 - \mu B_1 - B_0 =$$

$$= \sum_{k,m,n=1}^{\infty} [(\lambda_{kmn} - \lambda)\mu^4 + (\lambda_{kmn} - \lambda')\mu^2 - \alpha \left(\frac{\pi n}{c}\right)^2] < \varphi_{kmn}, \cdot > \varphi_{kmn},$$

In the case (i) when $\lambda \notin \sigma(\Delta)$ the A-spectrum of pencil $\vec{B}$ $\sigma^A(\vec{B}) = \{\mu^j_{rmn} : r, m, n \in \mathbb{N}, j = 1, ..., 4\}$, where μ^j_{rmn} are the roots of equation

$$(\lambda_{rmn} - \lambda)\mu^4 + (\lambda_{rmn} - \lambda')\mu^2 - \alpha \left(\frac{\pi n}{c}\right)^2 = 0. \tag{11}$$

In the case (ii) when $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda')$ the A-spectrum of pencil $\vec{B}$ $\sigma^A(\vec{B}) = \{\mu^j_{l,k} : k \in \mathbb{N}\}$, where $\mu^j_{l,k}$ the roots of equation (11) with $\lambda = \lambda_l$.

In the case (iii) when $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$ the A-spectrum of pencil $\vec{B}$ $\sigma^A(\vec{B}) = \{\mu^j_{l,k} : k \in \mathbb{N}, k \neq l\}$.

Lemma

Let (i) $\lambda \notin \sigma(\Delta))$ or (ii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda')$. Then pencil $\vec{B}$ is polynomially $(A, 0)$-bounded.

Mathematical Model III

Case (i) $P = \mathbb{I}$ and $Q = \mathbb{I}$. Case (ii)

$$P = \mathbb{I} - \sum_{\lambda=\lambda_{kmn}} < \varphi_{kmn}, \cdot > \varphi_{kmn},$$

and the projector Q has the same form but it is defined on space $\mathfrak{F}$. Construct the phase space

$$\mathfrak{M} = \{u \in \mathfrak{U} : \sum_{\lambda=\lambda_{kmn}} < \alpha \left(\frac{\pi n}{c}\right)^2 u + \Delta(u^3), \varphi_{kmn} > \varphi_{kmn} = 0\}.$$

By theorem 4 we have

Theorem

(i) Let $\lambda \notin \sigma(\Delta)$, $(u_0, u_1, \ldots, u_{n-1}) \in \mathfrak{U}^n$*. Then for some* $\tau = \tau(u_0, u_1, \ldots, u_{n-1}) > 0$ *there exists a unique solution* $u \in C^n\big((-\tau, \tau), \mathfrak{U}\big)$ *of problem (9), (10).*
(ii) Let $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda')$, $(u_0, u_1, \ldots, u_{n-1}) \in T^{n-1}\mathfrak{M}$ *and condition (8) be fulfilled. Then for some* $\tau = \tau(u_0, u_1, \ldots, u_{n-1}) > 0$ *there exists a unique solution* $u \in C^n\big((-\tau, \tau), \mathfrak{M}\big)$ *of problem (9), (10).*