SOBOLEV TYPE EQUATIONS OF HIGHER ORDER

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- Introduction
- Abstract theory of Sobolev type equations of higher order
- The Boussinesq – Love mathematical model: analytical and numerical investigation

$$Lu^{(n)} = Mu + f \tag{1}$$

$$Au^{(n)} = B_{n-1}u^{(n-1)} + ... + B_0 u + f \tag{2}$$

$$u^{(m)}(0) = u_m, m = 0, ..., n-1 \tag{3}$$

$$L(u^{(m)}(0) - u_m) = 0, m = 0, ..., n-1 \tag{4}$$

$$P(u^{(m)}(0) - u_m) = 0, m = 0, ..., n-1 \tag{5}$$

$$P_{in}(u^{(m)}(0) - u_m^0) = 0,\ P_{fin}(u^{(m)}(T) - u_m^T) = 0,\ m = 0, ..., n-1 \tag{6}$$

S.L. Sobolev

M.O. Korpusov, A.I. Kozhanov, G.V. Demidenko

N.A. Sidorov, B.V. Loginov, A.V. Sinitsin, M.V. Falaleev, I.V. Melnikova, A.I. Filinkov, M.A. Alshansky

Yu.E. Boyarintsev, V.F. Chistyakov, A.A. Sheglova
A.V. Keller

R. Showalter, A. Favini, A. Yagi

G.A. Sviridyuk, T.G. Sukacheva, V.E. Fedorov

The de Gennes mathematical model of acoustic waves in a smectic

$$\frac{\partial^2}{\partial t^2}\Delta_3 u = \alpha_1 \frac{\partial^2}{\partial z^2}\Delta_2 u,\ \ \alpha_1 > 0,$$

where $\Delta_3 = \Delta_2 + \frac{\partial^2}{\partial z^2},\ \ \Delta_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

$$u(x_1, x_2, z, t) = v(x_1, x_2, z)\exp(-i\omega t)$$

$$\frac{\partial^2}{\partial z^2}(\Delta_2 v + \alpha_2 v) + \alpha_2 \Delta_2 v = 0,\ \ \alpha_2 = \omega^2 \alpha_1^{-1}$$

The mathematical model of fluctuations in the DNA molecule

$$\begin{array}{ll} u(x,0) = u_0(x), & \dot{u}(x,0) = u_1(x), \\ v(x,0) = v_0(x), & \dot{v}(x,0) = v_1(x), \end{array} \quad x \in \Omega,$$

$$u(x,t) = v(x,t) = 0, (x,t) \in \partial\Omega \times \mathbb{R},$$

$$\begin{cases} (b+\Delta)\ddot{u} = a\Delta u + f(u,v) + w_1, \\ (b+\Delta)\ddot{v} = d\Delta v + g(u,v) + w_2. \end{cases}$$

The mathematical model of shallow water waves propagation

$$(\lambda - \Delta)\ddot{u} = \alpha^2 \Delta u + f,$$
$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = u_1(x), \quad x \in \Omega,$$
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}.$$

The mathematical model of ion-acoustic waves in plasma

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2} + \omega_{B_i}^2\right)(\Delta_3 \Phi - \frac{1}{r_D^2}\Phi) + \omega_{p_i}^2 \frac{\partial^2}{\partial t^2}\Delta_3 \Phi + \omega_{B_i}^2 \omega_{p_i}^2 \frac{\partial^2 \Phi}{\partial x_3^2} = 0$$

$$(\Delta - \lambda)v_{tttt} + (\Delta - \lambda')v_{tt} + \alpha \frac{\partial^2 u}{\partial x_3^2} = 0$$

$$\frac{\partial^2}{\partial t^2}(\Delta\Phi - \Phi) + \Delta\Phi = 0$$

The Boussinesq – Love model

$$(\lambda - \Delta)u_{tt} = \alpha(\Delta - \lambda^{'})u_t + \beta(\Delta - \lambda^{''})u + g$$

By $\vec{B}$ denote the pencil formed by operators $B_{n-1}, \ldots, B_0$.

$$\rho^A(\vec{B}) = \{\mu \in \mathbb{C} : (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}, \quad \sigma^{\mathrm{A}}(\vec{\mathrm{B}}) = \mathbb{C} \setminus \rho^{\mathrm{A}}(\vec{\mathrm{B}}).$$

$$R^A_\mu(\vec{B}) = (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1}.$$

Definition 1.

The pencil $\vec{B}$ is called *polynomially bounded with respect to operator* A (or *polynomially* A*-bounded*), if

$$\exists a \in \mathbb{R}_+ \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (R^A_\mu(\vec{B}) \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})).$$

$$\int_\gamma \mu^m R^A_\mu(\vec{B}) d\mu \equiv \mathbb{O}, m = \overline{0, n-2}, \qquad (A)$$

where $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$.

Lemma 1.

If the pencil $\vec{B}$ is polynomially A-bounded, and condition (A) is satisfied, then the following operators

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R^A_\mu(\vec{B})\mu^{n-1}Ad\mu, \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} \mu^{n-1}AR^A_\mu(\vec{B})d\mu$$

are projectors in $\mathfrak{U}$ and $\mathfrak{F}$, respectively.

$\mathfrak{U}^0 = \ker P$, $\mathfrak{F}^0 = \ker Q$, $\mathfrak{U}^1 = \text{im } P$, $\mathfrak{F}^1 = \text{im } Q$, $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$. By A^k (B^k_l) we denote the restriction of the operator A (B_l) to $\mathfrak{U}^k$, $k = 0, 1$; $l = \overline{0, n-1}$.

Theorem 1.

Let the assumptions of Lemma 1 be satisfied. Then
(i) $A^k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$;
(ii) $B^k_l \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$, $l = 0, 1, \dots, n-1$;
(iii) *there exists an operator* $(A^1)^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$;
(iv) *there exists an operator* $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$.

Let us construct the operators $H_0 = (B^0_0)^{-1}A^0$, $H_m = (B^0_0)^{-1}B^0_{n-m}$, $m = \overline{1, n-1}$, $S_m = (A^1)^{-1}B^1_m$, $m = \overline{0, n-1}$.

Definition 2.

Introduce the family of operators $\{K_q^1, K_q^2, \ldots, K_q^n\}$ as follows:

$$K_0^s = \mathbb{O}, s \neq n, K_0^n = \mathbb{I},\ K_1^1 = H_0,\ K_1^2 = -H_{n-1}, \ldots, K_1^s = -H_{n+1-s}, \ldots, K_1^n = -H_1,$$
$$K_q^1 = K_{q-1}^n H_0,\ K_q^2 = K_{q-1}^1 - K_{q-1}^n H_{n-1}, \ldots, K_q^s = K_{q-1}^{s-1} - K_{q-1}^n H_{n+1-s}, \ldots,$$
$$K_q^n = K_{q-1}^{n-1} - K_{q-1}^n H_1, q = 2, 3, \ldots$$

Definition 3.

The point ∞ is called:
(i) *a removable singular point of the* A-resolvent of pencil $\vec{B}$, if $K_1^1 = K_1^2 = \ldots = K_1^n \equiv \mathbb{O}$;
(ii) *a pole* of order $p \in \mathbb{N}$ of the A-resolvent of pencil $\vec{B}$, if $K_p^s \not\equiv \mathbb{O}$, for some s, but $K_{p+1}^s \equiv \mathbb{O}$ for any $s = \overline{1, n}$;
(iii) *an essentially singular point* of the A-resolvent of pencil $\vec{B}$, if $K_p^n \not\equiv \mathbb{O}$ for any $p \in \mathbb{N}$.

$$Au^{(n)} = B_{n-1}u^{(n-1)} + \ldots + B_0 u. \tag{7}$$

$$U_m^t = \frac{1}{2\pi i}\int\limits_{\gamma} R_\mu^A(\vec{B})(\mu^{n-m-1}A - \mu^{n-m-2}B_{n-1} - \ldots - B_{m+1})e^{\mu t}d\mu, \quad m = \overline{0, n-1}.$$

Lemma 2.

(i) *For every* $m = \overline{0, n-1}$ *the operator function* U_m^t *is a propagator of (7).*
(ii) *For every* $m = \overline{0, n-1}$ *the operator function* U_m^t *is an entire function.*
(iii)

$$\frac{d^l}{dt^l}U_m^t\bigg|_{t=0} = \begin{cases} P, \ l = m; \\ \mathbb{O}, \ l \neq m; \end{cases} \quad \text{for all } m = \overline{0, n-1}, \ l = 0, 1, \ldots .$$

Definition 4.

The set $\mathcal{P} \subset \mathfrak{U}$ is called a *phase space* of equation (7), if
(i) every solution $u = u(t)$ of (7) lies in $\mathcal{P}$, i.e. $u(t) \in \mathcal{P} \quad \forall t \in \mathbb{R}$.
(ii) for arbitrary $u_m \in \mathcal{P}$, $m = \overline{0, n-1}$, there exists a unique solution to (3), (7).

Theorem 2.

If the pencil $\vec{B}$ is polynomially A-bounded, condition (A) is satisfied and ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of A-resolvent, then the phase space of equation (7) coincides with the image of projector P.

$$Au^{(n)} = B_{n-1}u^{(n-1)} + \ldots + B_0 u + y. \tag{8}$$

$$\mathcal{M}_y^m = \{u \in \mathfrak{U} : \ (\mathbb{I} - P)u = -\sum_{l=0}^{p} K_l^n (B_0^0)^{-1} \frac{d^{l+m}}{dt^{l+m}} (\mathbb{I} - Q)y(0)\}, \quad m = \overline{0, n-1}.$$

Theorem 3.

If the pencil $\vec{B}$ is polynomially A-bounded, condition (A) is satisfied and ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of the A-resolvent of pencil $\vec{B}$, the function $y : [0, \tau) \to \mathfrak{F}$ is such that $y^0 = (\mathbb{I} - Q)y \in C^{p+n}([0, \tau); \mathfrak{F}^0)$ and $y^1 = Qy \in C([0, \tau); \mathfrak{F}^1)$, then for arbitrary $u_m \in \mathcal{M}_y^m$, $m = \overline{0, n-1}$ there exists a unique classical solution to problem (3), (8) for $t \in [0, \tau)$ given by

$$u(t) = -\sum_{q=0}^{p} K_q^n (B_0^0)^{-1} \frac{d^q}{dt^q} y^0(t) + \sum_{m=0}^{n-1} U_m^t P u_m + \int_0^t U_{n-1}^{t-s} (A^1)^{-1} y^1(s) ds. \tag{9}$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$ of class C^∞. In the cylinder consider the initial-boundary value problem

$$(\lambda - \Delta)u_{tt} = \alpha(\Delta - \lambda')u_t + \beta(\Delta - \lambda'')u + y, \tag{10}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \tag{11}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega. \tag{12}$$

Denote by $\sigma(\Delta) = \{\lambda_k\}$ the set of eigenvalues of operator Δ numbered in nonincreasing order taking into account multiplicities, and by $\{\varphi_k\}$ the set of corresponding eigenfunctions orthonormal with respect to the inner product $< \cdot, \cdot >$ in $L^2(\Omega)$.

Put $\mathfrak{U} = \{u \in W_2^{l+2}(\Omega) : u(x) = 0, x \in \partial\Omega\}, \mathfrak{F} = W_2^l(\Omega),\ A = \lambda - \Delta\ ,\ B_1 = \alpha(\Delta - \lambda'),$ $B_0 = \beta(\Delta - \lambda'').$

Lemma 2. *Let one of the following conditions be fulfilled:*
(i) $\lambda \notin \sigma(\Delta)$;
(ii) $(\lambda \in \sigma(\Delta) \wedge (\lambda \neq \lambda')$;
(iii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$.
Then for all $\alpha, \beta \in \mathbb{R}\setminus\{0\}$ *the pencil* $\vec{B}$ *is polynomially* A*-bounded.*

$$(\lambda - \lambda_k)\mu^2 + \alpha(\lambda' - \lambda_k)\mu + \beta(\lambda'' - \lambda_k) = 0,\ k \in \mathbb{N}. \tag{13}$$

Remark 1. If $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$ then condition (A) doesn't hold and ∞ is an essential singular point of the A-resolvent of $\vec{B}$.

Remark 2. If $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$ then ∞ is a removable singular point of the A-resolvent of $\vec{B}$.

Corollary 2. *Let the vector-function* $y \in C^1((0,\tau);\mathfrak{F}) \cap C([0,\tau];\mathfrak{F})$ *and*
(i) $\lambda \notin \sigma(\Delta)$. *Then for arbitrary* $u_0, u_1 \in \mathfrak{U}$ *there exists a unique solution to (10) – (12), given by*

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\mu_k^1(\lambda-\lambda_k)+\alpha(\lambda'-\lambda_k)}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)} e^{\mu_k^1 t} + \frac{\mu_k^2(\lambda-\lambda_k)+\alpha(\lambda'-\lambda_k)}{(\lambda-\lambda_k)(\mu_k^2-\mu_k^1)} e^{\mu_k^2 t} \right] < \varphi_k, u_0 > \varphi_k +$$

$$\sum_{k=1}^{\infty} \frac{e^{\mu_k^1 t} - e^{\mu_k^2 t}}{(\mu_k^1-\mu_k^2)} < \varphi_k, u_1 > \varphi_k + \sum_{k=1}^{\infty} \int_0^t \frac{e^{\mu_k^1 (t-s)} - e^{\mu_k^2 (t-s)}}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)} < \varphi_k, y(s) > \varphi_k ds, \ t \in (0,\tau).$$

(ii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$. *Then for arbitrary* $u_0, u_1 \in \mathfrak{U}$ *such that*

$$< \varphi_k, u_0 > + \frac{< \varphi_k, y(0) >}{\beta(\lambda''-\lambda_k)} = < \varphi_k, u_1 > + \frac{< \varphi_k, y'(0) >}{\beta(\lambda''-\lambda_k)} = 0, \ \lambda = \lambda_k$$

there exists a unique solution to (10) – (12), given by

$$u(t) = -\sum_{\lambda=\lambda_k} \frac{< \varphi_k, y(t) >}{\beta(\lambda''-\lambda_k)} \varphi_k + \sum{}' \left[\frac{\mu_k^1(\lambda-\lambda_k)+\alpha(\lambda'-\lambda_k)}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)} + \right.$$

$$\left. + \frac{\mu_k^2(\lambda-\lambda_k)+\alpha(\lambda'-\lambda_k)}{(\lambda-\lambda_k)(\mu_k^2-\mu_k^1)} e^{\mu_k^2 t} \right] < \varphi_k, u_0 > \varphi_k +$$

$$+ \sum{}' \frac{e^{\mu_k^1 t} - e^{\mu_k^2 t}}{(\mu_k^1-\mu_k^2)} < \varphi_k, u_1 > \varphi_k + \sum{}' \int_0^t \frac{e^{\mu_k^1 (t-s)} - e^{\mu_k^2 (t-s)}}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)} < \varphi_k, y(s) > \varphi_k ds, \ t \in (0,\tau),$$

where prime at the sum means the absence of summands with index k *such that* $\lambda = \lambda_k$.

APPROXIMATE SOLUTION

$$(\lambda - \Delta)u_{tt} = \alpha(\Delta - \lambda')u_t + \beta(\Delta - \lambda'')u + y, \quad (10)$$

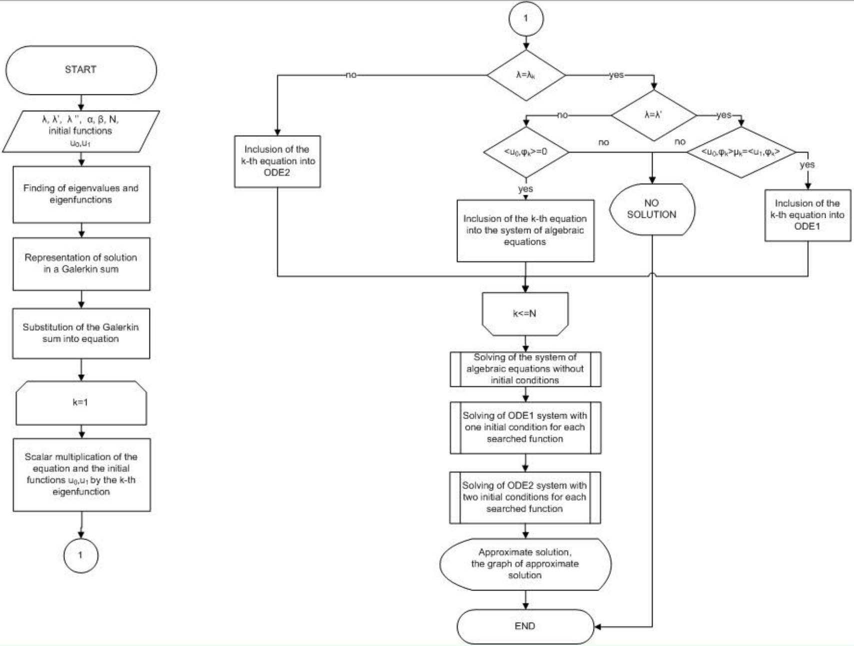
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \quad (11)$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega. \quad (12)$$

$$\tilde{u}(x,t) = u^N(x,t) = \sum_{k=1}^{N} u_k(t)\varphi_k(x),\ N \in \mathbb{N} : \lambda_N < \lambda.$$

where $N \in \mathbb{N}$ should be taken such that $\lambda_N < \lambda$ (to take into account effects of degeneracy).

The generalized scheme of numerical algorithm



Example 1.

Consider

$$(\lambda - \Delta)u_{tt} = \alpha(\lambda' - \Delta)u_t + \beta(\lambda'' - \Delta)u,$$

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,0) = x(\pi - x) - \pi/8 \sin x, u_t(x,0) = x(\pi - x) - \pi/8 \sin x,$$

where $\lambda = -1, \quad \lambda' = -1, \quad \lambda'' = 0, \quad \alpha = \beta = 1, N = 4 \quad x \in [0,\pi], \quad t \in [0,1].$

Since $\lambda \in \sigma(\Delta)$ and $\lambda = \lambda'$ then mathematical model is degenerate. Moreover, condition

$$< u_0, \varphi_1 >=< u_1, \varphi_1 >= 0 \tag{7}$$

holds, therefore u_0 and u_1 belong to the phase space of equation and the solution exists:

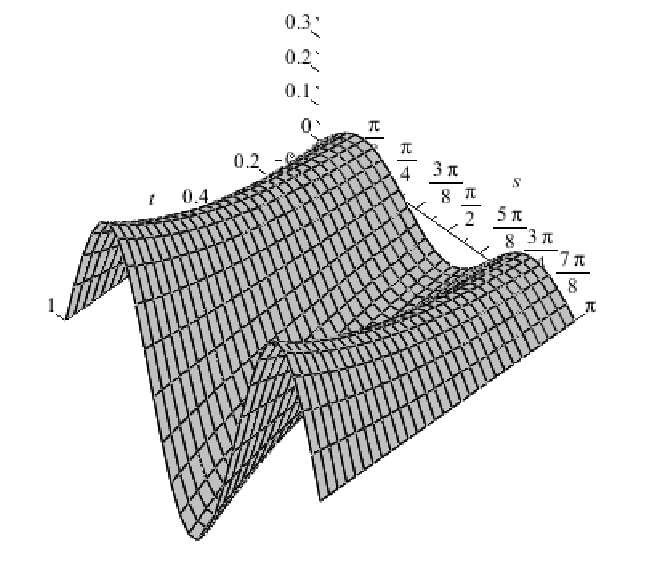
$$u(x,t) = [(2+\sqrt{22})e^{1/4(2+\sqrt{22})t} + (-2+\sqrt{22})e^{1/4(2-\sqrt{22})t}]\sin(3x)$$

Numerical accuracy

$\delta_{3,5}$	$\delta_{5,7}$	$\delta_{7,9}$	$\delta_{9,11}$	$\delta_{11,13}$	$\delta_{13,15}$
0.08259	0.03000	0.01409	0.00771	0.00467	0.00304

Here $\delta_{k,m} = \|u_k - u_m\|^2$ is a numerical accuracy.

Wave propagation for $t \in [0, 1]$.



Thank you for your attention!