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Lectures on positive semigroups and applications

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Notation and definitions

1 General Notation

The symbol ‘:=’ denotes ‘equal by definition’. The sets of all natural (not including 0), integer, real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, respectively. If $\lambda \in \mathbb{C}$, then we write $\Re\lambda$ for its real part, $\Im\lambda$ for its imaginary part, and $\bar{\lambda}$ for its complex conjugate. The symbols $[a, b]$, $]a, b[$ denote, respectively, closed and open intervals in $\mathbb{R}$. Moreover,

$$\mathbb{R}_+ := [0, \infty),$$
$$\mathbb{N}_0 := \{0, 1, 2, \ldots\}.$$

If there is a need to emphasise that we deal with multidimensional quantities, we use boldface characters, for example $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Usually we use the Euclidean norm in $\mathbb{R}^n$, denoted by

$$|\boldsymbol{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

If Ω is a subset of any topological space X, then by $\overline{\Omega}$ and $Int\,\Omega$ we denote, respectively, the closure and the interior of Ω with respect to X. If (X, d) is a metric space with metric d, we denote by

$$B_{x,r} := \{y \in X;\ d(x, y) \leq r\}$$

the closed ball with centre x and radius r. If X is also a linear space, then the ball with radius r, centred at the origin, is denoted by B_r.

Let f be a function defined on a set Ω and $x \in \Omega$. We use one of the following symbols to denote this function: f, $x \to f(x)$, and $f(\cdot)$. The symbol $f(x)$ is in general reserved to denote the value of f at x, however, occasionally, we abuse this convention and use it to denote the function itself.

If $\{x_n\}_{n\in\mathbb{N}}$ is a family of elements of some set, then the sequence of these elements, that is, the function $n \to x_n$, is denoted by $(x_n)_{n\in\mathbb{N}}$. However, for simplicity, we often abuse this notation and use $(x_n)_{n\in\mathbb{N}}$ also to denote $\{x_n\}_{n\in\mathbb{N}}$.

The derivative operator is usually denoted by ∂. However, as we occasionally need to distinguish different types of derivatives of the same function, we use other commonly accepted symbols for differentiation. To indicate the variable with respect to which we differentiate we write $\partial_t, \partial_x, \partial^2_{tx} \ldots$. If $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\partial_{\boldsymbol{x}} := (\partial_{x_1}, \ldots, \partial_{x_n})$ is the gradient operator.

If $\beta := (\beta_1, \ldots, \beta_n)$, $\beta_i \geq 0$ is a multi-index with $|\beta| := \beta_1 + \cdots + \beta_n = k$, then symbol $\partial^\beta_{\boldsymbol{x}} f$ is any derivative of f of order k. Thus, $\sum_{|\beta|=0}^{k} \partial^\beta f$ means the sum of all derivatives of f of order less than or equal to k.

2 Basic spaces

If $\Omega \subset \mathbb{R}^n$ is an open set, then for $k \in \mathbb{N}$ the symbol $C^k(\Omega)$ denotes the set of k times continuously differentiable functions in Ω. We denote by $C(\Omega) := C^0(\Omega)$ the set of all continuous functions in Ω and

$$C^\infty(\Omega) := \bigcap_{k=0}^{\infty} C^k(\Omega).$$

Functions from $C^k(\Omega)$ need not be bounded in Ω. If they are required to be bounded together with their derivatives up to the order k, then the corresponding set is denoted by $C_b^k(\Omega)$.

For a continuous function f, defined on Ω, we define the *support* of f as

$$\mathrm{supp} f = \overline{\{\boldsymbol{x} \in \Omega;\ f(\mathbf{x}) \neq 0\}}.$$

The set of all functions with compact support in Ω which have continuous derivatives of order smaller than or equal to k is denoted by $C_0^k(\Omega)$. As above, $C_0(\Omega) := C_0^0(\Omega)$ is the set of all continuous functions with compact support in Ω and

$$C_0^\infty(\Omega) := \bigcap_{k=0}^{\infty} C_0^k(\Omega).$$

Another important standard class of spaces are the spaces $L_p(\Omega)$, $1 \leq p \leq \infty$, of functions integrable with power p. To define them, let us establish some general notation and terminology. We begin with a *measure space* (Ω, Σ, μ), where Ω is a set, Σ is a σ-algebra of subsets of Ω, and μ is a σ-additive measure on Σ. We say that μ is *σ-finite* if Ω is a countable union of sets of finite measure.

In most applications in this book, $\Omega \subset \mathbb{R}^n$ and Σ is the σ-algebra of Lebesgue measurable sets. However, occasionally we need the family of *Borel sets* which, by definition, is the smallest σ-algebra which contains all open sets. The measure μ in the former case is called the Lebesgue measure and in the latter the Borel measure. Such measures are σ-finite.

A function $f : \Omega \to \mathbb{R}$ is said to be *measurable* (with respect to Σ, or with respect to μ) if $f^{-1}(B) \in \Sigma$ for any Borel subset B of $\mathbb{R}$. Because Σ is a σ-algebra, f is measurable if (and only if) preimages of semi-infinite intervals are in Σ.

Remark 1.1. The difference between Lebesgue and Borel measurability is visible if one considers compositions of functions. Precisely, if f is continuous and g is measurable on $\mathbb{R}$, then $g \circ f$ is measurable but, without any additional assumptions, $f \circ g$ is not. The reason for this is that the preimage of $\{x > a\}$ through f is open and preimages of open sets through Lebesgue measurable functions are measurable. On the other hand, preimage of $\{x > a\}$ through g is only a Lebesgue measurable set and preimages of such sets through continuous are not necessarily measurable. To have measurability of $f \circ g$ one has to assume that preimages of sets of measure zero through f are of measure zero (e.g., f is Lipschitz continuous).

We identify two functions which differ from each other on a set of μ-measure zero, therefore, when speaking of a function in the context of measure spaces, we usually mean a class of equivalence of functions. For most applications the distinction between a function and a class of functions is irrelevant.

The space of equivalence classes of all measurable real functions on Ω is denoted by $L_0(\Omega, d\mu)$ or simply $L_0(\Omega)$.

The integral of a measurable function f with respect to measure μ over a set Ω is written as

$$\int_{\Omega} f d\mu = \int_{\Omega} f(\boldsymbol{x}) d\mu_{\boldsymbol{x}},$$

where the second version is used if there is a need to indicate the variable of integration. If μ is the Lebesgue measure, we abbreviate $d\mu_{\boldsymbol{x}} = d\boldsymbol{x}$.

For $1 \leq p < \infty$, the spaces $L_p(\Omega)$ are defined as the subspaces of $L_0(\Omega)$ consisting of functions for which

$$\|f\|_p := \|f\|_{L_p(\Omega)} = \left(\int\limits_\Omega |f(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{1/p} < \infty. \tag{1.2.1}$$

The space $L_p(\Omega)$ with the above norm is a Banach space. It is customary to complete the scale of L_p spaces by the space $L_\infty(\Omega)$ defined to be the space of all Lebesgue measurable functions which are bounded almost everywhere in Ω, that is, bounded everywhere except possibly on a set of measure zero. The corresponding norm is defined by

$$\|f\|_\infty := \|f\|_{L_\infty(\Omega)} := \inf\{M;\ \mu(\{\boldsymbol{x} \in \Omega;\ |f(\boldsymbol{x})| > M\}) = 0\}. \tag{1.2.2}$$

The expression on the right-hand side of (1.2.2) is frequently referred to as the *essential supremum* of f over Ω and denoted $\operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})|$.

If $\mu(\Omega) < \infty$, then for $1 \leq p \leq p' \leq \infty$ we have

$$L_{p'}(\Omega) \subset L_p(\Omega) \tag{1.2.3}$$

and, for $f \in L_\infty(\Omega)$,

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p, \tag{1.2.4}$$

which justifies the notation. However,

$$\bigcap_{1\leq p<\infty} L_p(\Omega) \neq L_\infty(\Omega),$$

as demonstrated by the function $f(x) = \ln x,\ x \in (0, 1]$. If $\mu(\Omega) = \infty$, then neither (1.2.3) nor (1.2.4) hold.

Occasionally we need functions from $L_0(\Omega)$ which are L_p only on compact subsets of $\Omega \subset \mathbb{R}^n$. Spaces of such functions are denoted by $L_{p,loc}(\Omega)$. A function $f \in L_{1,loc}(\Omega)$ is called *locally integrable* (in Ω).

In many applications the set Ω is countable (or even finite). In such a case Ω will be identified with $\mathbb{N}$ (or $\mathbb{N}_0$) and will be equipped with a (possibly weighted) counting measure. Accordingly, we shall use the notation l_p to denote the space of sequences $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ equipped with the norm

$$\|\mathbf{x}\|_p = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} < \infty$$

and l_∞ is the sequences are bounded with $\|\mathbf{x}\|_\infty = \sup\limits_{n\in\mathbb{N}} |x_n|$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. It is clear that

$$C_0^\infty(\Omega) \subset L_p(\Omega)$$

for $1 \leq p \leq \infty$. If $p \in [1, \infty)$, then we have even more: $C_0^\infty(\Omega)$ is dense in $L_p(\Omega)$.

$$\overline{C_0^\infty(\Omega)} = L_p(\Omega), \tag{1.2.5}$$

where the closure is taken in the L_p-norm.

Example 1.2. Having in mind further applications, it is worthwhile to have some understanding of the structure of this result. Let us define the function

$$\omega(\boldsymbol{x}) = \begin{cases} \exp\left(\frac{1}{|\boldsymbol{x}|^2-1}\right) & \text{for } |\boldsymbol{x}| < 1, \\ 0 & \text{for } |\boldsymbol{x}| \geq 1. \end{cases} \tag{1.2.6}$$

This is a $C_0^\infty(\mathbb{R}^n)$ function with support B_1.

Using this function we construct the family

$$\omega_\epsilon(\boldsymbol{x}) = C_\epsilon \omega(\boldsymbol{x}/\epsilon),$$

where C_ϵ are constants chosen so that $\int_{\mathbb{R}^n} \omega_\epsilon(\boldsymbol{x}) d\boldsymbol{x} = 1$; these are also $C_0^\infty(\mathbb{R}^n)$ functions with support B_ϵ, often referred to as *mollifiers*. Using them, we define the *regularisation* (or *mollification*) of f by taking the convolution

$$(J_\epsilon * f)(\boldsymbol{x}) := \int\limits_{\mathbb{R}^n} f(\boldsymbol{x}-\boldsymbol{y})\omega_\epsilon(\boldsymbol{y})d\boldsymbol{y} = \int\limits_{\mathbb{R}^n} f(\boldsymbol{y})\omega_\epsilon(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{y}. \tag{1.2.7}$$

Precisely speaking, if $\Omega \neq \mathbb{R}^n$, we integrate outside the domain of definition of f. Thus, in such cases below, we consider f to be extended by 0 outside Ω.

Then, we have

Theorem 1.3. *With the notation above,*

1. *Let $p \in [1, \infty)$. If $f \in L_p(\Omega)$, then*
$$\lim_{\epsilon \to 0^+} \|J_\epsilon * f - f\|_p = 0.$$
2. *Let Ω be open. If $f \in C(\Omega)$, then $J_\epsilon * f \to f$ uniformly on any G such that $\bar{G} \subset \Omega$ ($G \Subset \Omega$).*
3. *If $\bar{\Omega}$ is compact and $f \in C(\bar{\Omega})$, then $J_\epsilon * f \to f$ uniformly on $\bar{\Omega}$.*

Remark 1.4. We observe that, if f is nonnegative, then f_ϵ are also nonnegative by (1.2.7) and hence any nonnegative $f \in L_p(\mathbb{R}^n)$ can be approximated by nonnegative, infinitely differentiable, functions with compact support.

3 Operators

Let X, Y be real or complex Banach spaces with the norm denoted by $\|\cdot\|$ or $\|\cdot\|_X$.

An *operator* from X to Y is a linear rule $A : D(A) \to Y$, where $D(A)$ is a linear subspace of X, called the *domain* of A. The set of operators from X to Y is denoted by $L(X, Y)$. Operators taking their values in the space of scalars are called *functionals*. We use the notation $(A, D(A))$ to denote the operator A with domain $D(A)$. If $A \in L(X, X)$, then we say that A (or $(A, D(A))$) is an operator in X.

By $\mathcal{L}(X, Y)$, we denote the space of all bounded operators between X and Y; $\mathcal{L}(X, X)$ is abbreviated as $\mathcal{L}(X)$. The space $\mathcal{L}(X, Y)$ can be made a Banach space by introducing the norm of an operator X by

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\|. \tag{1.3.8}$$

If $(A, D(A))$ is an operator in X and $Y \subset X$, then the *part* of the operator A in Y is defined as

$$A_Y y = Ay \tag{1.3.9}$$

on the domain

$$D(A_Y) = \{x \in D(A) \cap Y;\ Ax \in Y\}.$$

A *restriction* of $(A, D(A))$ to $D \subset D(A)$ is denoted by $A|_D$. For $A, B \in L(X, Y)$, we write $A \subset B$ if $D(A) \subset D(B)$ and $B|_{D(A)} = A$.

Two operators $A, B \in \mathcal{L}(X)$ are said to commute if $AB = BA$. It is not easy to extend this definition to unbounded operators due to the difficulties with defining the domains of the composition. The extension is usually done to the case when one of the operators is bounded. Thus, an operator $A \in L(X)$ is said to *commute* with $B \in \mathcal{L}(X)$ if

$$BA \subset AB. \tag{1.3.10}$$

This means that for any $x \in D(A)$, $Bx \in D(A)$ and $BAx = ABx$.

We define the *image* of A by

$$ImA = \{y \in Y;\ y = Ax \text{ for some } x \in D(A)\}$$

and the *kernel* of A by

$$KerA = \{x \in D(A);\ Ax = 0\}.$$

We note a simple result which is frequently used throughout the book.

Proposition 1.5. *Suppose that $A, B \in L(X,Y)$ satisfy: $A \subset B, KerB = \{0\}$, and $ImA = Y$. Then $A = B$.*

Proof. If $D(A) \neq D(B)$, we take $x \in D(B) \setminus D(A)$ and let $y = Bx$. Because A is onto, there is $x' \in D(A)$ such that $y = Ax'$. Because $x' \in D(A) \subset D(B)$ and $A \subset B$, we have $y = Ax' = Bx'$ and $Bx' = Bx$. Because $KerB = \{0\}$, we obtain $x = x'$ which is a contradiction with $x \notin D(A)$. □

Furthermore, the *graph* of A is defined as

$$G(A) = \{(x,y) \in X \times Y;\ x \in D(A), y = Ax\}. \tag{1.3.11}$$

We say that the operator A is *closed* if $G(A)$ is a closed subspace of $X \times Y$. Equivalently, A is closed if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$, if $\lim_{n\to\infty} x_n = x$ in X and $\lim_{n\to\infty} Ax_n = y$ in Y, then $x \in D(A)$ and $y = Ax$.

An operator A in X is *closable* if the closure of its graph $\overline{G(A)}$ is itself a graph of an operator, that is, if $(0,y) \in \overline{G(A)}$ implies $y = 0$. Equivalently, A is closable if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$, if $\lim_{n\to\infty} x_n = 0$ in X and $\lim_{n\to\infty} Ax_n = y$ in Y, then $y = 0$. In such a case the operator whose graph is $\overline{G(A)}$ is called the *closure* of A and denoted by $\overline{A}$.

By definition, when A is closable, then

$$\begin{aligned} D(\overline{A}) = \{x \in X;\ &\text{there is } (x_n)_{n\in\mathbb{N}} \subset D(A) \text{ and } y \in X \text{ such that} \\ &\|x_n - x\| \to 0 \text{ and } \|Ax_n - y\| \to 0\}, \\ \overline{A}x = y.& \end{aligned}$$

For any operator A, its domain $D(A)$ is a normed space under the *graph norm*

$$\|x\|_{D(A)} := \|x\|_X + \|Ax\|_Y. \tag{1.3.12}$$

The operator $A : D(A) \to Y$ is always bounded with respect to the graph norm, and A is closed if and only if $D(A)$ is a Banach space under (1.3.12).

3.1 The differentiation operator

One of the simplest and most often used unbounded, but closed or closable, operators is the operator of differentiation. If X is any of the spaces $C([0,1])$ or $L_p([0,1])$, then considering $f_n(x) := C_n x^n$, where $C_n = 1$ in the former case and $C_n = (np+1)^{1/p}$ in the latter, we see that in all cases $\|f_n\| = 1$. However,

$$\|\partial f_n\| = n\left(\frac{np+1}{np+1-p}\right)^{1/p}$$

in $L_p([0,1])$ and $\|\partial f_n\| = n$ in $C([0,1])$, so that the operator of differentiation is unbounded.

Let us define $Tf = f'$ as an unbounded operator on $D(T) = \{f \in X;\ Tf \in X\}$, where X is any of the above spaces. We can easily see that in $X = C([0,1])$ the operator T is closed. Indeed, let us take $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} Tf_n = g$ in X. This means that $(f_n)_{n\in\mathbb{N}}$ and $(f'_n)_{n\in\mathbb{N}}$ converge uniformly to, respectively, f and g, and from basic calculus f is differentiable and $\partial f = g$.

The picture changes, however, in L_p spaces. To simplify the notation, we take $p = 1$ and consider the sequence of functions

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{1}{2}, \\ \frac{n}{2}\left(x - \frac{1}{2}\right)^2 & \text{for } \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n}, \\ x - \frac{1}{2} - \frac{1}{2n} & \text{for } \frac{1}{2} + \frac{1}{n} < x \le 1. \end{cases}$$

These are differentiable functions and it is easy to see that $(f_n)_{n\in\mathbb{N}}$ converges in $L_1([0,1])$ to the function f given by $f(x) = 0$ for $x \in [0, 1/2]$ and $f(x) = x - 1/2$ for $x \in (1/2, 1]$ and the derivatives converge to $g(x) = 0$ if $x \in [0, 1/2]$ and to $g(x) = 1$ otherwise. The function f, however, is not differentiable and so T is not closed. On the other hand, g seems to be a good candidate for the derivative of f in some more general sense. Let us develop this idea further. First, we show that T is closable. Let $(f_n)_{n\in\mathbb{N}}$ and $(\partial f_n)_{n\in\mathbb{N}}$ converge in X to f and g, respectively. Then, for any $\phi \in C_0^\infty((0,1))$, we have, integrating by parts,

$$\int_0^1 \partial f_n(x)\phi(x)dx = -\int_0^1 f_n(x)\partial\phi(x)dx$$

and because we can pass to the limit on both sides, we obtain

$$\int_0^1 g(x)\phi(x)dx = -\int_0^1 f(x)\partial\phi(x)dx. \tag{1.3.13}$$

Using the equivalent characterization of closability, we put $f = 0$, so that

$$\int_0^1 g(x)\phi(x)dx = 0$$

for any $\phi \in C_0^\infty((0,1))$ which yields $g(x) = 0$ almost everywhere on $[0,1]$. Hence $g = 0$ in $L_1([0,1])$ and consequently T is closable.

The domain of $\overline{T}$ in $L_1([0,1])$ is called the Sobolev space $W_1^1([0,1])$ which is discussed below.

These considerations can be extended to hold in any $\Omega \subset \mathbb{R}^n$. In particular, we can use (1.3.13) to generalize the operation of differentiation in the following way: we say that a function $g \in L_{1,loc}(\Omega)$ is the *generalised (or distributional) derivative* of $f \in L_{1,loc}(\Omega)$ of order α, denoted by $\partial_{\boldsymbol{x}}^\alpha f$, if

$$\int_\Omega g(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x} = (-1)^{|\beta|}\int_\Omega f(\boldsymbol{x})\partial_{\boldsymbol{x}}^\beta\phi(\boldsymbol{x})d\boldsymbol{x} \tag{1.3.14}$$

for any $\phi \in C_0^\infty(\Omega)$.

This operation is well defined. This follows from the du Bois Reymond lemma.

From the considerations above it is clear that $\partial_{\boldsymbol{x}}^\beta$ is a closed operator extending the classical differentiation operator (from $C^{|\beta|}(\Omega)$). One can also prove that ∂^β is the closure of the classical differentiation operator.

Proposition 1.6. *If $\Omega = \mathbb{R}^n$, then ∂^β is the closure of the classical differentiation operator.*

Proof. We use Proposition 1.3. Indeed, let $f \in L_p(\mathbb{R}^n)$ and $g = \partial^\beta f \in L_p(\mathbb{R}^n)$. We consider $f_\epsilon := J_\epsilon * f \to f$ in L_p. By the Fubini theorem, we prove

$$\int_{\mathbb{R}^n} (J_\epsilon * f)(\boldsymbol{x})\partial^\beta \phi(\boldsymbol{x})d\boldsymbol{x} = \int_{\mathbb{R}^n} \omega_\epsilon(y) \int_{\mathbb{R}^n} f(\boldsymbol{x}-\boldsymbol{y})\partial^\beta \phi(\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{y}$$
$$= (-1)^{|\beta|} \int_{\mathbb{R}^n} \omega_\epsilon(y) \int_{\mathbb{R}^n} g(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{y}$$
$$= (-1)^{|\beta|} \int_{\mathbb{R}^n} (J_\epsilon * g)\phi(\boldsymbol{x})d\boldsymbol{x}$$

so that $\partial^\beta f_\epsilon = J_\epsilon * \partial^\beta f = J_\epsilon * g \to g$ as $\epsilon \to 0$ in $L_p(\mathbb{R}^n)$. This shows that action of the distributional derivative can be obtained as the closure of the classical derivation.

Otherwise the proof is more complicated since we do not know whether we can extend f outside Ω in such a way that the extension still will have the generalized derivative. We shall discuss it later.

Example 1.7. **A non closable operator.** Let us consider the space $X = L_2((0,1))$ and the operator $K : X \to Y$, $Y = X \times \mathbb{C}$ (with the Euclidean norm), defined by

$$Kv =< v, v(1) > \tag{1.3.15}$$

on the domain $D(K)$ consisting of continuous functions on $[0,1]$. We have the following lemma

Lemma 1.8. *K is not closable, but has a bounded inverse. ImK is dense in Y.*

Proof. Let $f \in C^\infty([0,1])$ be such that

$$f(x) = \begin{cases} 0 \text{ for } 0 \le x < 1/3 \\ 1 \text{ for } 2/3 < x \le 1. \end{cases}$$

To construct such a function, we can consider e.g. $J_\epsilon * \bar{f}$, where

$$\bar{f}(x) = \begin{cases} 1 \quad \text{for} \quad \frac{2}{3} - \epsilon < x \le 1, \\ 0 \text{ otherwise} \end{cases}$$

and $\epsilon < 1/3$. Let $v_n(x) = f(x^n)$ for $0 \le x \le 1$. Clearly, $v_n \in D(K)$ and $v_n \to 0$ in $L_2((0,1))$ as

$$\int_0^1 f^2(x^n)dx = \int_{3^{-1/n}}^1 f^2(x^n)dx = \frac{1}{n}\int_{1/3}^1 z^{-1+1/n} f^2(z)dz.$$

However, $Kv_n =< v_n, 1 >\to< 0, 1 >\neq< 0, 0 >$.

Further, K is one-to-one with $K^{-1}(v, v(1)) = v$ and

$$\|K^{-1}(v, v(1))\|^2 = \|v\|^2 \le \|v\|^2 + |v(1)|^2.$$

To prove that ImK is dense in Y, let $< y, \alpha >\in Y$. We know that $C_0^\infty((0,1)) \subset D(K)$ is dense in $Z = L_2((0,1))$. Let (ϕ_n) be sequence of C_0^∞-functions which approximate y in $L_2(0,1)$ and put $w_n = \phi_n + \alpha v_n$. We have $Kw_n =< w_n, \alpha >\to< y, \alpha >$.

3.2 Operators defined by bilinear forms

In Hilbert spaces there is a convenient way of defining operators through bilinear (sesquilinear) forms.

Definition 1.9. *Let V be a Hilbert space. A bilinear form $a : V \times V \to \mathbb{R}$ is said to be* *(i) continuous of there is a constant C such that*

$$|a(x,y)| \leq C\|x\|\|y\|, \qquad x,y \in V;$$

coercive if there is a constant $\alpha > 0$ such that

$$a(x,x) \geq \alpha\|x\|^2.$$

Note that in the complex case, coercivity means $\Re a(x,x) \geq \alpha\|x\|^2$.

If we fix $x \in V$, then $y \to a(x,y)$ is a continuous linear functional on V. Thus there is an operator $A : V \to V^*$ satisfying $a(x,y) = <Ax, y>_{V^*\times V}$. Clearly, A is linear and satisfies

$$\|Ax\|_{V^*} \leq C\|x\|_V, \tag{1.3.16}$$

$$<Ax, y>_{V^*\times V} \geq \alpha\|x\|_V^2. \tag{1.3.17}$$

Indeed,

$$\|Ax\| = \sup_{\|y\|=1} | <Ax, y>_{V^*\times V} | \leq C\|x\| \sup_{\|y\|=1} \|y\|,$$

and (1.3.17) is obvious.

The above construction often is used in the following context. Let V and H be two Hilbert spaces such that $V \subset H$ with continuous and dense embedding. Using the Riesz theorem, we identify H with H^* and then we can make identification

$$V \subset H \subset V^*.$$

Then a continuous bilinear form $a(\cdot,\cdot)$ defines an unbounded operator A in H by restricting the operator A defined above to

$$D(A) = \{u \in V;\ Au \in H\}.$$

An important result in this context is known as the Lax-Milgram lemma.

Theorem 1.10. *Assume that $a(\cdot,\cdot)$ is a continuous coercive bilinear form on a Hilbert space V. Then, given any $\phi \in V^*$, there exists a unique element $x \in V$ such that for any $y \in V$*

$$a(x,y) = < \phi, y >_{V^*\times V} \tag{1.3.18}$$

Moreover, if a is symmetric, then x is characterized by the property

$$x \in V \quad \text{and} \quad \frac{1}{2}a(x,x) - < \phi, x >_{V^*\times V} = \min_{y\in V} \frac{1}{2}a(y,y) - < \phi, y >_{V^*\times V}. \tag{1.3.19}$$

4 Sobolev spaces

With the notion of generalised derivative we can introduce a class of Banach spaces which will play crucial role in the theory of differential equations which will be used throughout the lectures: the Sobolev spaces.

Let $\Omega \subseteq \mathbb{R}^n$. For $p \in [1,\infty]$ we define the Sobolev spaces $W_p^m(\Omega)$ are defined as

$$W_p^m(\Omega) := \{u \in L_p(\Omega);\ \partial^\alpha u \in L_p(\Omega),\ |\alpha| \leq m\}.$$

These are Banach spaces with the norm

$$\|u\|_{W_2^m(\Omega)} := \left(\sum_{|\alpha|\leq m} \|\partial^\alpha u\|^p_{L_p(\Omega)}\right)^{\frac{1}{p}}.$$

For the particular case $p = 2$, $W_2^m(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{W_2^m(\Omega)} := \sum_{|\alpha| \le m} (\partial^\alpha u, \partial^\alpha v)_{L_2(\Omega)} .$$

From Proposition 1.6 we see that for $p \in [1, \infty[$ the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W_p^m(\mathbb{R}^n)$. For $\Omega \ne \mathbb{R}^n$ the problem is more difficult due to the behaviour of functions close to the boundary of Ω.

4.1 One dimensional case

A special role is played by Sobolev spaces in one dimension. This is due to the following theorem.

Theorem 1.11. *Assume that $u \in L_{p,loc}(\mathbb{R})$ and its generalised derivative ∂u also satisfies $\partial u \in L_{p,loc}(\mathbb{R})$. Then there is a continuous representation $\tilde{u}$ of u such that*

$$\tilde{u}(x) = C + \int_0^x \partial u(t) dt$$

for some constant C and thus u is differentiable almost everywhere.

In the proof we use two facts. First, that if $f \in L_{p,loc}(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} f \phi' dx = 0$$

for any $\phi \in C_0^\infty(\mathbb{R})$, then $f = const$ almost everywhere and, second, that the function $v(x) = \int_{x_0}^x f(y) dy$ with $f \in L_{p,loc}(\mathbb{R})$ is continuous and the generalized derivative of v, ∂v, equals f. The most difficult part, that v is differentiable almost everywhere, is proved in a real analysis courses.

This makes it easier to prove that that restrictions of $C_0^\infty(\mathbb{R})$ functions are dense in $W_p^m(I)$, where I is an interval. To be able to use Proposition 1.6, we begin by extending the functions to $\mathbb{R}$.

We focus our attention on $u \in W_p^1(I)$ with $I =]0, \infty[$. Then, defining

$$u^*(x) = \begin{cases} u(x) & \text{for } x > 0, \\ u(-x) & \text{for } x < 0 \end{cases}$$

we find that

$$v(x) = \begin{cases} \partial v(x) & \text{for } x > 0, \\ -\partial u(-x) & \text{for } x < 0 \end{cases}$$

satisfies $v \in L_p(\mathbb{R})$ and, using the fact that u^* is continuous,

$$u^*(x) - u^*(0) = \int_0^x v(s) ds.$$

Hence, by the previous theorem, $v \in W_p^1(\mathbb{R})$. In the case of $]a, b[$, we transform it to $I =]0, 1[$ and consider the decomposition

$$u = \eta u + (1 - \eta) u$$

where η is a C^∞ function equal to 1 for $x \in] - \infty, 1/4]$ and to 0 for $x \in [3/4, \infty[$. Then ηu can be extended by 0 to $[0, \infty[$ without changing its class and then by reflection as above. The second term is extended by 0 to $] - \infty, 1]$ and then by reflection (about 1). In this way we obtain $u^* \in W_p^1(\mathbb{R})$. For u^* we use Proposition 1.6 to find a sequence $(u_n)_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R})$ such that $u_n \to u^*$ and $\partial u_n \to \partial u^*$ in $L_p(\mathbb{R})$ and thus the sequence of restrictions $(u_n|_I)_{n \in \mathbb{N}}$ converges to u in $W_p^1(I)$.

In the case $u \in W_p^m(I)$, the extension across is done by the following reflection

$$u^*(x) = \begin{cases} u(x) & \text{for } x > 0 \\ \lambda_1 u(-x) + \lambda_2 u\left(-\frac{x}{2}\right) + \ldots + \lambda_m u\left(-\frac{x_n}{m}\right) & \text{for } x < 0, \end{cases}$$

where $\lambda_1, \ldots, \lambda_m$ is the solution of the system

$$\begin{aligned} \lambda_1 + \lambda_2 + \ldots + \lambda_m &= 1, \\ -(\lambda_1 + \lambda_2/2 + \ldots + \lambda_m/m) &= 1, \\ &\ldots \\ (-1)^m(\lambda_1 + \lambda_2/2^{m-1} + \ldots + \lambda_m/m^{m-1}) &= 1 \end{aligned}$$

These conditions ensure that the derivatives are continuous at $x = 0$.

4.2 Dirichlet problem

One of the classical problems in mathematics and physics is the Dirichlet problem for the Poisson equation in $\Omega \subset \mathbb{R}^n$. Find u which solves

$$-\Delta u = f \qquad \text{in} \quad \Omega, \tag{1.4.20}$$

$$u|_{\partial\Omega} = 0. \tag{1.4.21}$$

Assume that there is a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$. If we multiply (1.4.20) by a test function $\phi \in C_0^\infty(\Omega)$ and integrate by parts, then we obtain the problem

$$\int_\Omega \nabla u \cdot \nabla \phi dx = \int_\Omega f \phi d\boldsymbol{x}. \tag{1.4.22}$$

Let us consider the space $H = L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$ bounded, with the scalar product

$$(u, v)_0 = \int_\Omega u(x) v(x) d\boldsymbol{x}.$$

We know that $\overline{C_0^\infty(\Omega)}^H = H$. The relation (1.4.22) suggests that we should consider another scalar product, initially on $C_0^\infty(\Omega)$, given by

$$(u, v)_{0,1} = \int_\Omega \nabla u(x) \nabla v(x) d\boldsymbol{x}.$$

Note that due to the fact that u, v have compact supports, this is a well defined scalar product as

$$0 = (u, u)_{0,1} = \int_\Omega |\nabla u(x)|^2 d\boldsymbol{x}$$

implies $u_{x_i} = 0$ for all x_i, $i = 1, \ldots, n$ hence $u = \mathit{const}$ and thus $u \equiv 0$. Note that this is not a scalar product on a space $C^\infty(\bar{\Omega})$.

A fundamental role in the theory is played by the Zaremba - Poincarè - Friedrichs lemma.

Lemma 1.12. *There is a constant d such that for any $u \in C_0^\infty(\Omega)$*

$$\|u\|_0 \le d \|u\|_{0,1}. \tag{1.4.23}$$

We define $\overset{\circ}{W}{}_2^1(\Omega)$ as the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{0,1}$. We have

Theorem 1.13. *The space $\overset{\circ}{W}{}_2^1(\Omega)$ is a separable Hilbert space which can be identified with a subspace continuously and densely embedded in $L_2(\Omega)$. Every $v \in \overset{\circ}{W}{}_2^1(\Omega)$ has generalized derivatives $D_{x_i}v \in L_2(\Omega)$. Furthermore, the distributional integration by parts formula*

$$\int\limits_{\Omega} \partial_{x_i} v u d\boldsymbol{x} = -\int\limits_{\Omega} v \partial_{x_i} u d\boldsymbol{x} \tag{1.4.24}$$

is valid for any $u, v \in \overset{\circ}{W}{}_2^1(\Omega)$.

We note that Zaremba - Poincarè - Friedrichs lemma shows that on $\overset{\circ}{W}{}_2^1(\Omega)$ the norm $\|\cdot\|_{0,1}$ is equivalent to $\|\cdot\|_{W_2^1(\Omega)}$.

Consider now on $\overset{\circ}{W}{}_2^1(\Omega)$ the bilinear form

$$a(u,v) = \int\limits_{\Omega} \nabla u \nabla v d\boldsymbol{x}.$$

Clearly, by Cauchy-Schwarz inequality

$$|a(u,v)| \leq \|u\|_{0,1}\|v\|_{0,1}$$

and

$$a(u,u) = \int\limits_{\Omega} \nabla u \nabla u d\boldsymbol{x} = \|u\|_{0,1}^2$$

and thus a is a continuous and coercive bilinear form on $\overset{\circ}{W}{}_2^1(\Omega)$. Thus, if we take $f \in (\overset{\circ}{W}{}_2^1(\Omega))^* \supset L_2(\Omega)$ then there is a unique $u \in \overset{\circ}{W}{}_2^1(\Omega)$ satisfying

$$\int\limits_{\Omega} \nabla u \nabla v d\boldsymbol{x} = < f, v >_{(\overset{\circ}{W}{}_2^1(\Omega))^* \times \overset{\circ}{W}{}_2^1(\Omega)}$$

for any $v \in \overset{\circ}{W}{}_2^1(\Omega)$ or, equivalently, minimizing the functional

$$J(v) = \frac{1}{2}\int\limits_{\Omega} |\nabla v|^2 d\boldsymbol{x} - < f, v >_{(\overset{\circ}{W}{}_2^1(\Omega))^* \times \overset{\circ}{W}{}_2^1(\Omega)}$$

over $K = \overset{\circ}{W}{}_2^1(\Omega)$.

The question is what this solution represents. Clearly, taking $v \in C_0^\infty(\Omega)$ we obtain

$$-\Delta u = f$$

in the sense of distribution. However, to get a deeper understanding of the meaning of the solution and, in the particular, in what sense the boundary condition is satisfied, we have investigate the structure of $\overset{\circ}{W}{}_2^1(\Omega)$ which is beyond the scope of these lectures.

2

Motivation

1 Models and semigroups

Laws of physics and, increasingly, also those of other sciences are in many cases expressed in terms of differential or integro–differential equations. If one models systems evolving with time, then the variable describing time plays a special role, as the equations are built by balancing the change of the system in time against its ‘spatial’ behaviour. Such equations are called *evolution equations*. Such equations usually are formulated pointwise; that is, all the operations, such as differentiation and integration, are understood in the classical (calculus) sense and the equation itself is supposed to be satisfied for all values of the independent variables in the relevant domain:

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= [\mathcal{A}u(\cdot,t)](x), \quad x \in \Omega \\ u(0,t) &= \mathring{u}, \end{aligned} \tag{2.1.1}$$

where $\mathcal{A}$ is a certain expression, differential, integral, or functional, that can be evaluated at any point $x \in \Omega$ for all functions from a certain subset S.

However, when we are trying to solve (2.1.1), we change its meaning by imposing various *a priori* restrictions on the solution to make (2.1.1) amenable to particular techniques. Quite often (2.1.1) does not provide a complete description of the dynamics even if it looks complete from the modelling point of view. Then the obtained solution maybe be not what we have been looking for. This becomes particularly important if we cannot get our hands on the actual solution but use ’soft analysis’ to find relevant properties of it. These lecture notes are predominantly devoted to one particular way of looking at the evolution of a system in which we describe time changes as transitions from one state to another; that is, the evolution is described by a family of operators $\{G(t)\}_{t\geq 0}$, parameterised by time, that map an initial state of the system onto all subsequent states in the evolution; that is solutions are represented as

$$u(t) = G(t)u_0. \tag{2.1.2}$$

The family $\{G(t)\}_{t\geq 0}$ is called *semigroup* and u_0 is an initial state.

In this approach we place everything in some abstract space X, which is chosen partially for the relevance to the problem and partially for mathematical convenience. For example, if (2.1.1) describes the evolution of an ensemble of particles, then u is the particle density function and the natural space seems to be $L_1(\Omega)$ as in this case the norm of a nonnegative u, that is, the integral over Ω, gives the total number of particles in the ensemble. It is important to note that this choice is not unique but rather is a mathematical intervention into the model, which could change it in a quite dramatic way. For instance, staying with this case, we could choose the space of measures on Ω with the same interpretation of the norm. On the other hand, if we are interested in controlling the maximum concentration of particles, a more proper choice would be some reasonable space with a supremum norm, such as, for example, the space of bounded continuous functions on Ω, $C_b(\Omega)$.

Once we select our space, the right-hand side can be interpreted as an operator $A : D(A) \to X$, defined on some subset $D(A)$ of X (not necessarily equal to X), such that $x \to [\mathcal{A}u](x) \in X$. With this, (2.1.1) can be written as an ordinary differential equation in X:

$$\begin{aligned} \partial_t u &= Au, \qquad t > 0, \\ u(0) &= u_0 \in X. \end{aligned} \tag{2.1.3}$$

The domain $D(A)$ is also not uniquely defined by the model. Clearly, a minimum requirement for $D(A)$, apart from $[\mathcal{A}u](\cdot) \in X$ for $u \in D(A)$, is that the solution originating from $D(A)$ could be differentiated in X so that both sides of the equation would make sense. It is important to remember that the expression $\mathcal{A}$ usually has multiple realizations and finding an appropriate one, such that with $(A, D(A))$ the problem (2.1.3) is well posed (such a realisation is often called the generator of the process) is a very difficult task. Although throughout the lectures we assume that the underlying space is given, finding $D(A)$, on which we define the realisation A of the expression $\mathcal{A}$, is a more complicated thing and has major implications as to whether we are getting from the model what we bargained for.

As an example we mention the so-called maximal realization of the expression $\mathcal{A}$, $A_{\max}$ defined as the restriction of $\mathcal{A}$ to

$$D(A_{\max}) = \{u \in X;\ x \to [\mathcal{A}u](x) \in X\}.$$

The generator may be, or may be not, equal to $A_{\max}$.

What can go wrong?

As we shall see, semigroup theory in some sense forces $D(A)$ upon us, although it is not necessarily the optimal choice from the modelling point of view. To explain this, we note that following pathologies often occur.

Dishonesty. Models are based on certain laws coming from the applied sciences and we expect the solutions to equations of these models to return these laws. However, this is not always true: we will see models built on the basis of population conservation principles, solutions of which, for certain classes of parameters, do not preserve populations. Such models are called *dishonest*. Dishonesty could be a sign of a phase transition happening in the model, or simply indicate limits of validity of the model.

Multiple solutions. Even if all side conditions relevant to the modelled process seem to have been built into the model, we may find that the model does not provide full description of the dynamics; while for some classes of parameters the model gives uniquely determined solutions, for others there exist multiple solutions. This happens if $A \neq A_{\max}$.

Though we also discuss a general theory, our main focus is on models preserving some notion of *positivity*: non-negative inputs should give non-negative outputs (in a suitable sense of the word). We will see that methods based on positivity methods provide a comprehensive explanation of these two 'pathological' phenomena.

2 Models

In most models discussed in these lecture notes we will look at the evolution of populations of some objects (not necessarily live - could be particles or polymers). One way of describing such an evolution is by providing the rule how the density of the objects with respect to certain selected attributes changes in time. The density, say $u(x)$, is either the number of elements with an attribute x (if the number of possible attributes is finite or countable), or gives the quantity of elements with attributes in a set A, according to the formula

$$\int_A u(x)d\mu, \tag{2.2.4}$$

if x is a continuous variable. Here μ is a measure selected accordingly to the properties of the model which we analyze.

In many cases we are interested in tracking the total number of elements of the population which, for a given time t, is given by

$$\sum_{x\in\Omega} u(x,t), \tag{2.2.5}$$

if Ω is countable, and by

$$\int_{\Omega} u(x,t)\mathrm{d}\mu_x, \tag{2.2.6}$$

if Ω is a continuum. Thus, the natural spaces to study such objects are spaces l^1 in the former case (where Ω is identified with $\mathbb{N}$) or $L_1(\Omega)$ in the latter. However, there are models in which a more suitable are the spaces $L_p(\Omega)$ or $C_b(\Omega)$. Appropriate space will be defined when needed.

2.1 Evolution of countable ensembles of objects

We consider a countable system of objects labelled by the some attribute $n \in \mathbb{N}$. The state of the system is described by a vector $\mathbf{u} = (u_0, u_1, \ldots, u_n, \ldots)$, where u_n is the number of objects with attribute n. Note, that in probabilistic interpretation, u_n is the probability of finding an object with attribute n so that the coordinates of $\mathbf{u}$ add up to 1. Any object in the system can change its attribute by some mechanism and, in the simplest case discussed here, the only possible events are changing the attribute n either to $n+1$, or to $n-1$. We assume that the rates of change are given and are denoted by d_n and b_n for changes $n \to n-1$ and $n \to n+1$, respectively. In general, we can also include a mechanism that changes a number of objects with attribute n by, for example, removing them from the environment or, otherwise, introducing them. The rate of this mechanism is denoted by $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$.

Standard modelling procedure by balancing gain and loss of objects at each level yields

$$\begin{aligned} u_0' &= -a_0u_0 + d_1u_1, \\ &\vdots \\ u_n' &= -a_nu_n + d_{n+1}u_{n+1} + b_{n-1}u_{n-1}, \\ &\vdots \end{aligned} \tag{2.2.7}$$

where $c_n = b_n + d_n - a_n$.

The classical application of this system comes from population theory, where it is a particular case of a Kolmogorov system; in this case u_n is the probability that the described population consists of n individuals and its state can change by either the death or birth of an individual thus moving the population to the state $n-1$ or $n+1$, respectively, hence the name birth-and-death system. The classical birth-and-death system is formally conservative; this is equivalent to $a_n = d_n + b_n$. However, recently a number of other important applications have emerged. For instance, we can consider an ensemble of cancer cells structured by the number of copies of a drug-resistant gene they contain. Here, the number of cells with n copies of the gene can change due to mutations, but the cells also undergo division without changing the number of genes in their offspring which is modelled by a nonzero vector $\mathbf{c}$.

The most common setting for birth-and-death problems is the space l_1.

2.2 Transport and its variants

Possibly the simplest partial differential equation is the transport equation

$$\partial_t u(x,t) + c\partial_x u(x,t) = 0, \qquad x \in \mathbb{R}, \tag{2.2.8}$$

which is a conservation law and can describe for instance a flow of a substance with density u to the right if $c > 0$ or to the left if $c < 0$. The equation must be supplemented with the initial condition

$$u(x, 0) = u_0(x), \tag{2.2.9}$$

where $u_0(x)$ is a given initial density. If we do not consider the flow on the whole $\mathbb{R}$ we should specify a boundary condition at the starting point of the flow; if $c > 0$, then we can require e.g.

$$u(0, t) = \phi(t) \tag{2.2.10}$$

for some given function ϕ.

This equation has, however, many more applications.

McKendrick equation

Consider an age-structured population of individuals, where $n(a, t)$ is the density of the population of age a and time t. Contrary to the previous example, with just two age classes, here the age a of an individual is a continuous variable. In the simplest case it is described by the McKendrick model, [49], the density n is a solution of the following initial boundary value problem

$$\begin{aligned} \partial_t n(a, t) &= -\partial_a n(a, t) - \mu(a) n(a, t), \\ n(0, t) &= \int_0^\omega \beta(a) n(a, t) \mathrm{d}a, \\ n(a, 0) &= \overset{\circ}{n}(a), \end{aligned} \tag{2.2.11}$$

where $a \in [0, \omega[$ and $0 < \omega \leq \infty$ is the maximum age in the population. The first equation, where μ is the age specific death rate, is a conservation law. It shows that the rate of change of the number of individuals at any age is due to aging, $-\partial_a n(a, t)$, with the rate of aging equal to 1 and to the deaths of individuals. At the same time, newborns appear necessarily at the age $a = 0$ and thus they are modelled by the second equation in (2.2.11) which is the boundary condition. Here, $\beta(a)$ is the age specific per capita birth rate so that the right hand side of the second equation is the sum over all ages of the number of newborns produced in a unit time by the individuals; that is, it gives the total birth rate at time t of the population.

2.3 Age-structured epidemiological model

Consider an SIRS system

$$\begin{aligned} S' &= -\Lambda(I) S + \delta I, \\ I' &= \Lambda(I) S - (\delta + \gamma) I, \\ R' &= \gamma I, \end{aligned} \tag{2.2.12}$$

where S, I, R are, respectively, the number of susceptibles, infectives and recovered (with immunity) and γ, δ are recovery rates with and without immunity. For many diseases the rates of infection and recovery significantly vary with age. Thus the vital dynamics of the population and the infection mechanism can interact to produce a nontrivial dynamics. To model it, we assume that the total population in the absence of disease can be modelled by the linear McKendrick model describing the evolution in time of the density of the population with respect to age $a \in [0, \omega[$, $\omega \leq \infty$, denoted by $n(a, t)$. The evolution is driven by the processes of death and birth with vital rates $\mu(a)$ and $\beta(a)$, respectively. Due to the epidemics, we split the population into susceptibles, infectives and recovered,

$$n(a, t) = s(a, t) + i(a, t) + r(a, t),$$

so that the scalar McKendrick equation for n splits, according to (2.2.12), into the system

$$\begin{aligned}
\partial_t s(a,t) + \partial_a s(a,t) + \mu(a)s(a,t) &= -\Lambda(a, i(\cdot,t))s(a,t) + \delta(a)i(a,t),\\
\partial_t i(a,t) + \partial_a i(a,t) + \mu(a)i(a,t) &= \Lambda(a, i(\cdot,t))s(a,t) - (\delta(a)+\gamma(a))i(a,t),\\
\partial_t r(a,t) + \partial_a r(a,t) + \mu(a)r(a,t) &= \gamma(a)i(a,t),
\end{aligned} \tag{2.2.13}$$

where now the rates are age specific, see [40]. The function Λ is the infection rate (or the force of infection). In the so-called intercohort model, which will be analysed later in these lectures, we use

$$\Lambda(a, i(\cdot,t)) = \int\limits_0^\omega K(a,a')i(a',t)da', \tag{2.2.14}$$

where K is a nonnegative bounded function which accounts for the age dependence of the infections. For instance, for a typical childhood disease, K should be large for small a, a' and close to zero for large a or a' (not necessarily 0, as usually adults can contract them). System (2.2.13) is supplemented by the boundary conditions

$$\begin{aligned}
s(0,t) &= \int\limits_0^\omega \beta(a)(s(a,t) + (1-p)i(a,t) + (1-q)r(a,t))da,\\
i(0,t) &= p\int\limits_0^\omega \beta(a)i(a,t)da, \qquad r(0,t) = q\int\limits_0^\omega \beta(a)r(a,t)da,
\end{aligned} \tag{2.2.15}$$

where $p, q \in [0,1]$ are the vertical transmission parameters of infectiveness and immunity, respectively. Finally, we prescribe the initial conditions

$$s(a,0) = \mathring{s}(a), \quad i(a,0) = \mathring{i}(a), \quad r(a,0) = \mathring{r}(a). \tag{2.2.16}$$

2.4 Diffusion

Equations of random walks

Let us assume that particles jump on the real line with equal probabilities p to the right and to the left, by the distance δ every time interval θ. We define $v(x,t)\Delta x$ as the probability of finding a particle in a small interval of length Δx around x and at time t if it starts at $x = 0$ and $t = 0$. The function v satisfies the forward Kolmogorov equation

$$v(x, t+\theta) = p(x-\delta)v(x-\delta, t) + p(x+\delta)v(x+\delta, t). \tag{2.2.17}$$

Assuming v and σ to be twice differentiable, we expand (2.2.17) into the Taylor series

$$v + \theta\partial_t v = \frac{1}{2}\left(v - \delta\partial_x v + \frac{1}{2}\delta^2\partial_{xx}^2 v\right) + \frac{1}{2}\left(v + \delta\partial_x v + \frac{1}{2}\delta^2\partial_{xx}^2 v\right) + \mathcal{O}(\theta^2) + \mathcal{O}(\delta^3),$$

and, passing to the limits with $\delta, \theta \to 0$ in such a way that

$$D(x) := \lim_{\substack{\delta\to 0\\ \theta\to 0}} \frac{\delta^2}{\theta} \tag{2.2.18}$$

remains finite, we arrive at the diffusion (Fokker-Planck) equation satisfied by the probability distribution function v:

$$\partial_t v = \frac{1}{2}\partial_{xx}^2(Dv). \tag{2.2.19}$$

The equation (2.2.19) is to be supplemented by an initial condition

$$v(x,0) = \overset{\circ}{v}\,. \tag{2.2.20}$$

The requirement that the diffusion constant D is finite implies that the particle speed

$$\gamma := \lim_{\substack{\delta\to 0\\ \theta\to 0}} \frac{\delta}{\theta} \tag{2.2.21}$$

is infinite. This result clearly indicates that the diffusion equation does not supply an adequate description of the random walk but, nevertheless, it is commonly used in application due to good agreement (under certain restrictions) with experiments.

Diffusion equation is often used to describe dispersal of individuals from regions of higher density to regions of lower density (Fick's law) or temperature of a body. Typically it is considered in $L_1(\Omega)$ or $C_b(\Omega)$ spaces but there are important cases where the space $L_2(\Omega)$ becomes relevant.

2.5 Transport on networks

Let us consider a network with some substance flowing along the edges and being redistributed in the nodes. The process of the redistribution of the flow is the loss-gain process governed by the Kirchoff's law (flow-in = flow-out). The network under consideration is represented by a simple directed graph

$$G = (V(G), E(G)) = (\{v_1,\dots,v_n\}, \{e_1,\dots,e_m\})$$

with n vertices $v_1,\dots,v_n$ and m edges (arcs), $e_1,\dots,e_m$. We suppose that G is connected but not necessarily strongly connected, see e.g. [26, 33]. Each edge is normalized so as to be identified with $[0,1]$, with the head at 0 and the tail at 1. The outgoing incidence matrix, $\Phi^- = (\phi^-_{ij})_{1\le i\le n, 1\le j\le m}$, and the incoming incidence matrix, $\Phi^+ = (\phi^+_{ij})_{1\le i\le n, 1\le j\le m}$, of this graph are defined, respectively as

$$\phi^-_{ij} = \begin{cases} 1 \text{ if } v_i \overset{e_j}{\to} \\ 0 \text{ otherwise.}\end{cases} \qquad \phi^+_{ij} = \begin{cases} 1 \text{ if } \overset{e_j}{\to} v_i \\ 0 \text{ otherwise.}\end{cases}$$

If the vertex v_i has more than one outgoing edge, we place a non negative weight w_{ij} on the outgoing edge e_j such that for this vertex v_i,

$$\sum_{j\in E_i} w_{ij} = 1,$$

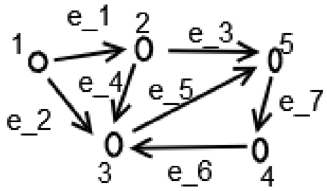
where E_i is defined by saying that $j \in E_i$ if the edge e_j is outgoing from v_i. Naturally, $w_{ij} = 1$ if $E_i = \{j\}$ and, to shorten notation, we adopt the convention that $w_{ij} = 1$ for any j if $E_i = \emptyset$. Then the weighted outgoing incidence matrix, Φ^-_w, is obtained from Φ^- by replacing each nonzero ϕ^-_{ij} entry by w_{ij}. If each vertex has an outgoing edge, then Φ^-_w is row stochastic, hence $\Phi^-\left(\Phi^-_w\right)^T = I_n$ (where the superscript T denotes the transpose). The (weighted) adjacency matrix $\mathbb{A} = (a_{ij})_{1\le i,j\le n}$ of the graph is defined by taking $a_{ij} = w_{jk}$ if there is e_k such that $v_j \overset{e_k}{\to} v_i$ and 0 otherwise; that is, $\mathbb{A} = \Phi^+(\Phi^-_w)^T$. An important role is played by the line graph Q of G. To recall, $Q = (V(Q), E(Q)) = (E(G), E(Q))$, where

$$\begin{aligned} E(Q) &= \{uv;\ u, v \in E(G), \text{the head of } u \text{ coincides with the tail of } v\} \\ &= \{\varepsilon_j\}_{1\le j\le k}. \end{aligned}$$

By $\mathbb{B}$ we denote the weighted adjacency matrix for the line graph; that is,

$$\mathbb{B} := (\Phi^-_w)^T \Phi^+. \tag{2.2.22}$$

If there is an outgoing edge at each vertex then, from the definition of $\mathbb{B}$, we see that it is column stochastic. A vertex v will be called a source if it has no incoming edge and a sink if there are no outgoing edge.



Example 2.1 *Consider the following graph. For this graph, the matrices Φ^-, Φ^+ are given below.*

$$\Phi^- = \begin{pmatrix} 1\,1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,1 \end{pmatrix}, \quad \Phi^+ = \begin{pmatrix} 0\,0\,0\,0\,0\,0\,0 \\ 1\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,1\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,1 \\ 0\,0\,1\,0\,1\,0\,0 \end{pmatrix},$$

while the adjacency matrix is given by

$$\begin{pmatrix} 0\,0\,0\,0\,0 \\ 1\,0\,0\,0\,0 \\ 1\,1\,0\,1\,0 \\ 0\,0\,0\,0\,1 \\ 0\,1\,1\,0\,0 \end{pmatrix}.$$

We are interested in a flow on a closed network G. Then the standard assumption is that the flow satisfies the Kirchoff law at the vertices

$$\sum_{j=1}^{m} \phi_{ij}^- c_j u_j(1,t) = w_{ij} \sum_{j=1}^{m} \phi_{ij}^+ c_j u_j(0,t), \quad t > 0, i \in 1, \ldots, n,$$

which, in this context, is the conservation of mass law: the total inflow of mass per unit time equals the total outflow at each node (vertex) of the network.

Let $u_j(x,t)$ be the density of particles at position x and at time $t \geq 0$ flowing along edge e_j for $x \in [0,1]$. The particles on e_j are assumed to move with velocity $c_j > 0$ which is constant for each j. We consider a generalization of Kirchoff's law by allowing for a decrease/amplification of the flow at the entrances and exits of each vertex. Then the flow is described by

$$\begin{cases} \partial_t u_j(x,t) &= c_j \partial_x u_j(x,t), \quad x \in (0,1), \quad t \geq 0, \\ u_j(x,0) &= f_j(x), \\ \phi_{ij}^- \xi_j c_j u_j(1,t) &= w_{ij} \sum\limits_{k=1}^{m} \phi_{ik}^+ (\gamma_k c_k u_k(0,t)), \end{cases} \tag{2.2.23}$$

where $\gamma_j > 0$ and $\xi_j > 0$ are the absorption/amplification coefficients at, respectively, the head and the tail of the edge e_j. If $\gamma_j = \xi_j = 1$ for all $j = 1, \cdots, m$, then we recover the Kirchoff law at the vertices.

Remark 2.1. We observe that the boundary condition in (2.2.23) takes a special form if v_i is either a sink or a source. If it is a sink, then $E_i = \emptyset$ and, by the convention above,

$$0 = \sum_{k=1}^{m} \phi_{ik}^+ (\gamma_k c_k u_k(0,t)), \quad t > 0, \tag{2.2.24}$$

and

$$\phi_{ij}^- \xi_j c_j u_j(1,t) = 0, \quad t > 0, j = 1, \ldots, m, \tag{2.2.25}$$

if it is a source. The last condition is nontrivial only if $j \in E_i$ as then $\phi_{ij}^- \neq 0$.

We consider (2.2.23) as an abstract Cauchy problem

$$\mathbf{u}_t = A\mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{f}, \tag{2.2.26}$$

in $X = (L_1([0,1]))^m$, where A is the realization of the expression $\mathsf{A} = (c_j\partial_x)_{1\leq j\leq m}$ on the domain

$$D(A) = \{\mathbf{u} \in (W_1^1([0,1]))^m;\ \mathbf{u}\ \text{satisfies the b. c. in (2.2.23)}\}. \tag{2.2.27}$$

We denote $\mathbb{C} = \operatorname{diag}(c_j)_{1\leq j\leq m}, \mathbb{K} = \operatorname{diag}(\xi_j)_{1\leq j\leq m}$ and $\mathbb{G} = \operatorname{diag}(\gamma_j)_{1\leq j\leq m}$. It can be proved, [23], that

$$D(A) = \{\mathbf{u} \in (W_1^1([0,1]))^m;\ \mathbf{u}(1) = \mathbb{K}^{-1}\mathbb{C}^{-1}\mathbb{B}\mathbb{G}\mathbb{C}\mathbf{u}(0)\} \tag{2.2.28}$$

where $\mathbb{B}$ is the adjacency matrix of the line graph of G.

Such conditions not only do arise from network flows. For instance, consider the Rotenberg-Rubinov-Lebovitz model. Let x be the independent variable that describes the degree of maturity of a cell ($0 \leq x \leq 1$). Cells are born at $x = 0$ and divide at $x = 1$. The maturation rate, or velocity, $v_i, i = 1,\ldots,n$, is a discrete, nonnegative parameter. Since a large number of individuals is being considered, the dependent variable is taken to be the number density $u_j(x,t) = u(x,v_j,t)$. Then $u_j(x,t) := u(x,v_j,t)$ is governed by the equation

$$\begin{aligned}\partial_t u_j(x,t) + v_j\partial_x u_j(x,t) &= -\mu_j u_j(x,t),\\ v_j u_j(0,t) &= \sum_i k_{ji}v_i u_i(1,t),\end{aligned}$$

for $x \in (0,1), \quad t \geq 0$, where where μ_j is the death rate. The meaning of the boundary condition is that the cell divides ending its individual life producing offspring of various maturation velocities according to a non-negative matrix $\mathcal{K} = \{k_{ji}\}_{1\leq i,j\leq n}$.

Hence, we see that there is a class of problems which can be written in the form

$$\begin{aligned}\mathbf{u}_t &= A\mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{f},\\ \mathbf{u}(0) &= \mathcal{K}\mathbf{u}(1),\end{aligned} \tag{2.2.29}$$

where A is a first order differential operator and $\mathcal{K}$ is an arbitrary (not necessarily nonnegative) matrix. Apart from the usual questions about well posedenss of the problem (including positivity of solutions), some new problems arise. For instance, can we always find a diagraph G such that (4.4.18) is a flow problem on a graph? By (3.1.1), we see that this problem is equivalent to the question: Under what conditions a nonnegative matrix $\mathcal{K}$ is a weighted incidence matrix of the line graph $L(G)$.

2.6 Fragmentation and coagulation processes

The name may seem very specific, but such processes occur in a wide range of applications, see [44].

- Chemical engineering: polymerization/depolimerization processes, with possible mass loss through dissolution, chemical reactions, oxidation etc, or mass growth due to the deposition of material on the clusters;
- Biology: Blood cells' coagulation and splitting, animal grouping, phytoplankton at the level of aggregates;
- Planetology: merging of planetesimals;
- Aerosol research: coagulation of smoke, smog and dust particles, droplets in clouds.

Also, they possibly are the most rewarding kinetic processes to study from the analytical point of view.

Fragmentation and coagulation may be discrete, when we assume that there is a minimal size of interacting particles and all clusters are finite ensembles of such fundamental building blocks, and continuous with the matter assumed to be a continuum. Here we only consider the continuous model. In the case of pure fragmentation a standard modelling process leads to the following equation:

$$\partial_t u(x,t) = -a(x)u(x,t) + \int\limits_x^\infty a(y)b(x|y)u(y,t)dy, \tag{2.2.30}$$

u is the density of particles of mass x, a is the fragmentation rate and b describes the distribution of particle masses x spawned by the fragmentation of a particle of mass y; that is, the expected number of particles of mass x resulting from fragmentation of a particle of mass y. Further

$$M(t) = \int\limits_0^\infty xu(x,t)dx \tag{2.2.31}$$

is the total mass at time t. Local conservation principle requires

$$\int\limits_0^y xb(x|y)dx = y, \tag{2.2.32}$$

while the expected number of particles produced by a particle of mass y is given by $n_0(y) = \int_0^y b(x|y)dx$.

Fragmentation can be supplemented by growth/decay, transport or diffusion processes, [16, 21, 18], but we will not discuss them here.

If we combine the fragmentation process with coagulation, we will get

$$\begin{aligned}\partial_t u(x,t) &= -a(x)u(x,t) + \int\limits_x^\infty a(y)b(x|y)u(y,t)dy \\ &\quad -u(x,t)\int\limits_0^\infty k(x,y)u(y,t)dy + \frac{1}{2}\int\limits_0^x k(x-y,y)u(x-y,t)u(y,t)dy.\end{aligned} \tag{2.2.33}$$

The coagulation kernel $k(x,y)$ represents the likelihood of a particle of size x attaching itself to a particle of size y and, for a moment, we assume that it is a symmetric nonnegative positive function.

Since the fragmentation and coagulation processes just rearrange the mass among the clusters, (2.2.31) implies that the natural space to analyse them is

$$X_1 = L_1(\mathbb{R}_+, xdx) = \left\{ u;\ \|u\|_1 = \int\limits_0^\infty |u(x)|xdx < +\infty \right\}.$$

However, for the coagulation processes it is important to control also the number of particles, or even some higher moments of the density. The best results are obtained in the scale of spaces $X_{1,m}$, $m \geq 1$, where

$$X_{1,m} = L_1(\mathbb{R}_+, (1+x^m)dx) = \left\{ u;\ \|u\|_{0,m} = \int\limits_0^\infty |u|(1+x^m)dx < +\infty \right\}.$$

3

Mathematical toolbox

1 First semigroups

As mentioned before, we are concerned with methods of finding solutions of the Cauchy problem:

Definition 3.1. *Given a complex or real Banach space and a linear operator $\mathcal{A}$ with domain $D(\mathcal{A})$ and range* $\mathrm{Ran}\mathcal{A}$ *contained in X, and also given an element $\mathring{u} \in X$, find a function $u(t) = u(t, \mathring{u})$ such that*

1. *$u(t)$ is continuous on $[0, \infty[$ and continuously differentiable on $]0, \infty[$ in the norm of X,*
2. *for each $t > 0$, $u(t) \in D(\mathcal{A})$ and*
$$\partial_t u(t) = \mathcal{A}u(t), \quad t > 0, \tag{3.1.1}$$
3. $$\lim_{t \to 0^+} u(t) = u_0 \tag{3.1.2}$$
in the norm of X.

A function satisfying all conditions above is called the classical solution of (3.1.1), (3.1.2). If $u(t) \in D(\mathcal{A})$ (and thus $u \in C^1([0, \infty[, X))$, then such a function is called a strict solution to (3.1.1), (3.1.2).

To shorten notation, we denote by $C^k(I, X)$ a space of functions which, for each $t \in I \subset \mathbb{R}$ satisfy $u(t) \in X$ and are continuously differentiable k times in t with respect to the norm of X. Thus, e.g. a classical solution u satisfies $u \in C([0, \infty[, X) \cap C^1(]0, \infty[, X)$.

1.1 Definitions and basic properties

If the solution to (3.1.1), (3.1.2) is unique, then we can introduce a family of operators $\{G(t)\}_{t \geq 0}$ such that $u(t, u_0) = G(t)u_0$. Ideally, $G(t)$ should be defined on the whole space for each $t > 0$, and the function $t \to G(t)u_0$ should be continuous for each $u_0 \in X$, leading to well-posedness of (3.1.1), (3.1.2). Moreover, uniqueness and linearity of $\mathcal{A}$ imply that $G(t)$ are linear operators. A fine-tuning of these requirements leads to the following definition.

Definition 3.2. *A family $\{G(t)\}_{t \geq 0}$ of bounded linear operators on X is called a C_0-semigroup, or a strongly continuous semigroup, if*

(i) $G(0) = I$;

(ii) $G(t + s) = G(t)G(s)$ *for all* $t, s \geq 0$;

(iii) $\lim_{t \to 0^+} G(t)x = x$ *for any* $x \in X$.

A linear operator A is called the (infinitesimal) generator of $\{G(t)\}_{t\geq 0}$ if

$$Ax = \lim_{h\to 0^+} \frac{G(h)x - x}{h}, \tag{3.1.3}$$

with $D(A)$ defined as the set of all $x \in X$ for which this limit exists. Typically the semigroup generated by A is denoted by $\{G_A(t)\}_{t\geq 0}$.

If $\{G(t)\}_{t\geq 0}$ is a C_0-semigroup, then the local boundedness and (ii) lead to the existence of constants $M > 0$ and ω such that for all $t \geq 0$

$$\|G(t)\|_X \leq Me^{\omega t} \tag{3.1.4}$$

(see, e.g., [54, p. 4]). We say that $A \in \mathcal{G}(M, \omega)$ if it generates $\{G(t)\}_{t\geq 0}$ satisfying (3.1.4). The *type* of $\{G(t)\}_{t\geq 0}$ is defined as

$$\omega_0(G) = \inf\{\omega;\ \text{there is } M \text{ such that (3.1.4) holds}\}. \tag{3.1.5}$$

Let $\{G_A(t)\}_{t\geq 0}$ be the semigroup generated by A. The following properties of $\{G_A(t)\}_{t\geq 0}$ are frequently used, [54, Theorem 2.4].

(a) For $x \in X$

$$\lim_{h\to 0} \frac{1}{h} \int\limits_t^{t+h} G_A(s)x ds = G_A(t)x. \tag{3.1.6}$$

(b) For $x \in X$, $\int_0^t G_A(s)x ds \in D(A)$ and

$$A \int\limits_0^t G_A(s)x ds = G_A(t)x - x. \tag{3.1.7}$$

(c) For $x \in D(A)$, $G_A(t)x \in D(A)$ and

$$\frac{d}{dt} G_A(t)x = AG_A(t)x = G_A(t)Ax. \tag{3.1.8}$$

(d) For $x \in D(A)$,

$$G_A(t)x - G_A(s)x = \int\limits_s^t G_A(\tau)Ax d\tau = \int\limits_s^t AG_A(\tau)x d\tau. \tag{3.1.9}$$

From (3.1.8) and condition (iii) of Definition 3.2 we see that if A is the generator of $\{G_A(t)\}_{t\geq 0}$, then for $\mathring{u} \in D(A)$ the function $t \to G_A(t) \mathring{u}$ is a classical solution of the following Cauchy problem,

$$\partial_t u(t) = A(u(t)), \quad t > 0, \tag{3.1.10}$$

$$\lim_{t\to 0^+} u(t) = \mathring{u}\,. \tag{3.1.11}$$

We note that ideally the generator A should coincide with $\mathcal{A}$ but in reality very often it is not so. However, for most of this chapter we are concerned with solvability of (3.1.10), (3.1.11); that is, with the case when $\mathcal{A}$ of (3.1.1) is the generator of a semigroup.

We noted above that for $\mathring{u} \in D(A)$ the function $u(t) = G_A(t) \mathring{u}$ is a classical solution to (3.1.10), (3.1.11). For $\mathring{u} \in X \setminus D(A)$, however, the function $u(t) = G_A(t) \mathring{u}$ is continuous but, in general, not differentiable, nor $D(A)$-valued, and, therefore, not a classical solution. Nevertheless, from (3.1.7), it follows that $v(t) = \int_0^t u(s)ds \in D(A)$ and therefore it is a strict solution of the integrated version of (3.1.10), (3.1.11):

$$\begin{aligned}\partial_t v &= Av + \overset{\circ}{u}, \quad t > 0\\ v(0) &= 0,\end{aligned} \tag{3.1.12}$$

or equivalently,

$$u(t) = A\int_0^t u(s)ds + \overset{\circ}{u}\,. \tag{3.1.13}$$

We say that a function u satisfying (3.1.12) (or, equivalently, (3.1.13)) is a *mild solution* or *integral solution* of (3.1.10), (3.1.11). It can be proved that $t \to G(t)\overset{\circ}{u}$, $\overset{\circ}{u} \in D(A)$, is the only solution of (3.1.10), (3.1.11) taking values in $D(A)$. Similarly, for $\overset{\circ}{u} \in X$, the function $t \to G(t)\overset{\circ}{u}$ is the only mild solution to (3.1.10), (3.1.11).

Thus, if we have a semigroup, we can identify the Cauchy problem of which it is a solution. Usually, however, we are interested in the reverse question; that is, in finding the semigroup for a given equation. The answer is given by the Hille–Yoshida theorem (or, more properly, the Feller–Miyadera–Hille–Phillips–Yosida theorem). Before, however, we need to recall some terminology related to the spectrum of an operator.

1.2 Interlude - the spectrum of an operator

Let us recall that the resolvent set of A is defined by

$$\rho(A) = \{\lambda \in \mathbb{C};\ (\lambda I - A)^{-1} \in \mathcal{L}(X)\}$$

and, for $\lambda \in \rho(A)$, we define the resolvent of A by

$$R(\lambda, A) = (\lambda I - A)^{-1}.$$

The complement of $\rho(A)$ in $\mathbb{C}$ is called the *spectrum* of A and denoted by $\sigma(A)$. In general, it is possible that either $\rho(A)$ or $\sigma(A)$ is empty. The spectrum is usually subdivided into several subsets.

- *Point spectrum* $\sigma_p(A)$ is the set of $\lambda \in \sigma(A)$ for which the operator $\lambda I - A$ is not one-to-one. In other words, $\sigma_p(A)$ is the set of all eigenvalues of A.
- *Residual spectrum* $\sigma_r(A)$ is the set of $\lambda \in \sigma(A)$ for which $\lambda I - A$ is one-to-one and $Im\ (\lambda I - A)$ is not dense in X.
- *Continuous spectrum* $\sigma_c(A)$ is the set of $\lambda \in \sigma(A)$ for which the operator $\lambda I - A$ is one-to-one and its range is dense in, but not equal to, X

The resolvent of any operator A satisfies the *resolvent identity*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \qquad \lambda, \mu \in \rho(A). \tag{3.1.14}$$

For any bounded operator the spectrum is a compact subset of $\mathbb{C}$ so that $\rho(A) \neq \emptyset$. If A is bounded, then the limit

$$r(A) = \lim_{n\to\infty} \sqrt[n]{\|A^n\|} \tag{3.1.15}$$

exists and is called *the spectral radius*. Clearly, $r(A) \leq \|A\|$. Equivalently,

$$r(A) = \sup_{\lambda\in\sigma(A)} |\lambda|. \tag{3.1.16}$$

To show that $\lambda \in \mathbb{C}$ belongs to the spectrum we often use the following result.

Theorem 3.3. *Let A be a closed operator. If $\lambda \in \rho(A)$, then $dist(\lambda, \sigma(A)) = 1/r(R(\lambda, A)) \geq 1/\|R(\lambda, A)\|$. In particular, if $\lambda_n \to \lambda$, $\lambda_n \in \rho(A)$, then $\lambda \in \sigma(A)$ if and only if $\{\|R(\lambda_n, A)\|\}_{n\in\mathbb{N}}$ is unbounded.*

For an unbounded operator A the role of the spectral radius often is played by the *spectral bound* $s(A)$ defined as

$$s(A) = \sup\{\Re\lambda;\ \lambda \in \sigma(A)\}. \tag{3.1.17}$$

1.3 Hille-Yosida theorem

Theorem 3.4. $A \in \mathcal{G}(M, \omega)$ *if and only if*

(a) A is closed and densely defined,

(b) there exist $M > 0, \omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and for all $n \geq 1, \lambda > \omega$,

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}. \tag{3.1.18}$$

If A is the generator of $\{G_A(t)\}_{t \geq 0}$, then

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} G_A(t) x dt, \quad \Re\lambda > \omega, \tag{3.1.19}$$

is valid for all $x \in X$.

Yosida's idea of proof was to use the following observations. If $(A, D(A))$ is a closed and densely defined operator satisfying $\rho(A) \supset (0, \infty)$ and $\|\lambda R(\lambda, A)\| \leq 1$ for all $\lambda > 0$, then

(i) for any $x \in X$,

$$\lim_{\lambda \to \infty} \lambda R(\lambda, A)x = x. \tag{3.1.20}$$

Indeed, first consider $x \in D(A)$. Then

$$\|\lambda R(\lambda, A)x - x\| = \|AR(\lambda, A)x\| = \|R(\lambda, A)Ax\| \leq \frac{1}{\lambda}\|Ax\| \to 0$$

as $\lambda \to \infty$. Since $D(A)$ is dense and $\|\lambda R(\lambda, A)\| \leq 1$ then by 3ϵ argument we extend the convergence to X.

(ii) $AR(\lambda, A)$ are bounded operators and for any $x \in D(A)$,

$$\lim_{\lambda \to \infty} \lambda AR(\lambda, A)x = Ax. \tag{3.1.21}$$

Boundedness follows from $AR(\lambda, A) = \lambda R(\lambda, A) - I$. Eq. (3.1.21) follows (3.1.20).

Then Yosida used the bounded operators

$$A_\lambda = \lambda AR(\lambda, A), \tag{3.1.22}$$

as an approximation of A for which one can define semigroups uniformly continuous semigroups $\{G_\lambda(t)\}_{t \geq 0}$ via the exponential series and then he proved the convergence of $\{G_\lambda(t)\}_{t \geq 0}$ to the semigroup generated by A. A widely used approximation formula, which can also be used in the generation proof, is the operator version of the well-known scalar formula. Precisely, [54, Theorem 1.8.3], if A is the generator of a C_0-semigroup $\{G_A(t)\}_{t \geq 0}$, then for any $x \in X$,

$$G_A(t)x = \lim_{n \to \infty} \left(I - \frac{t}{n}A\right)^{-n} x = \lim_{n \to \infty} \left(\frac{n}{t} R\left(\frac{n}{t}, A\right)\right)^n x \tag{3.1.23}$$

and the limit is uniform in t on bounded intervals.

1.4 Dissipative operators and contractive semigroups

Let X be a Banach space (real or complex) and X^* be its dual. From the Hahn–Banach theorem, for every $u \in X$ there exists $u^* \in X^*$ satisfying $<u^*, u>= \|u\|^2 = \|u^*\|^2$. Therefore the *duality set*

$$\mathcal{J}(u) = \{u^* \in X^*;\ <u^*, u>= \|u\|^2 = \|u^*\|^2\} \tag{3.1.24}$$

is nonempty for every $u \in X$.

Definition 3.5. *We say that an operator $(A, D(A))$ is* dissipative *if for every $u \in D(A)$ there is $u^* \in \mathcal{J}(u)$ such that*

$$\Re <u^*, Ax> \leq 0. \tag{3.1.25}$$

If X is a real space, then the real part in the above definition can be dropped. An important equivalent characterisation of dissipative operators, [54, Theorem 1.4.2], is that A is dissipative if and only if for all $\lambda > 0$ and $u \in D(A)$,

$$\|(\lambda I - A)u\| \geq \lambda\|u\|. \tag{3.1.26}$$

Combination of the Hille–Yosida theorem with the above property gives a generation theorem for dissipative operators, known as the Lumer–Phillips theorem ([54, Theorem 1.4.3] or [34, Theorem II.3.15]).

Theorem 3.6. *For a densely defined dissipative operator $(A, D(A))$ on a Banach space X, the following statements are equivalent.*

(a) The closure $\overline{A}$ generates a semigroup of contractions.

(b) $\overline{Im(\lambda I - A)} = X$ for some (and hence all) $\lambda > 0$.

If either condition is satisfied, then A satisfies (3.1.25) for any $u^* \in \mathcal{J}(u)$.

We observe that a densely defined dissipative operator is closable, [54, Theorem 1.4.5], so that the statement in item (a) makes sense. In other words, to prove that (the closure of) a dissipative operators generates a semigroup, we only need to show that the equation

$$\lambda u - Au = f \tag{3.1.27}$$

is solvable for f from a dense subset of X for some $\lambda > 0$. We do not need to prove that the solution to (3.1.27) defines a resolvent satisfying (3.1.18).

In particular, if we know that A is closed, then the density of $Im(\lambda I - A)$ is sufficient for A to be a generator. On the other hand, if we do not know a priori that A is closed, then $Im(\lambda I - A) = X$ yields A being closed and consequently that it is a generator.

Example 3.7. Let us have a look at the classical problem which often is incorrectly solved. Consider

$$Au = -\partial_x u, \quad x \in (0, 1),$$

on $D(A) = \{u \in W_1^1(I);\ u(0) = 0\}$, where $I =]0, 1[$. The state space is real $X = L_1(I)$. For a given $u \in X$, we have

$$\mathcal{J}(u) = \begin{cases} \|u\|\text{sign}u(x) & \text{if } u(x) \neq 0, \\ \alpha \in [-\|u\|, \|u\|] & \text{if } u(x) = 0. \end{cases}$$

Note that $\mathcal{J}$ is a multivalued function. Further, [29], any element of $W_1^1(I)$ can be represented by an absolutely continuous function on I.

Now, for $v \in \mathcal{J}(u)$ we have

$$
\begin{aligned}
<-\partial_x u, v> &= -\|u\| \int\limits_0^1 \partial_x u(x) \mathrm{sign} u(x) dx \\
&= -\|u\| \left(\int\limits_{\{x \in I;\ u(x)>0\}} \partial_x u(x) dx - \int\limits_{\{x \in I;\ u(x)<0\}} \partial_x u(x) dx \right).
\end{aligned}
$$

Since u is continuous, both sets $I_+ := \{x \in I;\ u(x) > 0\}$ and $I_- := \{x \in I;\ u(x) < 0\}$ are open. Then, see [2, p. 42],

$$
I_\pm = \bigcup_n]\alpha_n^\pm, \beta_n^\pm[
$$

where $]\alpha_n^\pm, \beta_n^\pm[$ are non overlapping open intervals. Then

$$
\int\limits_{I_\pm} \partial_x u(x) dx = \sum_n (u(\beta_n^\pm) - u(\alpha_n^\pm)) = \begin{cases} u(1) & \text{if } 1 \in I_\pm, \\ 0 & \text{if } 1 \notin I_\pm, \end{cases}
$$

as 1 only can be the right end of only one of the component intervals and we used $u(0) = 0$. Now, if $1 \in I_+$, then $u(1) > 0$, if $1 \in I_-$, then $u(1) < 0$, and if $1 \notin I_+ \cup I_-$, then $u(1) = 0$. In any case,

$$
<-\partial_x u, v> \leq 0
$$

and the operator $(A, D(A))$ is dissipative. Clearly, the solution of

$$
\lambda u + \partial_x u = f, \qquad u(0) = 0,
$$

is given by $u(x) = e^{-\lambda x} \int\limits_0^x e^{\lambda s} f(s) ds$ and, for $\lambda > 0$,

$$
\|u\| \leq \int\limits_0^1 e^{-\lambda x} \left(\int\limits_0^x e^{\lambda s} |f(s)| ds \right) dx \leq \|f\|,
$$

which gives solvability of (3.1.27) in X. We note that, of course, with a more careful integration we would be able to obtain the Hille-Yosida estimate (3.1.18). This additional work is, however, not necessary for dissipative operators.

1.5 Analytic semigroups

In the previous paragraph we noted that if a closed operator is dissipative, then we can prove that it generates a semigroup, provided (3.1.27) is solvable. It turns out that the solvability of (3.1.27) can be used to prove that A generates a semigroup without assuming that it is dissipative but then we must consider complex λ. Note that the considerations below are valid for an arbitrary Banach space.

Hence, let the inverse $(\lambda I - A)^{-1}$ exists in the sector

$$
S_{\frac{\pi}{2}+\delta} := \{\lambda \in \mathbb{C};\ |\arg \lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}, \tag{3.1.28}
$$

for some $0 < \delta < \frac{\pi}{2}$, and let there exist C such that for every $0 \neq \lambda \in S_{\frac{\pi}{2}+\delta}$ the following estimate holds:

$$
\|R(\lambda, A)\| \leq \frac{C}{|\lambda|}. \tag{3.1.29}
$$

Then A is the generator of a uniformly bounded semigroup $\{G_A(t)\}_{t\geq 0}$ (the constant M in (3.1.4) not necessarily equals C) and $\{G_A(t)\}_{t\geq 0}$ is given by

$$G_A(t) = \frac{1}{2\pi i}\int\limits_{\Gamma} e^{\lambda t}R(\lambda, A)\mathrm{d}\lambda, \tag{3.1.30}$$

where Γ is an unbounded smooth curve in $S_{\frac{\pi}{2}+\delta}$. The reason why $\{G_A(t)\}_{t\geq 0}$ is called analytic is that it extends to an analytic function on S_δ.

If A is the generator of an analytic semigroup $\{G_A(t)\}_{t\geq 0}$, then $t \to G_A(t)$ has derivatives of arbitrary order on $]0,\infty[$. This shows that $t \to G_A(t)\mathring{u}$ solves the Cauchy problem (3.1.2) for arbitrary $\mathring{u} \in X$. This is a significant improvement upon the case of C_0-semigroup, for which $\mathring{u} \in D(A)$ was required.

One dimensional Robin boundary value problem for the diffusion equation in the space of continuous functions

$$\begin{aligned} \partial_t u(x,t) &= \partial_{xx}u(x,t), \qquad (x,t) \in]0,1[\times\mathbb{R}_+,\\ \partial_x u(0,t) + a_0 u(0,t) &= 0, \qquad t>0,\\ \partial_x u(1,t) + a_1 u(1,t) &= 0, \qquad t>0,\\ u(x,0) &= \mathring{u}\,(x), \qquad x\in]0,1[. \end{aligned} \tag{3.1.31}$$

We consider $(A, D(A))$ defined by $Au = \partial_{xx}u$ on

$$D(A) = \{u \in C^2([0,1]);\ \partial_x u(0) + a_0 u(0) = \partial_x u(1) + a_1 u(1) = 0\}.$$

First we show that $D(A)$ is dense in $C([0,1])$. One can proceed as follows. The space $C_0^\infty(]0,1[) \subset D(A)$ but $\overline{C_0^\infty(]0,1[)} = X_0 := \{u \in C([0,1]);\ u(0) = u(1) = 0\}$. To ascertain this, we observe that any $u \in X_0$ can be uniformly approximated by (discontinuous) functions which are zero close to the endpoints and equal to u elsewhere. Precisely, for any ϵ there is $\delta > 0$ such that $|u(x)| < \epsilon$ on $[0,\delta[\cup]1-\delta,1]$. Then we let $u_\delta(x) = u(x)$ on $[\delta, 1-\delta]$ and zero elsewhere and we approximate u_δ by $J_\eta * u_\delta \in C_0^\infty(]0,1[)$ with $0 < \eta < \delta/2$, by Theorem 1.3.

For a given $u \in C([0,1])$, we can find polynomial $\psi(x) = ax^3 + bx^2 + cx + d$ such that $\psi'(0) + a_0\psi(0) = 0, \psi'(1) + a_1\psi(1) = 0, \psi(0) = u(0), \psi(1) = u(1)$. Indeed, then we must have

$$\begin{aligned} a(a_1+3) + b(a_1+2) + c(a_1+1) + a_1 &= 0,\\ c + a_0 d &= 0,\\ d &= u(0),\\ a+b+c+d &= u(1) \end{aligned}$$

and this system is solvable. Then $u - \psi \in \{u \in C([0,1]);\ u(0) = u(1) = 0\}$ and if $C_0^\infty(]0,1[) \ni \phi_n \to u - \psi$, then $D(A) \ni \phi_n + \psi \to u$.

Let us take $0 \neq \lambda = \mu^2 = |\lambda|e^{i\theta}$ with $\Re\mu > 0$. The general solution of the resolvent equation

$$\lambda u - \partial_{xx}u = f \tag{3.1.32}$$

is given by

$$u(x) = C_1 e^{-\mu x} + C_2 e^{\mu x} + U_\mu(x), \tag{3.1.33}$$

where

$$U_\mu(x) = \frac{1}{2\mu}\int\limits_0^1 e^{-\mu|x-s|}f(s)ds. \tag{3.1.34}$$

We have

$$\partial_x U_\mu(0) = \mu U_\mu(0), \qquad \partial_x U_\mu(1) = -\mu U_\mu(1).$$

The constants C_1, C_2 are determined from the system

$$\begin{aligned}(-\mu + a_0)C_1 + (\mu + a_0)C_2 + (\mu + a_0)U_\mu(0) &= 0,\\ (-\mu + a_1)e^{-\mu}C_1 + (\mu + a_1)e^{\mu}C_2 + (-\mu + a_1)U_\mu(1) &= 0\end{aligned}$$

The determinant of this system is given by

$$D(\mu) = (a_0 - \mu)(a_1 + \mu)e^{\mu} - (a_1 - \mu)(a_0 + \mu)e^{-\mu}$$

which shows that the spectrum is at most countable. If $D(\mu) \neq 0$, then

$$\begin{aligned}C_1 &= \frac{(a_0 + \mu)(a_1 - \mu)U_\mu(1) - (a_0 + \mu)(a_1 + \mu_1)U_\mu(0)e^{\mu}}{D(\mu)},\\ C_2 &= \frac{(\mu - a_0)(a_1 - \mu)U_\mu(1) + (a_0 + \mu)(a_1 - \mu)U_\mu(0)e^{-\mu}}{D(\mu)}.\end{aligned}$$

The estimate of U_μ can be easily achieved by the Young inequality (after extending f by 0 onto $\mathbb{R}$), or by the direct calculation; we have

$$\|U_\mu\| \leq \frac{\|f\|}{|\mu|\Re\mu} = \frac{\|f\|}{|\lambda|\cos\theta/2}. \tag{3.1.35}$$

Next we obtain

$$\sup_{x\in[0,1]} |e^{\mu x}| = e^{\Re\mu} \qquad \sup_{x\in[0,1]} |e^{-\mu x}| = 1$$

and

$$|U_\mu(0)| \leq \frac{\|f\|}{2|\mu|\Re\mu}, \qquad |U_\mu(1)| \leq \frac{\|f\|}{2|\mu|\Re\mu}.$$

We observe that the coefficients multiplying $U_\mu(0)$ and $U_\mu(1)$ are bounded for large $|\lambda|$ and hence, for λ in the sector $|\theta| \leq |\theta_0| < \pi$, we have

$$\|u\| \leq |C_1| + |C_2|e^{\Re\mu} + \frac{\|f\|}{|\lambda|\cos\theta_0/2} \leq \frac{M\|f\|}{|\lambda|\cos\theta_0/2}.$$

for sufficiently large $|\lambda|$. Consequently, A is also closed and thus $(A, D(A))$ generates an analytic semigroup.

Note that the Dirichlet problem in the space of continuous functions does not behave so well. The reason for this is that the corresponding domain

$$D(A) = \{u \in C^2([0,1]);\ u(0) = u(1) = 0\}$$

cannot be dense in $C([0,1])$. Thus we have to work with the part of A in $X_0 := \{u \in C([0,1]);\ u(0) = u(1) = 0\}$ where, indeed, it generates an analytic semigroup.

The Dirichlet problem for the diffusion equation in n dimensions.

Let $C = \Omega \times (0,\infty), \Sigma = \partial\Omega \times (0,\infty)$ where Ω is an open set in $\mathbb{R}^n$. We consider the problem

$$\partial_t u = \Delta u, \quad \text{in } \Omega \times [0,T], \tag{3.1.36}$$
$$u = 0, \quad \text{on } \Sigma, \tag{3.1.37}$$
$$u = u_0, \quad \text{on } \Omega. \tag{3.1.38}$$

The strategy is to consider (4.2.12–(4.2.14) as the abstract Cauchy problem

$$u' = Au, \quad u(0) = u_0$$

in $X = L_2(\Omega)$ where A is the unbounded operator defined by

$$Au = \Delta u$$

for

$$u \in D(A) = \{u \in \overset{\circ}{W}{}_2^1(\Omega); \Delta u \in L_2(\Omega)\}.$$

Then we have

Theorem 3.8. *The operator $(A, D(A))$ generates an analytic semigroup in $L_2(\Omega)$.*

Proof. First we observe that A is densely defined as $C_0^\infty(\Omega) \subset \overset{\circ}{W}{}_2^1(\Omega)$ and $\Delta C_0^\infty(\Omega) \subset L_2(\Omega)$. To prove that the semigroup is analytic, we have to consider the resolvent. To this end, we have to solve

$$a(u,v) = \int\limits_\Omega \nabla u \nabla v d\boldsymbol{x} + \lambda \int\limits_\Omega uv d\boldsymbol{x} = \int\limits_\Omega fv d\boldsymbol{x}, \quad v \in \overset{\circ}{W}{}_2^1(\Omega)$$

for $\lambda = \sigma + i\tau$. Observe that if we find the solution $u \in \overset{\circ}{W}{}_2^1(\Omega)$ of this problem, then this solution will automatically belong to $D(A)$. We have

$$\Re((\lambda I - A)u, u) = \|\nabla u\|_0^2 + \sigma\|u\|_0^2 \tag{3.1.39}$$

so, in particular,

$$\sigma\|u\|_0^2 \le \|\nabla u\|_0^2 + \sigma\|u\|_0^2 \le \|\lambda u - Au\|_0\|u\|_0$$

for $\sigma \ge 0$. This shows that $\lambda I - A$ is invertible and, in particular,

$$\|\sigma R(\lambda, A)\| \le 1 \tag{3.1.40}$$

which, by considering $\lambda = \sigma$ shows that A generates a contractive semigroup. Similarly, using the triangle inequality,

$$-\|\nabla u\|_0^2 + |\tau|\|u\|_0^2 \le |\|\nabla u\|_0^2 + \tau\|u\|_0^2| \le |\Im(\lambda u - Au, u)| \tag{3.1.41}$$

It follows that either

$$|\Im(\lambda u - Au, u)| \ge \frac{|\tau|}{2}\|u\|_0^2$$

or

$$\Re(\lambda u - Au, u) \ge \frac{|\tau|}{2}\|u\|_0^2.$$

Indeed, if the former does not hold; that is,

$$|\Im(\lambda u - Au, u)| < \frac{|\tau|}{2}\|u\|_0^2,$$

then (3.1.41) yields

$$\|\nabla u\|_0^2 \ge \frac{|\tau|}{2}\|u\|_0^2$$

and then (3.1.39) gives the latter. Hence

$$\|\tau R(\lambda, A)\| \le 2, \qquad \sigma \ge 0. \tag{3.1.42}$$

and, in particular, for $\mu = i\tau$,

$$\|R(\mu, A)\| \le \frac{2}{|\mu|}. \tag{3.1.43}$$

Now, for arbitrary $\lambda = \sigma + i\tau$ we have

$$\lambda I - A = (\lambda - \mu)I + (\mu I - A) = (\mu I - A)(I + (\lambda - \mu)(\mu - A)^{-1})$$

which formally is equivalent to

$$(\lambda I - A)^{-1} = (I + (\lambda - \mu)(\mu - A)^{-1})^{-1}(\mu I - A)^{-1};$$

that is, $(\lambda I - A)$ is invertible if and only if $(I + (\lambda - \mu)(\mu - A)^{-1})^{-1}$ exists. Using the Neuman expansion and (3.1.43), the latter holds if

$$\frac{2|(\lambda - \mu)|}{|\mu|} = \frac{2|\sigma|}{|\tau|} < 1$$

and then

$$\|\lambda R(\lambda, A)\| \leq \frac{2(|\sigma| + |\tau|)}{|\tau|} \frac{1}{1 - 2\frac{|\sigma|}{|\tau|}}$$

as long as $|\sigma|/|\tau| < 1/2$. At the same time, (3.1.40) and (3.1.42) gives

$$\|\lambda R(\lambda, A)\| \leq 3$$

for $\sigma \geq 0$ so that the estimate

$$\|\lambda R(\lambda, A)\| \leq C$$

for some constant C holds for $\lambda \in \{\lambda \in \mathbb{C};\ \frac{\pi}{2} - \tan^{-1}\frac{1}{2} < \arg\lambda < \frac{\pi}{2} + \tan^{-1}\frac{1}{2}\}$. □

The estimate (3.1.29) is sometimes awkward to prove as it requires the knowledge of the resolvent in the whole sector. The proof of the preceding theorem uses estimates along the imaginary axis to extend them to $\Re\lambda < 0$. This approach works in a general situation, [34, Theorem II 4.6], and can be summarized in the following theorem.

Theorem 3.9. *An operator $(A, D(A))$ on a Banach space X generates a bounded analytic semigroup $(G_A(z))_{z\in S_\delta}$ in a sector Σ_δ if and only if A generates a bounded strongly continuous semigroup $\{G_A(t)\}_{t\geq 0}$ and there exists a constant $C > 0$ such that*

$$\|R(r + is, A)\| \leq \frac{C}{|s|} \tag{3.1.44}$$

for all $r > 0$ and $0 \neq s \in \mathbb{R}$.

This result can be generalized to arbitrary analytic semigroups: $(A, D(A))$ generates an analytic semigroup $(G_A(z))_{z\in S_\delta}$ if and only if A generates a strongly continuous semigroup $\{G_A(t)\}_{t\geq 0}$ and there exist constants $C > 0, \omega > 0$ such that

$$\|R(r + is, A)\| \leq \frac{C}{|s|} \tag{3.1.45}$$

for all $r > \omega$ and $0 \neq s \in \mathbb{R}$.

2 Uniqueness and Nonuniqueness

Let us return to the general Cauchy problem (3.1.1), (3.1.2). If, for a given u_0, it has two solutions, then their difference is again a solution of (3.1.1) but corresponding to the null initial condition – it is called a *nul-solution*; see [39, Section 23.7]. We say that a solution is of normal type ω if

$$\limsup_{t\to\infty} t^{-1}\log\|u(t)\| = \omega < +\infty. \tag{3.2.46}$$

A solution $u(t)$ is said to be of *normal type* if it is of normal type ω for some $\omega < +\infty$. A standard argument shows that if the solution is exponentially bounded if and only if it is of normal type.

Theorem 3.10. *[39, Theorem 23.7.1] If $\mathcal{A}$ is a closed operator whose point spectrum is not dense in any right half-plane, then for each $u_0 \in X$ the Cauchy problem of Definition 3.1 has at most one solution of normal type.*

Proof. If there are two solutions of possibly different, normal type, then their difference, say u, is a nul-solution of some normal type, say ω. Let

$$\mathcal{L}(\lambda)u = \int\limits_0^\infty e^{-\lambda t}u(t)dt,$$

where the integral exists as the Bochner integral for $\Re\lambda > \omega$ where it defines a holomorphic function. For such λ and $0 < \alpha < \beta < +\infty$ we have

$$\int\limits_\alpha^\beta e^{-\lambda t}u'(t)dt = \int\limits_\alpha^\beta e^{-\lambda t}\mathcal{A}u(t)dt = \mathcal{A}\int\limits_\alpha^\beta e^{-\lambda t}u(t)dt,$$

where we used the closedness of $\mathcal{A}$. Integrating the first term by parts we have

$$\int\limits_\alpha^\beta e^{-\lambda t}u'(t)dt = e^{-\beta\lambda}u(\beta) - e^{-\alpha\lambda}u(\alpha) + \lambda\int\limits_\alpha^\beta e^{-\lambda t}u(t)dt$$

and the right-hand side converges to $\lambda L(\lambda, u)$ as $\alpha \to 0^+$ and $\beta \to \infty$ because $u(0) = 0$. Thus $\mathcal{A}\int_\alpha^\beta e^{-\lambda t}u(t)dt$ also converges and because the integral converges to $\mathcal{L}(\lambda)u$, from closedness of $\mathcal{A}$ we obtain

$$\mathcal{A}\mathcal{L}(\lambda)u = \lambda\mathcal{L}(\lambda)u.$$

Now, $\mathcal{L}(\lambda)u$ is not identically zero as the Laplace transform of a supposedly nonzero function and, being analytic, can be equal to zero on at most discrete set of points. Thus, $\mathcal{L}(\lambda)u$ is an eigenvector of $\mathcal{A}$ for all λ with $\Re\lambda > \omega$ except possibly for a discrete set of λ. Thus the point spectrum is dense, contrary to the assumption. □

Theorem 3.11. *[39, Theorem 23.7.2] Let $\mathcal{A}$ be a closed operator. The Cauchy problem (3.1.1), (3.1.2) has a nul-solution of normal type $\leq \omega$ if and only if the eigenvalue problem*

$$\mathcal{A}y(\lambda) = \lambda y(\lambda) \tag{3.2.47}$$

has a solution $y(\lambda) \neq 0$ that is a bounded and holomorphic function of λ in each half-plane $\Re\lambda \geq \omega + \epsilon$, $\epsilon > 0$.

Proof. The necessity follows from the previous theorem. To prove sufficiency, assume that $y_0(\lambda)$ is bounded and holomorphic for $\Re\lambda \geq \omega + \epsilon$ for some $\epsilon > 0$. Because the solution to (3.2.47) can be multiplied by an arbitrary numerical function and still be a solution, we consider $y(\lambda) = (\lambda + 1 - \omega)^{-3}y_0(\lambda)$ and take the inverse Laplace transform,

$$u(t) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}y(\lambda)d\lambda, \quad \gamma > \omega. \tag{3.2.48}$$

Thanks to the regularising factor, the integrand is bounded by an integrable function locally uniformly with respect to $t \in (-\infty, +\infty)$. Thus it is absolutely convergent to a function continuous in t on the whole real line, which satisfies the estimate

$$\|u(t)\| \leq 2\sup_{-\infty<r<\infty}\|y_0(\gamma + ir)\|e^{\gamma t}(\gamma - \omega + 1)^2.$$

The estimate is independent of γ due to properties of complex integration and therefore, for $t < 0$, we obtain that $y(t) = 0$ by moving γ to ∞. From the above we also obtain that the type of $u(t)$ does not exceed ω. Using closedness of $\mathcal{A}$ we obtain

$$\mathcal{A}u(t) = \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \mathcal{A} y(\lambda) d\lambda = \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \lambda y(\lambda) d\lambda.$$

Due to the fact that the regularising factor behaves as $(\Im\lambda)^{-3}$, the last integral is still absolutely convergent and equals $u'(t)$. Thus it follows that $u(t)$ is a nul-solution of type $\leq \omega$. Clearly, $u(t)$ cannot be identically zero as it has a nonzero Laplace transform $y(\lambda)$. □

Similar considerations can be carried also for mild (or integral) solutions.

Now we investigate a relation between Cauchy problems (3.1.1), (3.1.2) and (3.1.10), (3.1.11). Let $(A, D(A))$ be the generator of a C_0-semigroup $\{G(t)\}_{t\geq 0}$ on a Banach space X. To simplify notation we assume that $\{G(t)\}_{t\geq 0}$ is a semigroup of contractions, hence $\{\lambda;\ Re\lambda > 0\} \subset \rho(A)$.

Let us further assume that there exists an extension $\mathcal{A}$ of A defined on the domain $D(\mathcal{A})$. We have the following basic result.

Lemma 3.12. *Under the above assumptions, for any λ with $Re\lambda > 0$,*

$$D(\mathcal{A}) = D(A) \oplus Ker(\lambda I - \mathcal{A}). \tag{3.2.49}$$

If we equip $D(\mathcal{A})$ with the graph norm, then $D(A)$ is a closed subspace of $D(\mathcal{A})$ and the projection of $D(\mathcal{A})$ onto $D(A)$ along $Ker(\lambda I - \mathcal{A})$ is given by

$$x = Px' = R(\lambda, A)(\lambda I - \mathcal{A})x', \quad x' \in D(\mathcal{A}). \tag{3.2.50}$$

Proof. Let us fix λ with $Re\,\lambda > 0$. Because $A \subset \mathcal{A}$, then

$$\lambda I - A \subset \lambda I - \mathcal{A}, \tag{3.2.51}$$

and therefore $Im(\lambda I - \mathcal{A}) = X$ for $Re\lambda > 0$. Because A is the generator of a contraction semigroup, for any $x' \in D(\mathcal{A})$ there exists a unique $x \in D(A)$ such that

$$(\lambda I - A)x = (\lambda I - \mathcal{A})x'.$$

Denote $P = R(\lambda, A)(\lambda I - \mathcal{A})$. By (3.2.51) it is a linear surjection onto $D(A)$, bounded as an operator from $D(\mathcal{A})$ into $D(\mathcal{A})$ equipped with the graph norm. Moreover, again by (3.2.51),

$$\begin{aligned} P^2 &= R(\lambda, A)(\lambda I - \mathcal{A})R(\lambda, A)(\lambda I - \mathcal{A}) = R(\lambda, A)(\lambda I - A)R(\lambda, A)(\lambda I - \mathcal{A}) \\ &= R(\lambda, A)(\lambda I - \mathcal{A}) = P, \end{aligned}$$

thus it is a projection. Clearly, for $e_\lambda \in Ker(\lambda I - \mathcal{A})$ we have $Pe_\lambda = 0$, hence this is a projection parallel to $Ker(\lambda I - \mathcal{A})$. By [42, p. 155], $D(A)$ is a closed subspace of $D(\mathcal{A})$ and the decomposition (3.2.49) holds. □

The next corollary links Theorem 3.11 with Lemma 3.12.

Corollary 3.13. *If $D(\mathcal{A}) \setminus D(A) \neq \emptyset$, then $\sigma_p(\mathcal{A}) \supseteq \{\lambda \in \mathbb{C};\ Re\lambda > 0\}$. Moreover, there exists a holomorphic (in the norm of X) function $\{\lambda \in \mathbb{C};\ Re\lambda > 0\} \ni \lambda \to e_\lambda$ such that for any λ with $Re\,\lambda > 0$, $e_\lambda \in Ker(\lambda I - \mathcal{A})$, which is also bounded in any closed half-plane, $\{\lambda \in \mathbb{C};\ Re\lambda \geq \gamma > 0\}$.*

Proof. Let $u \in D(\mathcal{A}) \setminus D(A)$ and $\mathcal{A}u = f$. For any λ with $Re\,\lambda > 0$, denote $g_\lambda = \lambda u - \mathcal{A}u$ and $v = R(\lambda, A)g_\lambda$, then by (3.2.51) $e'_\lambda = u - v \in Ker(\lambda I - \mathcal{A})$.

A quick calculation gives

$$\begin{aligned} e'_\lambda &= u - v = u - R(\lambda, A)(-f + \lambda u) = u - \lambda R(\lambda, A)u + R(\lambda, A)f \\ &= -AR(\lambda, A)u + R(\lambda, A)f. \end{aligned}$$

Taking the representation $e'_\lambda = u - \lambda R(\lambda, A)u + R(\lambda, A)f$ we see that because $\lambda \to R(\lambda, A)$ is holomorphic for $\Re\lambda > 0$, $\lambda \to e_\lambda$ is also holomorphic there. From the Hille–Yosida theorem we have the estimate $\|R(\lambda, A)\| \leq 1/\Re\lambda$ for $\Re\lambda > 0$. For any scalar function $C(\lambda)$, the element $e_\lambda = C(\lambda)e'_\lambda \in Ker(\lambda I - \mathcal{A})$ for each $\Re\lambda > 0$. Thus taking, for example, $C(\lambda) = \lambda^{-1}$, we obtain e_λ that satisfies the required conditions. □

Proposition 3.14. *If for some $\lambda > 0$ the null-space $Ker(\lambda I - \mathcal{A})$ is closed in X, then $\mathcal{A}$ is closed. In particular, $\mathcal{A}$ is closed if $Ker(\lambda I - \mathcal{A})$ is finite-dimensional.*

Proof. We know that $\mathcal{A}$ is closed if and only if $\lambda I - \mathcal{A}$ is closed, so we prove the closedness of $\lambda I - \mathcal{A}$. Let $x'_n \to x'$ and $(\lambda I - \mathcal{A})x'_n \to y$ in X. Operating on x'_n with the projector (3.2.50) we obtain that $x_n = R(\lambda, A)(\lambda I - \mathcal{A})x'_n$ converges to some $x \in D(A)$ (both in X and in $D(A)$). Thus $e_{\lambda,n} = x'_n - x_n \in Ker(\lambda I - \mathcal{A})$ also converges in X and, by assumption,

$$e_\lambda = \lim_{n\to\infty} e_{\lambda,n} \in Ker(\lambda I - \mathcal{A}).$$

Thus

$$x' = x + e_\lambda$$

and because both $D(A)$ and $Ker(\lambda I - \mathcal{A})$ are subspaces of $D(\mathcal{A})$, we have $x' \in D(\mathcal{A})$. Moreover, because $(\lambda I - \mathcal{A})x'_n \to y$ in X, we have $x_n = R(\lambda, A)(\lambda I - \mathcal{A})x'_n \to R(\lambda, A)y$; thus $R(\lambda, A)y = x$ and $(\lambda I - \mathcal{A})x = (\lambda I - \mathcal{A})R(\lambda, A)y = (\lambda I - A)R(\lambda, A)y = y$. This finally yields

$$(\lambda I - \mathcal{A})x' = (\lambda I - \mathcal{A})x + (\lambda I - \mathcal{A})e_\lambda = y$$

and $\mathcal{A}$ is closed. □

Example 3.15. Let us consider the Dirichlet problem for the heat equation

$$\begin{aligned} \partial_t u &= \Delta u, \quad \text{in } \Omega,\ t > 0 \\ u|_{\partial\Omega} &= 0, \\ u|_{t=0} &= \mathring{u}, \end{aligned} \tag{3.2.52}$$

where Ω is a plane domain with a polygonal boundary, [38]. We consider this problem in the space $L_2(\Omega)$. By the earlier example, the semigroup for the above problem is generated by the restriction A_2 of the distributional Laplacian to the domain

$$D(A_2) = \{u \in \mathring{W}^1_2(\Omega);\ \Delta u \in L_2(\Omega)\}.$$

Because Ω is bounded, $(A_2, D(A_2))$ is an isomorphism from $D(A_2)$ onto $L_2(\Omega)$, [38, Theorem 2.2.2.3]. Let us denote

$$D := \mathring{W}^1_2(\Omega) \cap W^2_2(\Omega).$$

If Ω is convex, $D(A_2) = D$ and we have the maximum possible regularity. On the other hand, if the angle α at one corner of Ω satisfies, say, $\pi < \alpha \leq 2\pi$, then $D = \mathring{W}^1_2(\Omega) \cap W^2_2(\Omega)$ is a proper subspace of $D(A_2)$ of codimension 1; see [38, Theorem 4.4.3.3], [11, 12]. In other words,

$$\dim L_2(\Omega)/A_2(D) = 1. \tag{3.2.53}$$

We introduce the maximal operator $A_{2,max}$ defined to be the distributional Laplacian Δ restricted to the domain

$$D(A_{2,\max}) = L_{2,0}(\Omega, \Delta) = \{u \in L_2(\Omega);\ \Delta u \in L_2(\Omega), \gamma u = 0\},$$

where the trace γu is well-defined by means of Green's theorem (see, e.g., [12]). We have the following theorem, [12].

Theorem 3.16. *The operator $A_{2,\max} : L_{2,0}(\Omega, \Delta) \to L_2(\Omega)$ is surjective and the kernel $Ker(A_{2,\max})$ in $L_{2,0}(\Omega, \Delta)$ is isomorphic to $L_2(\Omega)/A_2(D)$.*

The significance of this theorem is that because the generator $A_2 : D(A_2) \to L_2(\Omega)$ is an isomorphism, $Ker(A_{2,\max})$ is not trivial by (3.2.53) and functions from $Ker(A_{2,\max}) \subset D(A_{2,\max})$ do not belong to $D(A_2)$. Therefore $D(A_{2,\max}) \neq D(A_2)$ and by Theorem 3.11 and Corollary 3.13, there exist differentiable $L_2(\Omega)$-valued nul-solutions to (3.2.52).

4

Positivity

1 Basic positivity concepts

The common feature of the introduced models is that the solution originating from a nonnegative density should stay nonnegative; that is, the solution operator should be a 'positive' operator. Since we are talking about general Banach spaces, we have to define what we mean by a nonnegative element of a Banach space. Though in all cases discussed here our Banach space is an $L_1(\Omega, \mu)$ space, where the nonnegativity of a function f is understood as $f(x) \geq 0$ μ-almost everywhere, it is more convenient to work in a more abstract setting.

Defining Order

In a given vector space X an order can be introduced either geometrically, by defining the so-called *positive cone* (in other words, what it means to be a *positive element* of X), or through the axiomatic definition:

Definition 4.1. *Let X be an arbitrary set. A partial order (or simply, an order) on X is a binary relation, denoted here by '$\geq$', which is reflexive, transitive, and antisymmetric, that is,*

(1) $x \geq x$ for each $x \in X$;

(2) $x \geq y$ and $y \geq x$ imply $x = y$ for any $x, y \in X$;

(3) $x \geq y$ and $y \geq z$ imply $x \geq z$ for any $x, y, z \in X$.

The *supremum* of a set is its least upper bound and the *infimum* is the greatest lower bound. The supremum and infimum of a set need not exist. For a two-point set $\{x, y\}$ we write $x \wedge y$ or $\inf\{x, y\}$ to denote its infimum and $x \vee y$ or $\sup\{x, y\}$ to denote supremum.

We say that X is a *lattice* if every pair of elements (and so every finite collection of them) has both supremum and infimum.

From now on, unless stated otherwise, any vector space X is real.

Definition 4.2. *An ordered vector space is a vector space X equipped with partial order which is compatible with its vector structure in the sense that*

(4) $x \geq y$ implies $x + z \geq y + z$ for all $x, y, z \in X$;

(5) $x \geq y$ implies $\alpha x \geq \alpha y$ for any $x, y \in X$ and $\alpha \geq 0$.

The set $X_+ = \{x \in X;\ x \geq 0\}$ is referred to as the positive cone of X.

If the ordered vector space X is also a lattice, then it is called a vector lattice or a Riesz space.

For an element x in a Riesz space X we can define its positive and negative part, and its absolute value, respectively, by

$$x_+ = \sup\{x, 0\}, \quad x_- = \sup\{-x, 0\}, \quad |x| = \sup\{x, -x\},$$

which are called lattice operations. We have

$$x = x_+ - x_-, \qquad |x| = x_+ + x_-. \tag{4.1.1}$$

The absolute value has a number of useful properties that are reminiscent of the properties of the scalar absolute value.

As the next step, we investigate the relation between the lattice structure and the norm, when X is both a normed and an ordered vector space.

Definition 4.3. *A norm on a vector lattice X is called a lattice norm if*

$$|x| \leq |y| \quad \text{implies} \quad \|x\| \leq \|y\|. \tag{4.1.2}$$

A Riesz space X complete under the lattice norm is called a Banach lattice.

Property (4.1.2) gives the important identity:

$$\|x\| = \||x|\|, \qquad x \in X. \tag{4.1.3}$$

AM- and *AL*-spaces

Two classes of Banach lattices playing here a significant role are AL- and AM- spaces.

Definition 4.4. *We say that a Banach lattice X is*

(i) an AL-space if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X_+$,

(ii) an AM-space if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X_+$.

Example 4.5. Standard examples of AM-spaces are offered by the spaces $C(\overline{\Omega})$, where $\overline{\Omega}$ is either a bounded subset of $\mathbb{R}^n$, or in general, a compact topological space. Also the space $L_\infty(\Omega)$ is an AM-space. On the other hand, most known examples of AL-spaces are the spaces $L_1(\Omega)$. These examples exhaust all (up to a lattice isometry) cases of AM- and AL-spaces. However, particular representations of these spaces can be very different.

1.1 Positive Operators

Definition 4.6. *A linear operator A from a Banach lattice X into a Banach lattice Y is called positive, denoted $A \geq 0$, if $Ax \geq 0$ for any $x \geq 0$.*

An operator A is positive if and only if $|Ax| \leq A|x|$. This follows easily from $-|x| \leq x \leq |x|$ so, if A is positive, then $-A|x| \leq Ax \leq A|x|$. Conversely, taking $x \geq 0$, we obtain $0 \leq |Ax| \leq A|x| = Ax$.

Positive operators are fully determined by their behaviour on the positive cone. Precisely speaking, [16, Theorem 2.64]),

Theorem 4.7. *If $A : X_+ \to Y_+$ is additive, then A extends uniquely to a positive linear operator from X to Y. Keeping the notation A for the extension, we have, for each $x \in X$,*

$$Ax = Ax_+ - Ax_-. \tag{4.1.4}$$

A frequently used property of positive operators is given in

Theorem 4.8. *If A is an everywhere defined positive operator from a Banach lattice to a normed Riesz space, then A is bounded.*

Proof. If A were not bounded, then we would have a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $\|x_n\| = 1$ and $\|Ax_n\| \geq n^3$, $n \in \mathbb{N}$. Because X is a Banach space, $x := \sum_{n=1}^{\infty} n^{-2}|x_n| \in X$. Because $0 \leq |x_n|/n^2 \leq x$, we have $\infty > \|Ax\| \geq \|A(|x_n|/n^2)\| \geq \|A(x_n/n^2)\| \geq n$ for all n, which is a contradiction. $\square$

The norm of a positive operator can be evaluated by

$$\|A\| = \sup_{x\geq 0,\, \|x\|\leq 1} \|Ax\|. \tag{4.1.5}$$

Indeed, since $\|A\| = \sup_{\|x\|\leq 1} \|Ax\| \geq \sup_{x\geq 0, \|x\|\leq 1} \|Ax\|$, it is enough to prove the opposite inequality. For each x with $\|x\| \leq 1$ we have $|x| = x_+ + x_- \geq 0$ with $\|x\| = \||x|\| \leq 1$. On the other hand, $A|x| \geq |Ax|$, hence $\|A|x|\| \geq \||Ax|\| = \|Ax\|$. Thus $\sup_{\|x\|\leq 1} \|Ax\| \leq \sup_{x\geq 0, \|x\|\leq 1} \|Ax\|$ and the statement is proved.

As a consequence, we note that

$$0 \leq A \leq B \;\Rightarrow\; \|A\| \leq \|B\|. \tag{4.1.6}$$

Moreover, it is worthwhile to emphasize that if $A \geq 0$ and there exists K such that $\|Ax\| \leq K\|x\|$ for $x \geq 0$, then this inequality holds for any $x \in X$. Indeed, by (4.1.5) we have $\|A\| \leq K$ and using the definition of the operator norm, we obtain the desired statement.

1.2 Relation Between Order and Norm

There is a useful relation between the order, norm (absolute value) and the convergence of sequences if we are in $\mathbb{R}$ – any monotonic sequence which is bounded (in absolute value), converges. One would like to have a similar result in Banach lattices. It turns out to be not so easy.

Existence of an order in some set X allows us to introduce in a natural way the notion of (order) convergence. Proper definitions of order convergence require nets of elements but we do not need to go to such details.

For a non-increasing sequence $(x_n)_{n\in\mathbb{N}}$ we write $x_n\!\downarrow x$ if $\inf\{x_n;\ n \in \mathbb{N}\} = x$. For a non-decreasing sequence $(x_n)_{n\in\mathbb{N}}$ the symbol $x_n \uparrow x$ have an analogous meaning. Then we say that an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ is order convergent to x if it can be sandwiched between two monotonic sequences converging to x. We write this as $x_n \overset{o}{\to} x$. One of the basic results is:

Proposition 4.9. *Let X be a normed lattice. Then:*

(1) The positive cone X_+ is closed.

(2) If $(x_n)_{n\in\mathbb{N}}$ is nondecreasing and $\lim_{n\to\infty} x_n = x$ in the norm of X, then

$$x = \sup\{x_n;\ n \in \mathbb{N}\}.$$

Analogous statement holds for nonincreasing sequences.

In general, the converse of Proposition 4.9(2) is false; that is, we may have $x_n\!\uparrow x$ but $(x_n)_{n\in\mathbb{N}}$ does not converge in norm. Indeed, consider $\boldsymbol{x}_n = (1, 1, 1 \ldots, 1, 0, 0, \ldots) \in l_\infty$, where 1 occupies only the n first positions. Clearly, $\sup_{n\in\mathbb{N}} \boldsymbol{x}_n = \boldsymbol{x} := (1, 1, \ldots, 1, \ldots)$ but $\|\boldsymbol{x}_n - \boldsymbol{x}\|_\infty = 1$.

However, such a converse holds in a special class of Banach lattices, called Banach lattices with order continuous norm.

Definition 4.10. *We say that a Banach lattice X has order continuous norm if for any net $(x_\alpha)_{\alpha\in\Delta}$, $x_\alpha\downarrow 0$ implies $\|x_\alpha\|\downarrow 0$.*

Theorem 4.11. *[1, Theorem 12.9] For a Banach lattice X, the statements below are equivalent.*

(1) X has order continuous norm;

(2) If $0\leq x_n\uparrow\leq x$ holds in X, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence;

(3) X is σ-order complete and $x_n\downarrow 0$ implies $\|x_n\|\to 0$.

Moreover, every Banach lattice with order continuous norm is order complete.

All Banach lattices $L_p(\Omega)$ with $1\leq p<\infty$ have order continuous norms. On the other hand, neither $L_\infty(\Omega)$ nor $C(\bar{\Omega})$ (if Ω does not consist of isolated points) has order continuous norm.

The requirement that $(x_n)_{n\in\mathbb{N}}$ must be order dominated often is too restrictive. The spaces we are mostly concerned belong to a class which have a stronger property.

Definition 4.12. *We say that a Banach lattice X is a KB-*space *(Kantorovič–Banach space) if every increasing norm bounded sequence of elements of X_+ converges in norm in X.*

Example 4.13. We observe that if $x_n\uparrow x$, then $\|x_n\|\leq\|x\|$ for all $n\in\mathbb{N}$ and thus any KB-space has order continuous norm by Theorem 4.11. Hence, spaces which do not have order continuous norm cannot be KB-spaces. This rules out the spaces of continuous functions, l_∞ and $L_\infty(\Omega)$.

To see that the KB-class is indeed strictly smaller, let us consider the space c_0. First we prove that it has order continuous norm. It is clearly σ-order complete. Let the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$, given by $\mathbf{x}_n=(x^n_k)_{k\in\mathbb{N}}$, satisfy $\mathbf{x}_n\downarrow 0$. For a given $\epsilon>0$, we find k_0 such that $|x^1_k|<\epsilon$ for all $k\geq k_0$. Because $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is decreasing, we also have $|x^n_k|<\epsilon$ for all $k\geq k_0$ and $n\geq 1$. Then, we find n_0 such that $|x^n_k|<\epsilon$ for all $n\geq n_0$ and $1\leq k\leq k_0$ and combining these estimates we see that $\|\mathbf{x}_n\|<\epsilon$ for all $n\geq n_0$ so $\|\mathbf{x}_n\|\to 0$.

On the other hand, let us again take the sequence $\mathbf{x}_n=(1,1,\ldots,1,0,0,\ldots)$ where 1 occupies n first positions. It is clearly norm bounded and increasing, but it does not converge in norm to any element of c_0. Hence, c_0 is not a KB-space.

Any reflexive Banach space is a KB-space, [16, Theorem 2.82]. That AL-spaces (so, in particular, all L_1 spaces) are also KB-spaces follows from the following simple argument.

Theorem 4.14. *Any AL-space is a KB-space.*

Proof. If $(x_n)_{n\in\mathbb{N}}$ is an increasing and norm bounded sequence, then for $0\leq x_n\leq x_m$, we have

$$\|x_m\|=\|x_m-x_n\|+\|x_n\|$$

as $x_m-x_n\geq 0$ so that

$$\|x_m-x_n\|=\|x_m\|-\|x_n\|=|\|x_m\|-\|x_n\||\,.$$

Because, by assumption, $(\|x_n\|)_{n\in\mathbb{N}}$ is monotonic and bounded, and hence convergent, we see that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and thus converges. □

1.3 Complexification

Due to the construction, solutions to all our models must be real. Thus, our problems should be posed in real Banach spaces. However, to take full advantage of the tools of functional analysis, such as the spectral theory, it is worthwhile to extend our spaces to include also complex valued functions, so that they become complex Banach spaces. While the algebraic and metric structure of Banach spaces can be easily extended to the complex setting, the extension of the order structure must be done with more care. This is done by the procedure called *complexification*.

Definition 4.15. *Let X be a real vector lattice. The complexification X_C of X is the set of pairs $(x, y) \in X \times X$ where, following the scalar convention, we write $(x, y) = x + iy$. Vector operations are defined as in scalar case while the partial order is defined by*

$$x_0 + iy_0 \leq x_1 + iy_1 \quad \text{if and only if} \quad x_0 \leq x_1 \text{ and } y_0 = y_1. \tag{4.1.7}$$

Remark 4.16. Note that from the definition it follows that $x \geq 0$ in X_C is equivalent to $x \in X$ and $x \geq 0$ in X. In particular, X_C with partial order (4.1.7) is not a lattice.

Example 4.17. Any positive linear operator A on X_C is a real operator; that is, $A : X \to X$. In fact, let $X \ni x = x_+ - x_-$. By definition, $Ax_+ \geq 0$ and $Ax_- \geq 0$ so $Ax_+, Ax_- \in X$ and thus $Ax = Ax_+ - Ax_- \in X$.

It is a more complicated task to introduce a norm on X_C because standard product norms, in general, fail to preserve the homogeneity of the norm, see [16, Example 2.88].

Since X_C is not a lattice, we cannot define the modulus of $z = x+iy \in X_C$ in a usual way. However, following an equivalent definition of the modulus in the scalar case, for $x + iy \in X_C$ we define

$$|x + iy| = \sup_{\theta \in [0,2\pi]} \{x \cos\theta + y \sin\theta\}.$$

It can be proved that this element exists.

Such a defined modulus has all standard properties of the scalar complex modulus. Thus, one can define a norm on the complexification X_C by

$$\|z\|_c = \|x + iy\|_c = \||x + iy|\|. \tag{4.1.8}$$

Properties (a)–(c) and $|x| \leq |z|$, $|y| \leq |z|$ imply that $\|\cdot\|_c$ is a norm on X_C, which is equivalent to the Euclidean norm on $X \times X$, denoted by $\|\cdot\|_C$. As the norm $\|\cdot\|$ is a lattice norm on X, we have $\|z_1\|_c \leq \|z_2\|_c$, whenever $|z_1| \leq |z_2|$, and $\|\cdot\|_c$ becomes a lattice norm on X_C.

Definition 4.18. *A complex Banach lattice is an ordered complex Banach space X_C that arises as the complexification of a real Banach lattice X, according to Definition 4.15, equipped with the norm (4.1.8).*

Remember: a complex Banach lattice is not a Banach lattice!

Any linear operator A on X can be extended to X_C according to

$$A_C(x + iy) = Ax + iAy.$$

We observe that if A is a positive operator between real Banach lattices X and Y then, for $z = x+iy \in X_C$, we have

$$(Ax)\cos\theta + (Ay)\sin\theta = A(x\cos\theta + y\sin\theta) \leq A|z|,$$

therefore $|A_C z| \leq A|z|$. Hence for positive operators

$$\|A_C\|_c = \|A\|. \tag{4.1.9}$$

There are, however, examples where $\|A\| < \|A_C\|_c$.

Note that the standard $L_p(\Omega)$ and $C(\Omega)$ norms are of the type (4.1.8). These spaces have a nice property of preserving the operator norm even for operators which are not necessarily positive, see [16, p. 63].

Remark 4.19. If for a linear operator A, we prove that it generates a semigroup of say, contractions, in X, then this semigroup will be also a semigroup of contractions on X_C, hence, in particular, A is a dissipative operator in the complex setting. Due to this observation we confine ourselves to real operators in real spaces.

Series of Positive Elements in Banach Lattices

We note the following two results which are series counterparts of the dominated and monotone convergence theorems in Banach lattices.

Theorem 4.20. *[15] Let $(x_n(t))_{n\in\mathbb{N}}$ be family of nonnegative sequences in a Banach lattice X, parameterized by a parameter $t \in T \subset \mathbb{R}$, and let $t_0 \in \overline{T}$.*

(i) If for each $n \in \mathbb{N}$ the function $t \to x_n(t)$ is non-decreasing and $\lim\limits_{t\nearrow t_0} x_n(t) = x_n$ in norm, then

$$\lim_{t\nearrow t_0} \sum_{n=0}^{\infty} x_n(t) = \sum_{n=0}^{\infty} x_n, \tag{4.1.10}$$

irrespective of whether the right hand side exists in X or $\|\sum\limits_{n=0}^{\infty} x_n\| := \sup\{\|\sum\limits_{n=0}^{N} x_n\|;\ N \in \mathbb{N}\} = \infty$. In the latter case the equality should be understood as the norms of both sides being infinite.

(ii) If $\lim\limits_{t\to t_0} x_n(t) = x_n$ in norm for each $n \in \mathbb{N}$ and there exists $(a_n)_{n\in\mathbb{N}}$ such that $x_n(t) \le a_n$ for any $t \in T, n \in \mathbb{N}$ with $\sum\limits_{n=0}^{\infty} \|a_n\| < \infty$, then (4.1.10) holds as well.

Remark 4.21. Note that if X is a KB-space, then $\lim_{t\nearrow t_0} \sum_{n=0}^{\infty} x_n(t) \in X$ implies convergence of $\sum_{n=0}^{\infty} x_n$. In fact, since $x_n \ge 0$ (by closedness of the positive cone), $N \to \sum_{n=0}^{N} x_n$ is non-decreasing, and hence either $\sum_{n=0}^{\infty} x_n \in X$, or $\|\sum_{n=0}^{\infty} x_n\| = \infty$ and, in the latter case, $\|\lim_{t\nearrow t_0} \sum_{n=0}^{\infty} x_n(t)\| = \infty$.

2 Positive Semigroups

Definition 4.22. *Let X be a Banach lattice. We say that the semigroup $\{G(t)\}_{t\ge 0}$ on X is positive if for any $x \in X_+$ and $t \ge 0$,*

$$G(t)x \ge 0.$$

We say that an operator $(A, D(A))$ is resolvent positive if there is ω such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) \ge 0$ for all $\lambda > \omega$.

Remark 4.23. In this section, because we address several problems related to spectral theory, we need complex Banach lattices. Let us recall, Definitions 4.15 and 4.18, that a complex Banach lattice is always a complexification X_C of an underlying real Banach lattice X. In particular, $x \ge 0$ in X_C if and only if $x \in X$ and $x \ge 0$ in X.

Theorem 4.24. *A strongly continuous semigroup $\{G_A(t)\}_{t\ge 0}$ is positive if and only if the resolvent of its generator A is positive for large λ.*

Proof. The positivity of the resolvent for $\lambda > \omega$ follows from (3.1.19) and closedness of the positive cone; see Proposition 4.9. Conversely, the latter together with the exponential formula (3.1.23) shows that resolvent positive generators generate positive semigroups.

In practice, it is important to characterise positivity of the semigroup directly through a property of its generator.

2.1 Examples of positive semigroups

Finite dimensional case

Proposition 4.25. *The solution* $\mathbf{y}(t)$ *of*

$$\frac{\mathrm{d}\,\mathbf{y}}{\mathrm{d}\,t} = \mathcal{A}\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{y}_0$$

satisfies $\mathbf{y}(t) \geq 0$ *for any* $t > 0$ *for arbitrary* $\mathbf{y}_0 \geq 0$ *if and only if* $\mathcal{A}$ *has non-negative off-diagonal entries.*

Proof. First let us consider $\mathcal{A} \geq 0$. Then, using the exponential formula

$$e^{t\mathcal{A}} = \mathcal{I} + t\mathcal{A} + \frac{t^2}{2}\mathcal{A}^2 + \frac{t^3}{3!}\mathcal{A}^3 + \ldots + \frac{t^k}{k!}\mathcal{A}^k + \ldots,$$

and the fact that the powers of a nonnegative matrix are nonnegative, we see that $e^{t\mathcal{A}} \geq 0$ for $t \geq 0$. Next, we observe that for any real a and $0 \leq \overset{\circ}{\mathbf{y}} \in \mathbb{R}^n$, the function $\mathbf{y}(t) = e^{at}e^{t\mathcal{A}}\mathbf{y}_0 \geq 0$ if and only if $e^{t\mathcal{A}}\mathbf{y}_0 \geq 0$ and satisfies the equation

$$\frac{\mathrm{d}\,\mathbf{y}}{\mathrm{d}\,t} = a\mathbf{y} + \mathcal{A}\mathbf{y} = (a\mathcal{I} + \mathcal{A})\mathbf{y}.$$

Hence, if the diagonal entries of $\mathcal{A}$, a_{ii}, are negative then, denoting $r = \max_{1\leq i\leq n}\{-a_{ii}\}$, we find that $\tilde{\mathcal{A}} = r\mathcal{I} + \mathcal{A} \geq 0$. Using the first part of the proof, we see that

$$e^{t\mathcal{A}} = e^{-rt}e^{t\tilde{\mathcal{A}}} \geq 0. \tag{4.2.11}$$

To prove the converse, let us write

$$e^{t\mathcal{A}} = \mathcal{E}(t) = \begin{pmatrix} \epsilon_{11}(t) & \ldots & \epsilon_{1n}(t) \\ \vdots & & \vdots \\ \epsilon_{n1}(t) & \ldots & \epsilon_{nn}(t) \end{pmatrix},$$

so that $\epsilon_{ij}(t) \geq 0$ for all $i, j = 1, \ldots, n$, and consider $\mathcal{E}(t)\mathbf{e}_i = (\epsilon_{1i}(t), \ldots, \epsilon_{ni}(t))$. Then

$$\begin{aligned} (a_{1i}, \ldots, a_{ii}, \ldots, a_{ni}) &= \mathcal{A}\mathcal{E}(t)\mathbf{e}_i|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{E}(t)\mathbf{e}_i\right|_{t=0} \\ &= \lim_{h\to 0^+}\left(\frac{\epsilon_{1i}(h)}{h}, \ldots, \frac{\epsilon_{ii}(h) - 1}{h}, \ldots, \frac{\epsilon_{ni}(h)}{h}\right), \end{aligned}$$

so that $a_{ji} \geq 0$ for $j \neq i$.

The diffusion semigroup in $L_2(\Omega)$

Often we have to resort to PDE specific techniques.

Consider the diffusion problem (4.2.12)–(4.2.14)

$$\partial_t u = \Delta u, \quad \text{in } \Omega \times [0, T], \tag{4.2.12}$$
$$u = 0, \quad \text{on } \Sigma, \tag{4.2.13}$$
$$u = u_0, \quad \text{on } \Omega \tag{4.2.14}$$

and its variational solution u which satisfies $u \in C([0, \infty[, L_2(\Omega)) \cap C^\infty(]0, \infty[, L_2(\Omega))$.

We us the method of barrier function introduced by G. Stampacchia. We assume that Ω is bounded and $u_0 \geq 0$.

Let G be a bounded $C^1(\mathbb{R})$ function satisfying $G(s) = 0$ for $s \geq 0$, $G'(s) < 0$ for $s < 0$ (so that $G(s) > 0$ for $s < 0$) and $|G'(s)| \leq M$ for $s \in \mathbb{R}$. Let us define

$$H(s) = \int_0^s G(\sigma)d\sigma$$

and

$$\phi(t) = \int_\Omega H(u(\boldsymbol{x}, t))d\boldsymbol{x}.$$

We observe that for any $u \in \overset{\circ}{W}{}_2^1(\Omega)$, $G(u(\cdot)) \in \overset{\circ}{W}{}_2^1(\Omega)$. Indeed, if $\phi \in C_0^\infty(\Omega)$, then supp $G(\phi(\cdot)) = \{\boldsymbol{x} \in \mathbb{R}^n;\ \phi(\boldsymbol{x}) < 0\} \subset$ supp ϕ. If $C_0^\infty(\Omega) \ni \phi_n \to u \in \overset{\circ}{W}{}_2^1(\Omega)$ then, without changing notation, we can use the subsequence converging almost everywhere. It is easy to see that

$$\|\partial G(\phi_n)\partial\phi_n - \partial G(u)\partial u\|_0^2 \leq \int_\Omega |G'(\phi_n)|^2|\partial\phi_n - \partial u|^2 d\boldsymbol{x} + \int_\Omega |G'(\phi_n) - G'(u)|^2|\partial u|^2 d\boldsymbol{x}$$

and the convergence follows since $|G'| \leq M$ (in the second term by the dominated convergence theorem and the fact that $G'(\phi_n) \to G'(u)$ almost everywhere). Now, we observe that $\phi \in C([0, \infty[) \cap C^1(]0, \infty[)$. Indeed

$$|\phi(t + h) - \phi(t)| \leq L \int_\Omega |u(x, t + h) - u(x, t)|d\boldsymbol{x} \leq L\sqrt{\mu(\Omega)}\|u(\cdot, t + h) - u(\cdot, t)\|_0.$$

Similarly, using the formal derivative

$$\partial_t\phi(t) = \int_\Omega G(u(\boldsymbol{x}, t))\partial_t u(\boldsymbol{x}, t)d\boldsymbol{x}$$

and the fact that H is twice differentiable

$$\begin{aligned}\left|\frac{\phi(t + h) - \phi(t) - h\partial_t\phi(t)}{h}\right| &\leq \int_\Omega |G(u(\boldsymbol{x}, t))| \left|\frac{u(\boldsymbol{x}, t + h) - u(\boldsymbol{x}, t) - h\partial_t u(t, \boldsymbol{x})}{h}\right| d\boldsymbol{x} \\ &\quad + \frac{1}{2h}\int_\Omega |\partial G(\tilde{u}(\boldsymbol{x}, t)||u(\boldsymbol{x}, t + h) - u(\boldsymbol{x}, t)|^2 d\boldsymbol{x},\end{aligned}$$

where $\tilde{u}$ is some intermediate point. Since G and G' are bounded, the first integral converges to zero as $h \to 0$ by L_2 differentiability of u for $t > 0$ and for the second term we have

$$\left|\frac{1}{2h}\int_\Omega |\partial G(\tilde{u}(\boldsymbol{x}, t)||u(\boldsymbol{x}, t + h) - u(\boldsymbol{x}, t)|^2 d\boldsymbol{x}\right| \leq \frac{hM}{2}\frac{\|u(\cdot, t + h) - u(\cdot, t)\|_0^2}{h^2}$$

which also converges to zero as $h \to 0$ and $t > 0$. Using continuity, we see that $\phi(0) = 0$ as, due to $u_0 \geq 0$, the integral of G is zero. Moreover, the integral is nonzero only if $u(\boldsymbol{x}, t) < 0$ which makes H negative and so $\phi(t) \leq 0$ for $t \geq 0$. Using the properties of ϕ, we find

$$\partial_t\phi(t) = \int_\Omega G(u(\boldsymbol{x}, t))\partial_t u(\boldsymbol{x}, t)d\boldsymbol{x} = -\int_\Omega \partial G(u(\boldsymbol{x}, t))|\nabla u(\boldsymbol{x}, t)|^2 d\boldsymbol{x} \geq 0.$$

Since, however, $\phi(0) = 0$ and $\phi(t) \leq 0$ we find $\phi(t) = 0$ for all $t \geq 0$. This implies $H(u(\boldsymbol{x}, t)) = 0$ almost everywhere which, in turn, gives $u(\boldsymbol{x}, t) \geq 0$ for almost any $\boldsymbol{x}$.

Positive maximum principle

For semigroups on the space of of continuous functions, a powerful tool for recognizing positive semigroups is offered by the maximum principle.

Definition 4.26. *We say that an (unbounded) operator A on $X = C(K)$, where K is compact, satisfies the positive maximum principle if for every $0 \leq u \in D(A)$ and $x \in K$, $u(x) = 0$ implies $(Au)(x) \geq 0$.*

Then we have

Theorem 4.27. *A semigroup $\{G(t)\}_{t\geq 0}$ on $C(K)$ generated by A is positive if and only if A satisfies the positive maximum principle.*

Proof. If $\{G(t)\}_{t\geq 0}$ is positive, then $G(t)u \geq 0$ for any $0 \leq u \in D(A)$ and $t \geq 0$. But, from the definition of the generator,

$$Au = \lim_{h\to 0^+} \frac{G(h)u - u}{h}$$

uniformly in x on K, and hence pointwise. If $u(x) = 0$, then

$$(Au)(x) = \lim_{h\to 0^+} \frac{[G(h)u](x)}{h} \geq 0$$

and A satisfies the positive maximum principle.

Conversely, let A satisfies positive maximum principle. Our aim is to show that it is resolvent positive for large λ. Let $s = \inf\{\lambda \in \mathbb{R};\ [\lambda, \infty[\subset \rho(A)\}$. Then $s < \infty$ by Hille-Yosida theorem. Let us take a strictly positive $C(K) \ni u >> 0$. Since $\lim\limits_{\mu\to\infty} \mu R(\mu)u = u$ by (3.1.20) and, since this limit is uniform, we can consider

$$\lambda_0 = \inf\{\lambda > s;\ R(\mu, A)u >> 0 \text{ for all } \mu \in]\lambda, \infty[\} < \infty.$$

If $\lambda_0 \neq s$, then $\lambda_0 > s$ and $\lambda_0 \in \rho(A)$. Moreover, $R(\lambda_0, A)u$ is not strictly positive, therefore there exists $x \in K$ for which $[R(\lambda_0, A)u](x) = 0$. Then the positive maximum principle implies $[AR(\lambda_0, A)u](x) \geq 0$. But then

$$0 < u(x) = \lambda_0 R(\lambda_0, A)u](x) - [AR(\lambda_0, A)u](x) \leq 0$$

which is a contradiction. Thus, $R(\lambda, A)u >> 0$ for all $u >> 0$ and $\lambda > s$. Since $\{u \in C(K);\ u >> 0\}$ is dense in $C(K)_+$, we see that $R(\lambda, A) \geq 0$ for $\lambda > s$. □

With this result in hand, we can prove that the semigroup for the diffusion problem with the Robin boundary conditions, (3.1.31), is positive. To use the positive maximum principle, consider $0 \leq u \in \{C^2([0, 1]);\ \partial_x u(0) + a_0 u(0) = \partial_x u(1) + a_1 u(1) = 0\}$ and $x \in [0, 1]$ such that $u(x) = 0$. If $x \in]0, 1[$, then u attains local minimum at x and thus $[Au](x) = \partial_{xx} u(x) \geq 0$. If $x = 0$, then $\partial_x u(0) = 0$. But, if $u(0) = \partial_x u(0) = 0$, then the even extension u^* of u to $[-1, 0]$ is twice differentiable on $[-1, 1]$ and thus $\partial_{xx} u^*(0) = \partial_{xx} u(0) = 0$. Similar argument works if $u(1) = 0$.

3 Arendt-Batty-Robinson theorem

Positivity of a semigroup allows for strengthening of several results pertaining to the spectrum of its generator.

Theorem 4.28. *[53] Let $\{G_A(t)\}_{t\geq 0}$ be a positive semigroup on a Banach lattice, with the generator A. Then*

$$R(\lambda, A)x = \int\limits_0^\infty e^{-\lambda t} G_A(t)x dt \qquad (4.3.15)$$

for all $\lambda \in \mathbb{C}$ with $\Re\lambda > s(A)$. Furthermore, *(i) Either* $s(A) = -\infty$ *or* $s(A) \in \sigma(A)$*;*

(ii) For a given $\lambda \in \rho(A)$*, we have* $R(\lambda, A) \geq 0$ *if and only if* $\lambda > s(A)$*;*

(iii) For all $\Re\lambda > s(A)$ *and* $x \in X$*, we have* $|R(\lambda, A)x| \leq R(\Re\lambda, A)|x|$*.*

We conclude this section by briefly describing an approach of [3] which leads to several interesting results for resolvent positive operators. To fix attention, assume for the time being that $\omega < 0$ (thus, in particular, A is invertible and $-A^{-1} = R(0, A)$) and $\lambda > 0$. The resolvent identity

$$-A^{-1} = (\lambda - A)^{-1} + \lambda(\lambda - A)^{-1}(-A^{-1}),$$

can be extended by induction to

$$-A^{-1} = R(\lambda, A) + \lambda R(\lambda, A)^2 + \cdots + \lambda^n R(\lambda, A)^n(-A^{-1}). \tag{4.3.16}$$

Now, because all terms above are nonnegative, we obtain

$$\sup_{n\in\mathbb{N},\lambda>\omega} \{\lambda^n \|(\lambda - A)^{-n}(-A^{-1})\|_X\} = M < +\infty.$$

This is 'almost' the Hille–Yosida estimate and allows us to prove that the Cauchy problem (3.1.10), (3.1.11) has a mild Lipschitz continuous solution for $\mathring{u} \in D(A^2)$. If, in addition, A is densely defined, then this mild solution is differentiable, and thus it is a strict solution (see, e.g., [4] and [6, pp. 191–200]). These results are obtained by means of the integrated, or regularised, semigroups, which are beyond the scope of this lecture and thus we do not enter into details of this very rich field. We mention, however, an interesting consequence of (4.3.16) for the semigroup generation, which has already found several applications and which we use later.

Theorem 4.29. *[4, 27] Let* A *be a densely defined resolvent positive operator. If there exist* $\lambda_0 > s(A), c > 0$ *such that for all* $x \geq 0$*,*

$$\|R(\lambda_0, A)x\|_X \geq c\|x\|_X, \tag{4.3.17}$$

then A *generates a positive semigroup* $\{G_A(t)\}_{t\geq 0}$ *on* X*.*

Proof. Let us take $s(A) < \omega \leq \lambda_0$ and set $B = A - \omega I$ so that $s(B) < 0$. Because $R(0, B) = R(\omega, A) \geq R(\lambda_0, A)$, it follows from (4.3.17) and (4.1.6) that

$$\|R(0, B)x\|_X \geq \|R(\lambda_0, A)x\|_X \geq c\|x\|_X$$

for $x \geq 0$. Using (4.3.16) for B and $x = \lambda^n R(\lambda, B)^n g$, $g \geq 0$, we obtain, by (4.3.17),

$$\|\lambda^n R(\lambda, B)^n g\|_X \leq c^{-1}\|R(0, B)\lambda^n R(\lambda, B)^n g\| \leq M\|g\|_X,$$

for $\lambda > 0$. Again, by (4.1.6), we can extend the above estimate onto X proving the Hille–Yosida estimate. Because B is densely defined, it generates a bounded positive semigroup and thus $\|G_A(t)\| \leq Me^{\omega t}$. □

4 Application – transport on graphs

We return to the problem Hence, we see that there is a class of problems which can be written in the form

$$\begin{aligned} \mathbf{u}_t = A\mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{f}, \\ \mathbf{u}(1) = \mathbb{K}\mathbf{u}(0), \end{aligned} \tag{4.4.18}$$

where $A = \mathrm{diag}\{c_j\partial_x\}_{1\leq j\leq m}$ and $\mathbb{K}$ is an arbitrary (not necessarily nonnegative) matrix. We consider A on the domain

$$D(A) = \{\mathbf{u} \in (W_1^1([0, 1]))^m;\ \mathbf{u}(1) = \mathbb{K}\mathbf{u}(0)\} \tag{4.4.19}$$

and by $\mathbb{C}$ we denote $\mathrm{diag}\{c_j\}_{1\leq j\leq m}$.

Theorem 4.30. *The operator $(A, D(A))$ generates a C_0 semigroup on $(L_1([0,1]))^m$. The semigroup is positive of and only if $\mathbb{K} \geq 0$.*

Proof. Clearly, $(C_0^\infty((0,1)))^m \subset D(A)$ and hence $D(A)$ is dense in X. Let us consider the resolvent equation for A. We have to solve

$$\lambda u_j - c_j \partial_x u_j = f_j, \quad j = 1, \ldots, m, \quad x \in (0,1),$$

with $\mathbf{u} \in D(A)$. Integrating, we find the general solution

$$c_j u_j(x) = c_j e^{\frac{\lambda}{c_j}x} v_j + \int\limits_x^1 e^{\frac{\lambda}{c_j}(x-s)} f_j(s) ds, \tag{4.4.20}$$

where $\mathbf{v} = (v_1, \ldots, v_m)$ is an arbitrary vector. Let $\mathbb{E}_\lambda(s) = \mathrm{diag}\left\{e^{\frac{\lambda}{c_j}s}\right\}_{1\leq j\leq m}$. Then (4.4.20) takes the form

$$\mathbb{C}\mathbf{u}(x) = \mathbb{C}\mathbb{E}_\lambda(x)\mathbf{v} + \int\limits_x^1 \mathbb{E}_\lambda(x-s)\mathbf{f}(s)ds.$$

To determine $\mathbf{v}$ so that $\mathbf{u} \in D(A)$, we use the boundary conditions. At $x = 1$ and at $x = 0$ we obtain, respectively

$$\mathbb{C}\mathbf{u}(1) = \mathbb{C}\mathbb{E}_\lambda(1)\mathbf{v}, \qquad \mathbb{C}\mathbf{u}(0) = \mathbb{C}\mathbf{v} + \int\limits_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds.$$

Let us first consider $\mathbb{K} \geq 0$. Then we also take $\mathbf{f} \geq 0$.

$$\mathbb{C}\mathbb{E}_\lambda(1)\mathbf{v} = \mathbb{C}\mathbf{u}(1) = \mathbb{C}\mathbb{K}\mathbf{u}(0) = \mathbb{C}\mathbb{K}\left(\mathbf{v} + \mathbb{C}^{-1}\int\limits_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds\right),$$

which can be written as

$$(\mathbb{I} - \mathbb{E}_\lambda(-1)\mathbb{K})\mathbf{v} = \mathbb{E}_\lambda(-1)\mathbb{K}\mathbb{C}^{-1}\int\limits_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds. \tag{4.4.21}$$

Since the norm of $\mathbb{E}_\lambda(-1)$ can be made as small as one wishes by taking large λ, we see that $\mathbf{v}$ is uniquely defined by the Neumann series provided λ is sufficiently large and hence the resolvent of A exists. We need to find an estimate for it. First we observe that the Neumann series expansion ensures that A is a resolvent positive operator and hence the norm estimates can be obtained using only nonnegative entries. Adding together the rows in

$$\mathbb{E}_\lambda(1)\mathbf{v} = \mathbb{K}\left(\mathbf{v} + \mathbb{C}^{-1}\int\limits_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds\right),$$

we obtain

$$\sum_{j=1}^m e^{\frac{\lambda}{c_j}} v_j = \sum_{j=1}^m \kappa_j v_j + \sum_{j=1}^m \frac{\kappa_j}{c_j}\int\limits_0^1 e^{-\frac{\lambda}{c_j}s} f_j(s)ds,$$

where $\kappa_j = \sum\limits_{i=1}^m k_{ij}$. By (4.4.20), we can evaluate, for $j \in \{1, \ldots, m\}$,

$$\begin{aligned}\int\limits_0^1 u_j(x)dx &= v_j \int\limits_0^1 e^{\frac{\lambda}{c_j}x}dx + \frac{1}{c_j}\int\limits_0^1\int\limits_x^1 e^{\frac{\lambda}{c_j}(x-s)} f_j(s)dsdx \\ &= \frac{v_j c_j}{\lambda}\left(e^{\frac{\lambda}{c_j}} - 1\right) + \frac{1}{\lambda}\int\limits_0^1 \left(1 - e^{-\frac{\lambda}{c_j}s}\right) f_j(s)ds\end{aligned}$$

so that, renorming X with the norm $\|\mathbf{u}\|_c = \sum_{j=1}^m c_j^{-1}\|u_j\|_{L_1([0,1])}$, we have

$$\begin{aligned}\|\mathbf{u}\|_c &= \sum_{j=1}^m c_j^{-1}\int_0^1 u_j(x)dx \\ &= \frac{1}{\lambda}\sum_{j=1}^m v_j\left(e^{\frac{\lambda}{c_j}}-1\right) + \frac{1}{\lambda}\sum_{j=1}^m c_j^{-1}\int_0^1 \left(1-e^{-\frac{\lambda}{c_j}s}\right)f_j(s)ds \\ &= \frac{1}{\lambda}\sum_{j=1}^m v_j(\kappa_j-1) + \frac{1}{\lambda}\sum_{j=1}^m \frac{\kappa_j-1}{c_j}\int_0^1 e^{-\frac{\lambda}{c_j}s}f_j(s)ds + \frac{1}{\lambda}\sum_{j=1}^m\frac{1}{c_j}\int_0^1 f_j(s)ds.\end{aligned} \tag{4.4.22}$$

We consider three cases.

(a) $\kappa_j \leq 1$ for $j = 1,\ldots,m$. This condition means that the l_1 matrix norm of $\mathbb{K}$ is at most 1, hence $\|\mathbb{E}_\lambda(-1)\mathbb{K}\| < 1$ and thus $\mathbf{v}$ is defined for any $\lambda > 0$. Therefore $R(\lambda, A)$ is defined and positive for any $\lambda > 0$. Under the assumption of this item, by dropping two first terms in the second line, (4.4.22) gives

$$\|\mathbf{u}\|_c \leq \frac{1}{\lambda}\sum_{j=1}^m\frac{1}{c_j}\int_0^1 f_j(s)ds = \frac{1}{\lambda}\|\mathbf{f}\|_c, \quad \lambda > 0.$$

Since $D(A)$ is dense in X, $(A, D(A))$ generates a positive semigroup of contractions in $(X, \|\cdot\|_c)$.

(b) $\kappa_j \geq 1$ for $j = 1,\ldots,m$. Then (4.4.22) implies that for some $\lambda > 0$ and $c = 1/\lambda$ we have

$$\|R(\lambda, A)\mathbf{f}\|_c \geq c\|\mathbf{f}\|_c$$

and, by density of $D(A)$, the application of the Arendt-Batty-Robinson theorem, Theorem 4.29, gives the existence of a positive semigroup generated by A in $(X, \|\cdot\|_c)$. Since, however, the norm $\|\cdot\|_c$ and the standard norm $\|\cdot\|$ are equivalent, we see that A generates a positive semigroup in X.

(c) $\kappa_j < 1$ for $j \in I_1$ and $\kappa_j \geq 1$ for $j \in I_2$, where $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{1,\ldots,m\}$. Let $\mathbb{L} = (l_{ij})_{1\leq i,j\leq m}$, where $l_{ij} = k_{ij}$ for $j \in I_2$ and $l_{ij} = 1$ for $j \in I_1$. Thus, denoting by $A_{\mathbb{L}}$ the operator given by the differential expression $\mathrm{diag}\{c_j\partial_x\}_{1\leq j\leq m}$ restricted to

$$D(A_{\mathbb{L}}) = \{\mathbf{u} \in (W_1^1([0,1]))^m;\ \mathbf{u}(1) = \mathbb{L}\mathbf{u}(0)\},$$

we see, by (4.4.21), that

$$0 \leq R(\lambda, A) \leq R(\lambda, A_{\mathbb{L}}) \tag{4.4.23}$$

for any λ for which $R(\lambda, A_{\mathbb{L}})$ exists. But, by item (b), $A_{\mathbb{L}}$ generates a positive semigroup and thus satisfies the Hille–Yosida estimates. Since clearly (4.4.23) yields $R^k(\lambda, A) \leq R^k(\lambda, A_{\mathbb{L}})$ for any $k \in \mathbb{N}$, for some $\omega > 0$ and $M \geq 1$ we have

$$\|R^k(\lambda, A)\| \leq \|R^k(\lambda, A_{\mathbb{L}})\| \leq M(\lambda-\omega)^{-k}, \qquad \lambda > \omega, k \in \mathbb{N},$$

and hence we obtain the generation of a semigroup by A.

Assume now that $\mathbb{K}$ is non positive. The analysis up to (4.4.21) remains valid. Then (4.4.21) can be expanded as

$$\mathbf{v} = (\mathbb{I} - \mathbb{E}_\lambda(-1)\mathbb{K})^{-1}\,\mathbb{E}_\lambda(-1)\mathbb{K}\mathbb{C}^{-1}\int_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds = \sum_{n=0}^\infty(\mathbb{E}_\lambda(-1)\mathbb{K})^n\mathbb{E}_\lambda(-1)\mathbb{K}\mathbb{C}^{-1}\int_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds \tag{4.4.24}$$

and

$$\mathbf{u}(x) = \mathbb{E}_\lambda(x)\sum_{n=0}^\infty(\mathbb{E}_\lambda(-1)\mathbb{K})^n\mathbb{E}_\lambda(-1)\mathbb{K}\mathbb{C}^{-1}\int_0^1 \mathbb{E}_\lambda(-s)\mathbf{f}(s)ds + \mathbb{C}^{-1}\int_x^1 \mathbb{E}_\lambda(x-s)\mathbf{f}(s)ds. \tag{4.4.25}$$

Denoting now $|\mathbf{u}| = (|u_1|, \ldots, |u_n|)$ and $|\mathbb{K}| = (|k_{ij}|)_{1\leq i,j\leq m}$ and using the fact that only $\mathbb{K}$ may have non positive entries, we find

$$|\mathbf{u}(x)| \leq \mathbb{E}_\lambda(x)\sum_{n=0}^{\infty}(\mathbb{E}_\lambda(-1)|\mathbb{K}|)^n\mathbb{E}_\lambda(-1)|\mathbb{K}|\mathbb{C}^{-1}\int\limits_0^1 \mathbb{E}_\lambda(-s)|f|(s)ds + \mathbb{C}^{-1}\int\limits_x^1 \mathbb{E}_\lambda(x-s)|f|(s)ds = R(\lambda, A_{|\mathbb{K}|})|f|,$$

where, similarly, $A_{|\mathbb{K}|}$ denotes the transport operator restricted to $W_1^1([0,1])$ functions satisfying $\mathbf{u}(1) = |\mathbb{K}|\mathbf{u}(0)$. So, we can write

$$|R(\lambda, A_{\mathbb{K}})f| \leq R(\lambda, A_{|\mathbb{K}|})|f|$$

and, iterating,

$$|R(\lambda, A_{\mathbb{K}})^2 f| = |R(\lambda, A_{\mathbb{K}})R(\lambda, A_{\mathbb{K}})f| \leq R(\lambda, A_{|\mathbb{K}|})|R(\lambda, A_{\mathbb{K}})f| \leq R(\lambda, A_{|\mathbb{K}|})^2|f|$$

so that inductively

$$|R(\lambda, A_{\mathbb{K}})^k f| \leq R(\lambda, A_{|\mathbb{K}|})^k|f|.$$

Using the fact that in Banach lattices taking the modulus does not change its norm, we find

$$\|R(\lambda, A_{\mathbb{K}})^k f\| \leq \|R(\lambda, A_{|\mathbb{K}|})^k|f|\| \frac{M}{\lambda-\omega}\|f\|.$$

with M and ω following from the Hille-Yosida estimates for $A_{|\mathbb{K}|}$.

The fact that $\mathbb{K} \geq 0$ yields the positivity of the semigroup follows from the first part of the proof. To prove the converse, let $k_{ij} < 0$ and consider the initial condition $\mathbf{f}(x) = (f_1(x), \ldots, f_m(x))$ with $f_k = 0$ for $k \neq j$ and $f_j \in C^1([0,1])$ with $f_j(0) = f_j(1) = 0$ so that $\mathbf{f} \in D(A)$. Then, at least for $t < \min_{1\leq j\leq m}\{1/c_j\}$, u_i satisfies

$$\partial_t u_i = c_i\partial_x u_i, \quad u_i(x,0) = 0, u_i(1,t) = k_{ij}f_j(c_jt);$$

that is,

$$u_i(x,t) = k_{ij}f_j\left(c_j\frac{x-c_it-1}{c_i}\right), \qquad t \geq \frac{1-x}{c_i}$$

and we see that the solution is not non negative. This ends the proof. $\square$

5

Perturbation methods

Verifying conditions of the Hille–Yosida, or even of the Lumer–Phillips, theorems for a concrete problem is quite often a formidable task. On the other hand, in many cases the operator appearing in the evolution equation at hand is built as a combination of much simpler operators that are relatively easy to analyse. The question now is to what extent the properties of these simpler operators are inherited by the full equation. More precisely, we are interested in the problem:

> **Problem P.** *Let $(A, D(A))$ be a generator of a C_0-semigroup on a Banach space X and $(B, D(B))$ be another operator in X. Under what conditions does $A+B$, or an extension K of $A+B$, generates a C_0-semigroup on X?*

We note that the situation when $K = A + B$ is quite rare. Usually at best we can show that there is an extension of $A + B$ (another realization of $\mathcal{K} = \mathcal{A} + \mathcal{B}$) which is the generator. The reason for this is that, unless B is in some sense strictly subordinated to A, adding B to A may significantly alter some vital properties of A. The identification of K in such cases usually is a formidable task.

1 A Spectral Criterion

Usually the first step in establishing whether $A + B$, or some of its extensions, generates a semigroup is to find if $\lambda I - (A + B)$ (or its extension) is invertible for all sufficiently large λ.

In all cases discussed here we have the generator $(A, D(A))$ of a semigroup and a perturbing operator $(B, D(B))$ with $D(A) \subseteq D(B)$.

We note that B is A-bounded; that is, for some $a, b \geq 0$ we have

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \qquad x \in D(A) \tag{5.1.1}$$

if and only if $BR(\lambda, A) \in \mathcal{L}(X)$ for $\lambda \in \rho(A)$.

In what follows we denote by K an extension of $A + B$. We now present an elegant result relating the invertibility properties of $\lambda I - K$ to the properties of 1 as an element of the spectrum of $BR(\lambda, A)$, first derived in [36].

Theorem 5.1. *Assume that $\Lambda = \rho(A) \cap \rho(K) \neq \emptyset$.*

(a) $1 \notin \sigma_p(BR(\lambda, A))$ for any $\lambda \in \Lambda$;

(b) $1 \in \rho(BR(\lambda, A))$ for some/all $\lambda \in \Lambda$ if and only if $D(K) = D(A)$ and $K = A + B$;

(c) $1 \in \sigma_c(BR(\lambda, A))$ for some/all $\lambda \in \Lambda$ if and only if $D(A) \subsetneq D(K)$ and $K = \overline{A + B}$;

(d) $1 \in \sigma_r(BR(\lambda, A))$ for some/all $\lambda \in \Lambda$ if and only if $K \supsetneq \overline{A + B}$.

Corollary 5.2. *Under the assumptions of Theorem 5.1, $K = A + B$ if one of the following criteria is satisfied: for some $\lambda \in \Lambda$ either*

(i) $BR(\lambda, A)$ is compact (or, if $X = L_1(\Omega, d\mu)$, weakly compact), or

(ii) the spectral radius $r(BR(\lambda, A)) < 1$.

Proof. If (ii) holds, then obviously $I - BR(\lambda, A)$ is invertible by the Neumann series theorem:

$$(I - BR(\lambda, A))^{-1} = \sum_{n=0}^{\infty} (BR(\lambda, A))^n, \tag{5.1.2}$$

giving the thesis by Theorem 5.1 (b). Additionally, we obtain

$$R(\lambda, A + B) = R(\lambda, A)(I - BR(\lambda, A))^{-1} = R(\lambda, A) \sum_{n=0}^{\infty} (BR(\lambda, A))^n. \tag{5.1.3}$$

If (i) holds, then either $BR(\lambda, A)$ is compact or, in L_1 setting, $(BR(\lambda, A))^2$ is compact, [32, p. 510], and therefore, if $I - BR(\lambda, A)$ is not invertible, then 1 must be an eigenvalue, which is impossible by Theorem 5.1(a). □

If we write the resolvent equation

$$(\lambda I - (A + B))x = y, \quad y \in X, \tag{5.1.4}$$

in the (formally) equivalent form

$$x - R(\lambda, A)Bx = R(\lambda, A)y, \tag{5.1.5}$$

then we see that we can hope to recover x provided the Neumann series

$$R(\lambda)y := \sum_{n=0}^{\infty} (R(\lambda, A)B)^n R(\lambda, A)y = \sum_{n=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^n y. \tag{5.1.6}$$

is convergent. Clearly, if (5.1.2) converges, then we can factor out $R(\lambda, A)$ from the series above getting again (5.1.3). However, $R(\lambda, A)$ inside acts as a regularising factor and (5.1.6) converges under weaker assumptions than (5.1.2) and this fact is frequently used to construct the resolvent of an extension of $A + B$ (see e.g. Theorem 5.12 and, in general, results of Section 2).

The most often used perturbation theorem is the Bounded Perturbation Theorem, see e.g. [34, Theorem III.1.3]

Theorem 5.3. *Let $(A, D(A)) \in \mathcal{G}(M, \omega)$ for some $\omega \in \mathbb{R}, M \geq 1$. If $B \in \mathcal{L}(X)$, then $(K, D(K)) = (A + B, D(A)) \in \mathcal{G}(M, \omega + M\|B\|)$.*

In many cases the Bounded Perturbation Theorem gives insufficient information. Then it can be combined with the Trotter product formula, [34, 54]. Assume K_0 is of type $(1, \omega_0)$, $\omega \in \mathbb{R}$, and K_1 is of type $(1, \omega_1)$. If $(K, D(K_0) \cap D(K_1)) := (K_0 + K_2, D(K_0) \cap D(K_1))$ generates a semigroup, then

$$G_K(t)x = \lim_{n \to \infty} \left(G_{K_0}(t/n) G_{K_1}(t/n)\right)^n x, \quad x \in X, \tag{5.1.7}$$

uniformly in t on compact intervals and K is of type $(1, \omega)$ with $\omega = \omega_0 + \omega_1$. Moreover, if both semigroups $\{G_{K_0}(t)\}_{t\geq 0}$ and $\{G_{K_1}(t)\}_{t\geq 0}$ are positive, then $\{G_K(t)\}_{t\geq 0}$ is positive.

The assumption of boundedness of B, however, is often too restrictive. Another frequently used result uses special structure of dissipative operators.

Theorem 5.4. *Let A and B be linear operators in X with $D(A) \subseteq D(B)$ and $A + tB$ is dissipative for all $0 \leq t \leq 1$. If*

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \tag{5.1.8}$$

for all $x \in D(A)$ with $0 \leq a < 1$ and for some $t_0 \in [0, 1]$ the operator $(A + t_0 B, D(A))$ generates a semigroup (of contractions), then $A + tB$ generates a semigroup of contractions for every $t \in [0, 1]$.

Proof. The proof consists in showing, by using Neumann expansion, that if $I-(A+t_0B)$ is invertible, then $I-(A+tB)$ is invertible provided $|t-t_0|<1-a/(2a+b)$. Since the length of the interval on which $I-(A+tB)$ is invertible is independent of the starting point t_0, by using finitely many successive steps, we can cover the whole interval $[0,1]$. Thus $(A+tB, D(A))$ is a dissipative operator such that $I-(A+tB)$ is surjective for all $t\in[0,1]$. It is also densely defined because $D(A)$ is dense and so $(A+tB, D(A))$ generates a semigroup of contractions. □

The fact that $a<1$ in the previous theorem is crucial and a lot of work has been done to change $<$ to $=$. One result, in general setting, is given below. Some others, employing positivity, are discussed further on.

Theorem 5.5. *Let A be the generator of a semigroup of contractions and B, with $D(A)\subset D(B)$, is such that $A+tB$ is dissipative for all $t\in[0,1]$. If*

$$\|Bx\|\leq\|Ax\|+b\|x\|, \tag{5.1.9}$$

for $x\in D(A)$ and B^* is densely defined, then $\overline{A+B}$ is the generator of a contractive semigroup. In particular, if B is closable and X reflexive, then B^* is densely defined.

2 Positive perturbations of positive semigroups

Perturbation results can be significantly strengthened in the framework of positive semigroups. This approach goes back to the work of T. Kato [43]. His results were extended in [56, 16] and recently, in a more abstract setting, in [9, 50]. The presented results are based on the exposition of [16] which is sufficient for our purposes.

We have seen in (5.1.2) that the condition $r(BR(\lambda,A))<1$ implies invertibility of $\lambda I-(A+B)$. It turns out that this condition is equivalent to invertibility for positive perturbations of resolvent positive operators.

Theorem 5.6. *[57] Assume that X is a Banach lattice. Let A be a resolvent positive operator in X and $\lambda>s(A)$. Let $B: D(A)\to X$ be a positive operator. Then the following are equivalent,*

(a) $r(B(\lambda I-A)^{-1})<1$;

(b) $\lambda\in\rho(A+B)$ and $(\lambda I-(A+B))^{-1}\geq 0$.

If either condition is satisfied, then

$$(\lambda I-A-B)^{-1}=(\lambda I-A)^{-1}\sum_{n=0}^{\infty}(B(\lambda I-A-B)^{-1})^n\geq(\lambda I-A)^{-1}. \tag{5.2.10}$$

2.1 Desch theorem

Theorem 5.7. *Let A be the generator of a positive C_0-semigroup in $X=L_1(\Omega)$ and let $B\in\mathcal{L}(D(A),X)$ be a positive operator. If for some $\lambda>s(A)$ the operator $\lambda I-A-B$ is resolvent positive, then $(A+B, D(A))$ generates a positive C_0-semigroup on X.*

2.2 Arendt-Rhandi theorem

Theorem 5.8. *[5] Assume that X is a Banach lattice, $(A, D(A))$ is a resolvent positive operator which generates an analytic semigroup and $(B, D(A))$ is a positive operator. If $(\lambda_0 I-(A+B), D(A))$ has a nonnegative inverse for some λ_0 larger than the spectral bound $s(A)$ of A, then $(A+B, D(A))$ generates a positive analytic semigroup.*

Proof. The proof is an application of Theorem 5.6. Under assumptions of this theorem, we obtain that $r(BR(\lambda_0, A)) < 1$. In particular, the series $\sum\limits_{n=0}^{\infty}(BR(\lambda_0, A))^n$ converges in the uniform operator topology. Next, by Theorem 4.28, $R(\lambda, A) \geq 0$ if and only if $\rho(A) \ni \lambda > s(A)$. Thus, using the resolvent identity we have

$$R(\lambda, A) = R(\lambda_0, A) - (\lambda - \lambda_0)R(\lambda_0, A)R(\lambda, A) \leq R(\lambda_0, A)$$

whenever $\lambda \geq \lambda_0$. Since $BR(\lambda, A)$ is bounded in X, see Theorem 4.8, $B : D(A) \to X$ is bounded in the graph norm of $D(A)$. Let us now take $\lambda \in \mathbb{C}$ with $\Re\lambda \geq \lambda_0$, $\mathbb{R} \ni \mu > \lambda_0$ and $f \in D(A)$. Then $\mu R(\mu, A)R(\lambda, A)f \to R(\lambda, A)f$ as $\mu \to \infty$ in the graph norm of $D(A)$, see e.g., [54, Lemmas 1.3.2 and 1.3.3] and we have, for $f \in D(A)$,

$$\begin{aligned}|BR(\lambda, A)f| &= \lim_{\mu\to\infty} |\mu BR(\mu, A)R(\lambda, A)| \leq \lim_{\mu\to\infty} \mu BR(\mu, A)R(\Re\lambda, A)|f| \\ &= BR(\mu, A)R(\Re\lambda, A)|f|,\end{aligned}$$

where we used $|R(\lambda, A)f| \leq R(\Re\lambda, A)|f|$ for $\Re\lambda > s(A)$, see Theorem 4.28. Thus, by density,

$$|BR(\lambda, A)f| \leq BR(\Re\lambda, A)|f| \tag{5.2.11}$$

for all $f \in X$ and therefore

$$r(BR(\lambda, A)) \leq r(BR(\lambda_0, A)) < 1$$

for any $\lambda \in \mathbb{C}$ with $\Re\lambda \geq \lambda_0$. In particular, $\sum\limits_{n=0}^{\infty}(BR(\lambda, A))^n$ converges to a bounded linear operator with

$$\left\|\sum_{n=0}^{\infty}(BR(\lambda, A))^n f\right\| \leq \left\|\sum_{n=0}^{\infty}(BR(\lambda_0, A))^n |f|\right\| \leq M_{\lambda_0}\|f\|,$$

uniformly for $\lambda \in \mathbb{C}$ with $\Re\lambda > \lambda_0$. This, in particular, shows that $A + B$ generates a C_0-semigroup. Indeed, from (5.2.10) and the above estimate we see that

$$\|R(\lambda, A + B)f\| \leq M_{\lambda_0}\|R(\lambda, A)\|\|f\|$$

for $\lambda > \lambda_0$, so the claim follows since A satisfies the Hille-Yosida estimates (3.1.18) there.

Next we consider the analyticity issue. Using Theorem 3.9, for the operator A, there are ω_A and M_A such that

$$\|R(r + is, A)\| \leq \frac{M_A}{|s|}$$

for $r > \omega_A$. Taking now $\omega > \max\{\lambda_0, \omega_A\}$ we have, by (5.2.10),

$$\begin{aligned}\|R(r + is, A + B)f\| &= \left\|R(\lambda, A)\sum_{n=0}^{\infty}(BR(\lambda, A))^n f\right\| \leq \frac{M_A}{|s|}\left\|\sum_{n=0}^{\infty}(BR(\lambda, A))^n f\right\| \\ &\leq \frac{M_A M_{\lambda_0}}{|s|}\|f\|, \qquad f \in X,\end{aligned}$$

for all $r > \omega$. Therefore $(A + B, D(A))$ generates an analytic semigroup. □

2.3 Kato-Voigt type results

In most perturbation theorems of the previous chapter an essential role was played by a strict inequality in some condition comparing A and B (or $\{G_A(t)\}_{t\geq 0}$ and B). This provided some link between the generator and both operators A and B, and ensured that the semigroup was generated by $A + B$ or, at worst, by $\overline{A + B}$. In many cases of practical importance, however, this inequality becomes a weak inequality or even an equality. We show that in such a case we can still get existence of a semigroup albeit we usually lose control over its generator which can turn to be a larger extension of $A + B$ than $\overline{A + B}$. In such a case the resulting semigroup has properties that are not 'contained' in A and B alone; these are discussed in the next chapter. Here we provide the generation theorem, obtained in [16], which is a generalisation of Kato's result from 1954, [43], as well as some of its consequences.

Theorem 5.9. *Let X be a KB-space. Let us assume that we have two operators $(A, D(A))$ and $(B, D(B))$ satisfying:*

(A1) A *generates a positive semigroup of contractions* $\{G_A(t)\}_{t\geq 0}$,

(A2) $r(BR(\lambda, A)) \leq 1$ *for some* $\lambda > 0 (= s(A))$,

(A3) $Bx \geq 0$ *for* $x \in D(A)_+$,

(A4) $<x^*, (A+B)x> \leq 0$ *for any* $x \in D(A)_+$, *where* $<x^*, x> = \|x\|$, $\quad x^* \geq 0$.

Then there is an extension $(K, D(K))$ of $(A+B, D(A))$ generating a C_0-semigroup of contractions, say, $\{G_K(t)\}_{t\geq 0}$. The generator K satisfies

$$R(\lambda, K)x = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x, \quad \lambda > 0. \tag{5.2.12}$$

Proof. We define operators K_r, $0 \leq r < 1$ by $K_r = A + rB$, $D(K_r) = D(A)$. We see that, as by (A2) the spectral radius of $rBR(\lambda, A)$ does not exceed $r < 1$, the resolvent $(\lambda I - (A + rB))^{-1}$ exists and is given by

$$R(\lambda, K_r) := (\lambda I - (A + rB))^{-1} = R(\lambda, A) \sum_{n=0}^{\infty} r^n \left(BR(\lambda, A)\right)^n, \tag{5.2.13}$$

where the series converges absolutely and each term is positive. Hence,

$$\|R(\lambda, K_r)y\| \leq \lambda^{-1}\|y\| \tag{5.2.14}$$

for all $y \in X$. Therefore, by the Lumer–Phillips theorem, for each $0 \leq r < 1$, $(K_r, D(A))$ generates a contraction semigroup which we denote $\{G_r(t)\}_{t\geq 0}$. The net $(R(\lambda, K_r)x)_{0\leq r<1}$ is increasing as $r \uparrow 1$ for each $x \in X_+$ and $\{\|R(\lambda, K_r)x\|\}_{0\leq r<1}$ is bounded, so by assumption that X is a KB-space, there is an element $y_{\lambda,x} \in X_+$ such that

$$\lim_{r\to 1^-} R(\lambda, K_r)x = y_{\lambda,x}$$

in X. By the Banach–Steinhaus theorem we obtain the existence of a bounded positive operator on X, denoted by $R(\lambda)$, such that $R(\lambda)x = y_{\lambda,x}$. We use the Trotter–Kato theorem to obtain that $R(\lambda)$ is defined for all $\lambda > 0$ and it is the resolvent of a densely defined closed operator K which generates a semigroup of contractions $\{G_K(t)\}_{t\geq 0}$. Moreover, for any $x \in X$,

$$\lim_{r\to 1^-} G_r(t)x = G_K(t)x, \tag{5.2.15}$$

and the limit is uniform in t on bounded intervals and, provided $x \geq 0$, monotone as $r \uparrow 1$. By the monotone convergence theorem, [16, Theorem 2.91],

$$R(\lambda, K)x = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x, \qquad x \in X \tag{5.2.16}$$

and we can prove that $R(\lambda, K)(\lambda I - (A+B))x = x$ which shows that $K \supseteq A + B$. $\square$

The semigroup $\{G_K(t)\}_{t\geq 0}$ obtained in Theorem 5.9 is the smallest in the following sense.

Proposition 5.10. *Let D be a core of A. If $\{G(t)\}_{t\geq 0}$ is another positive semigroup generated by an extension of $(A+B, D)$, then $G(t) \geq G_K(t)$.*

The assumption (A2) of Theorem 5.9 is stronger than the assumption that B is A-bounded, used in Theorem 5.5. Thus, it is worthwhile to compare Theorem 5.9 with Theorems 5.5 and 5.4.

Proposition 5.11. *[16, Proposition 5.5] Let $\{G(t)\}_{t\geq 0}$ be the semigroup generated by $A+B$ or $\overline{A+B}$ under conditions of Theorems 5.4 or 5.5, respectively. If A is a resolvent positive operator and B is positive, then $\{G(t)\}_{t\geq 0}$ is positive.*

Thus, if X is reflexive and B is closable, then Theorem 5.5 is evidently stronger than Theorem 5.9 as the former requires positivity of neither $\{G_A(t)\}_{t\geq 0}$ nor of B. Moreover, in Theorem 5.5, we obtain the full characterisation of the generator as the closure of $A+B$. However, checking the closability of the operator B in particular applications is often difficult, whereas the positivity is often obvious. Also, there is a large class of nonclosable operators which can nevertheless be positive, for example, finite-rank operators (in particular, functionals) are closable if and only if they are bounded. Moreover, Theorem 5.9 gives a constructive formula (5.2.12) for the resolvent of the generator, which seems to be unavailable in general case, and this, in turn, allows other representation results that are discussed below. Also, what is possibly the most important fact, in nonreflexive spaces Theorem 5.9 refers to a substantially different class of phenomena because, as we show in the next chapter, in many cases covered by this theorem the generator does not coincide with the closure of $A+B$. Arguments used in the proof of Theorem 5.9 are very powerful and can be generalized in many ways. Theorem 5.9 is most often used in L_1-setting, where it can be significantly simplified:

Corollary 5.12. *Let $X = L_1(\Omega)$ and suppose that the operators A and B satisfy*

1. *$(A, D(A))$ generates a substochastic semigroup $\{G_A(t)\}_{t\geq 0}$;*
2. *$D(B) \supset D(A)$ and $Bu \geq 0$ for $u \in D(B)_+$;*
3. *for all $u \in D(A)_+$*

$$\int_\Omega (Au + Bu)d\mu \leq 0. \tag{5.2.17}$$

Then the assumptions of Theorem 5.9 are satisfied.

Proof. First, assumption (5.2.17) gives us assumption (A4), that is, dissipativity on the positive cone. Next, let us take $u = R(\lambda, A)x = (\lambda I - A)^{-1}x$ for $x \in X_+$ so that $u \in D(A)_+$. Because $R(\lambda, A)$ is a surjection from X onto $D(A)$, by

$$(A+B)u = (A+B)R(\lambda, A)x = -x + BR(\lambda, A)x + \lambda R(\lambda, A)x,$$

we have

$$-\int_\Omega x\, d\mu + \int_\Omega BR(\lambda, A)x\, d\mu + \lambda \int_\Omega R(\lambda, A)x\, d\mu \leq 0. \tag{5.2.18}$$

Rewriting the above in terms of the norm, we obtain

$$\lambda\|R(\lambda, A)x\| + \|BR(\lambda, A)x\| - \|x\| \leq 0, \qquad x \in X_+, \tag{5.2.19}$$

from which $\|BR(\lambda, A)\| \leq 1$; that is, assumption (A2) is satisfied. □

3 Substochastic semigroups and generator identification

The identification of K is a much more difficult task and is related to both honesty of the semigroup and to the existence of multiple solutions. Let us consider the general equation in $X = L_1(\Omega, d\mu)$.

$$\partial_t u = \mathcal{A}u + \mathcal{B}u, \qquad u(0) = \overset{\circ}{u}, \tag{5.3.20}$$

where $\mathcal{A}$ and $\mathcal{B}$ are some differential, integral or other expressions acting of functions from, respectively, $D(\mathcal{A}), D(\mathcal{B}) \subset X$. Typically we assume that $D(\mathcal{B}) \supset D(\mathcal{A})$. In our applications, $\mathcal{A}$ describes loss of particles while $\mathcal{B}$ describes their gain and thus $\mathcal{A}$ and $\mathcal{B}$ have 'opposite' signs. We are looking for a realization K of $\mathcal{A}+\mathcal{B}$ which generates a semigroup.

First we note that the pair of expressions $\mathcal{A}$ and $\mathcal{B}$ in a natural way defines two operators: the *minimal operator* $K_{\min} = A + B$, where $A = \mathcal{A}|_{D(A)}$, $B = \mathcal{B}|_{D(A)}$ and

$$D(K_{\min}) = D(A) = \{u \in X;\ \mathcal{A}u \in X\},$$

and the *maximal operator* $K_{\max} = \mathcal{A} + \mathcal{B}$ defined on

$$D(K_{\max}) = \{u \in X;\ \mathcal{A}u + \mathcal{B}u \in X\},$$

where the evaluation of $\mathcal{A}$ and $\mathcal{B}$ is understood in the sense of (2.1.1). The difference between these two operators is that in the minimal operator both summands $\mathcal{A}u$ and $\mathcal{B}u$ must belong to X but in the maximal operator the summands are supposed to be defined just a.e. on Ω and only their sum is required to belong to X. In general, $D(K_{\min}) = D(A) \subsetneq D(K_{\max})$ as there may be cancellation of singularities in $\mathcal{A}u$ and $\mathcal{B}u$ due to their opposite signs.

3.1 Why $D(K)$ is related to honesty

In the discussed context it can be proved that the generator K of the semigroup associated with the problem (5.3.20) satisfies $K_{\min} \subset K \subset K_{\max}$. Where K is situated on this scale determines the well-posedness of the problem. To explain why honesty of $\{G_K(t)\}_{t\geq 0}$ should be determined by this, let us suppose for simplicity that the model is formally conservative; that is, for sufficiently regular u, say $u \in D(A)$,

$$\int\limits_{\Omega} (\mathcal{A} + \mathcal{B})ud\mu = \int\limits_{\Omega} \mathcal{A}u + \int\limits_{\Omega} \mathcal{B}ud\mu 0. \tag{5.3.21}$$

If A generates a substochastic semigroup and B is positive, then by Corollary 5.12, there is an extension K of $A + B$ generating a semigroup of contractions, say $\{G_K(t)\}_{t\geq 0}$.

Assume now that the semigroup $\{G_K(t)\}_{t\geq 0}$ is generated by $K = K_{\min} = A + B$ on $D(K) = D(A)$. Then the solution $u(t) = G_K(t)u_0$, emanating from $u_0 \in D(K)_+$, satisfies $u(t) \in D(A)_+$ and, therefore, because

$$\frac{d}{dt}u(t) = Ku(t) = Au(t) + Bu(t),$$

we obtain that for any $t \geq 0$

$$\frac{d}{dt}\|u(t)\| = \int\limits_{\Omega} \frac{du(t)}{dt}d\mu = \int\limits_{\Omega} (Au(t) + Bu(t))d\mu = 0, \tag{5.3.22}$$

so that $\|u(t)\| = \|u_0\|$ for any $t \geq 0$ and the solutions are indeed conservative. If $K = \overline{A + B}$, then the same result can be obtained by limit argument so that the solutions are conservative as well. That $K = \overline{A + B}$ is also the necessary condition is not that clear but can be proved, see Theorem 5.16.

3.2 Substochastic semigroups

To make the above considerations precise and general, let us recall that a semigroup $\{G(t)\}_{t\geq 0}$ is said to be a *substochastic semigroup* if for any $t \geq 0$ and $f \geq 0$, $G(t)f \geq 0$ and $\|G(t)f\| \leq \|f\|$, and a *stochastic semigroup* if additionally $\|G(t)f\| = \|f\|$ for $f \in X_+$. We consider linear operators in $X = L_1(\Omega, d\mu)$: A and B, that satisfy the assumptions of Corollary 5.12: $(A, D(A))$ generates a substochastic semigroup $\{G_A(t)\}_{t\geq 0}$, $D(B) \supset D(A)$ and $Bu \geq 0$ for $u \in D(B)_+$ and for all $u \in D(A)_+$,

$$\int\limits_{\Omega} (Au + Bu)\, d\mu = -c(u) \leq 0, \tag{5.3.23}$$

where c is a positive functional on a domain $D \supset D(A)$. We assume that c has a monotone convergence property: $c(u_n) \to c_u$ for $u_n \uparrow u$, then $u \in D$ and $c_u = c(u)$. For instance, c may be an integral functional of the form

$$c(f) = \int\limits_{\Omega} f(x)\, d\mu'_x, \tag{5.3.24}$$

where μ' is a measure which can be even concentrated on a subsets of Ω of lower dimension. We note that a more general version of this assumption is considered in [9, 50].

Under these assumptions, Corollary 5.12, Theorem 5.9, and other results of the previous chapter, give the existence of a smallest substochastic semigroup $\{G_K(t)\}_{t\geq 0}$ generated by an extension K of the operator $A+B$. This semigroup, for arbitrary $f \in D(K)$ and $t > 0$, satisfies

$$\frac{d}{dt} G_K(t)f = KG_K(t)f. \tag{5.3.25}$$

It is important to distinguish the class of semigroups corresponding to $c \neq 0$, as such semigroups cannot be stochastic but their substochasticity is built into the model and not caused by the dishonesty of it.

Definition 5.13. *A positive semigroup $\{G_K(t)\}_{t\geq 0}$ generated by an extension K of the operator $A+B$ is said to be strictly substochastic if (5.3.23) holds with $c \neq 0$. The semigroup $\{G_K(t)\}_{t\geq 0}$ is honest if c extends to $D(K)$ and for any $0 \leq f \in D(K)$ the solution $u(t) = G_K(t)f$ of (5.3.25) satisfies*

$$\frac{d}{dt} \int\limits_{\Omega} u(t)\, d\mu = \frac{d}{dt} \|u(t)\| = -c\,(u(t))\,. \tag{5.3.26}$$

It can be proved that (5.3.26) is equivalent to its 'integrated' version: $\{G_K(t)\}_{t\geq 0}$ is honest if and only if for any $f \in X_+$ and $t \geq 0$,

$$\|G_K(t)f\| = \|f\| - c\left(\int\limits_0^t G_K(s) f ds\right). \tag{5.3.27}$$

This result allows for the introduction of the defect function

$$\eta_f(t) = \|G_K(t)f\| - \|f\| + \int\limits_0^t c\,(G_K(s)f)\, ds \tag{5.3.28}$$

for $f \in X_+$ and $t \geq 0$. It follows that η_f is a nonpositive and nonincreasing function for $t \geq 0$. Arguing as in (5.2.19) we obtain that condition (5.3.23) is equivalent to

$$-c(L_\lambda f) = \lambda \|R(\lambda, A)f\| + \|BR(\lambda, A)f\| - \|f\|, \qquad f \in X_+. \tag{5.3.29}$$

The following theorem is fundamental for characterizing the generator of the semigroup.

Theorem 5.14. *For any fixed $\lambda > 0$, there is $0 \leq \beta_\lambda \in X^*$ with $\|\beta_\lambda\| \leq 1$ such that for any $f \in X_+$,*

$$\lambda \|R(\lambda, K)f\| = \|f\| - <\beta_\lambda, f> - c\,(R(\lambda, K)f) \tag{5.3.30}$$

and c extends to a nonnegative continuous linear functional on $D(K)$.

Proof. Let us fix $f \in X_+$. From (5.2.12) and nonnegativity we obtain

$$\lambda \|(\lambda I - K)^{-1} f\| = \lim_{N\to\infty} \sum_{n=0}^{N} \lambda \|L_\lambda (BL_\lambda)^n f\|.$$

By (5.3.29) we get

$$\sum_{n=0}^{N}\lambda\|L_\lambda(BL_\lambda)^n f\| = \|f\| - \|(BL_\lambda)^{N+1}f\| - c\left(\sum_{n=0}^{N}L_\lambda(BL_\lambda)^n f\right).$$

By non-negativity, the monotone convergence theorem gives

$$\lim_{N\to\infty} c\left(\sum_{n=0}^{N}L_\lambda(BL_\lambda)^n f\right) = c(R(\lambda,K)f) < +\infty.$$

This shows that c extends to a finite functional on $D(K)$, which is continuous in the graph topology. Similarly, $\|(BL_\lambda)^{N+1}f\|$ converges to some $\beta_\lambda(f) \geq 0$ and, by a similar argument, β_λ extends to a continuous linear functional on X with the norm not exceeding 1. □

By taking the Laplace transform of η_f, we obtain

$$<\beta_\lambda, f> = -\lambda\int_0^\infty e^{-\lambda t}\eta_f(t)dt$$

for $f \in X_+$ and hence the following result is true

Theorem 5.15. *$\{G_K(t)\}_{t\geq 0}$ is honest if and only if $\beta_\lambda \equiv 0$ for $\lambda > 0$.*

A central result on the characterization of honesty is:

Theorem 5.16. *[8] The semigroup $\{G_K(t)\}_{t\geq 0}$ is honest if and only if one of the following holds:*
(a) $K = \overline{A+B}$.
(b) $\int_\Omega Ku\,d\mu \geq -c(u), \quad u \in D(K)_+$.

Proof. (a) implies honesty as in (5.3.22) and considerations below it – properties of the functional c allow for passing to the limit. Conversely, if $\{G_K(t)\}_{t\geq 0}$ is honest, then $\beta_\lambda \equiv 0$ for any $\lambda > 0$, which means, by the proof of Theorem 5.8, that $\lim_{n\to\infty}(BL_\lambda)^n f = 0$. Hence the series in (5.2.12) converges to $R(\lambda, \overline{T+B})$.

If $\{G_K(t)\}_{t\geq 0}$ is honest, then the part (a) gives (b) with the equality sign. Conversely, for $u = R(\lambda,K)f$, $f \in X_+$, we have

$$\int_\Omega Ku\,d\mu = -\|f\| + \lambda\|R(\lambda,K)f\| = -c(u) - <\beta_\lambda, f>,$$

which implies $<\beta_\lambda, f> \leq 0$ for all $f \in X_+$, thus $\beta_\lambda = 0$. □

Unfortunately, typically we do not know K and thus condition (b) has a limited practical value. There are two important theorems providing conditions for honesty and dishonesty in terms of known operators. The first is based on Theorem 5.1 which, combined with Theorem 5.16, shows that $\{G_K(t)\}_{t\geq 0}$ is honest if and only if $1 \notin \sigma_p((BR(\lambda,A)^*)$. In particular, using the definition of β_λ, we see that

$$<\beta_\lambda, BR(\lambda,T)f> = \lim_{n\to\infty}\|(BR(\lambda,T))^{n+1}f\| = <\beta_\lambda, f>,$$

$f \in X_+$, so that

$$(BR(\lambda,T))^*\beta_\lambda = \beta_\lambda. \tag{5.3.31}$$

The other set of results is based on the fact that we know at least one extension of the generator K, namely K_{max}. Let $\mathcal{K}$ be any extension of K.

Theorem 5.17. *[8] (a) If $\int_\Omega \mathcal{K}u\,d\mu \geq -c(u)$ for all $u \in D(\mathcal{K})_+$, then $\{G_K(t)\}_{t\geq 0}$ is honest.*
(b) $\{G_K(t)\}_{t\geq 0}$ is not honest if there is $u \in D(\mathcal{K})_+ \cap X$ and $\lambda > 0$
(i) $\lambda u(x) - [\mathcal{K}u](x) = g(x) \geq 0, \quad$ a.e.,
(ii) $c(u)$ is finite and

$$\int_\Omega \mathcal{K}u\,d\mu < -c(u). \tag{5.3.32}$$

Proof. The statement (a) is obvious from Theorem 5.16(b) as $\mathcal{K}$ contains K. In practice, however, we are interested to use the smallest possible extension since taking a too large one could spoil the inequality. Similarly, (b) uses Theorem 5.16(b) but here the function $u \in D(\mathcal{K})$, satisfying (5.3.32) may fail to belong to $D(K)$; the other two conditions allow one to prove that there is an element of $D(K)$ satisfying (5.3.32), thus proving dishonesty of $\{G_K(t)\}_{t\geq 0}$. □

3.3 Extension techniques

For further reference we briefly sketch a particularly effective extension technique, originally introduced in [7]. We embed $X = L_1(\Omega, d\mu)$ in the set of μ-measurable functions that are defined on Ω and take values in the extended set of real numbers, denoted by E; by E_f we denote the subspace of E consisting of functions that are finite a. e. E is a lattice with respect to the usual relation: '$\leq$ a. e', $X \subset \mathsf{E}_f \subset \mathsf{E}$ with X and E_f being sublattices of E.

Let $\mathsf{F} \subset \mathsf{E}$ be defined by the condition: $f \in \mathsf{F}$ if and only if for any nonnegative and nondecreasing sequence $(f_n)_{n\in\mathbb{N}}$ satisfying $\sup_{n\in\mathbb{N}} f_n = |f|$ we have $\sup_{n\in\mathbb{N}}(I-A)^{-1}f_n \in X$. We define mapping $\mathsf{L} : \mathsf{F}_+ \to X_+$ by

$$\mathsf{L}f := \sup_{n\in\mathbb{N}} R(1,A)f_n, \qquad f \in \mathsf{F}_+,$$

where $0 \leq f_n \leq f_{n+1}$ for any $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} f_n = f$ and extend it to a positive linear operator on the whole F, Theorem 4.7.

In most applications $(I-A)^{-1}$ is an integral (or even multiplication) operator with a positive kernel so that, by monotone convergence theorem, F is the set of measurable functions for which the integral exists and is a function in $L_1(\Omega)$. In the same way we define B on $D(\mathsf{B})$. It turns out that L is one-to-one therefore we can define the operator T with $D(\mathsf{A}) = \mathsf{LF} \subset X$ by

$$\mathsf{A}u = u - \mathsf{L}^{-1}u, \tag{5.3.33}$$

so that A is an extension of A. The central theorem of this paragraph reads:

Theorem 5.18. *If $(A, D(A))$ and $(B, D(B))$ are operators in X such that $(A, D(A))$ generates a substochastic semigroup $\{G_A(t)\}_{t\geq 0}$ on X, $D(B) \supset D(A)$, $Bu \geq 0$ for $u \in D(B)_+$, and*

$$\int_\Omega (Au + Bu)\, d\mu \leq 0, \tag{5.3.34}$$

for all $u \in D(A)_+$, then the extension K of $A+B$, that generates the smallest substochastic semigroup on X described by Corollary 5.12, is given by

$$\begin{aligned} Ku &= \mathsf{A}u + \mathsf{B}u, \\ D(K) &= \{u \in D(\mathsf{A}) \cap D(\mathsf{B}) : \mathsf{A}u + \mathsf{B}u \in X, \text{ and } \lim_{n\to+\infty} ||(\mathsf{LB})^n u|| = 0\}. \end{aligned} \tag{5.3.35}$$

This notion allows to give a more focused version of Theorem 5.17(a).

Theorem 5.19. *If for any $g \in \mathsf{F}_+$ such that $-g + \mathsf{BL}g \in X$ and $c(\mathsf{L}g)$ exists,*

$$\int_\Omega \mathsf{L}g\, d\mu + \int_\Omega (-g + \mathsf{BL}g)\, d\mu \geq -c\,(\mathsf{L}g), \tag{5.3.36}$$

then $K = \overline{A+B}$.

Remark 5.20. It makes sense to consider 'pointwise in space' honesty and say that $\{G_K(t)\}_{t\geq 0}$ is honest along the trajectory $\{G_K(t)f\}_{t\geq 0}$ if (5.3.27) holds for this particular f and for all $t \geq 0$. Accordingly, such a trajectory is called an honest trajectory. Thus $\{G_K(t)\}_{t\geq 0}$ is honest if and only if each trajectory $\{G_K(t)f\}_{t\geq 0}$ is honest. Moreover, honesty can also be considered to be a 'pointwise in time' phenomenon. Indeed, if $u(t_0) \in D(\overline{T+B})$ for some $t_0 > 0$ then, as by (5.3.22) and limiting argument

$$\frac{d}{dt}\|u\|\bigg|_{t=t_0} = -c(u(t_0)),$$

and therefore we can say that the trajectory $\{G_K(t)f\}_{t\geq 0}$ is honest over a time interval I if and only if $G_K(t)f \in D(\overline{T+B})$ for $t \in I$.

In general, our theory cannot determine, in general, whether a given system $\{G_K(t)\}_{t\geq 0}$ can be dishonest along some trajectories and honest along the others. Using specific properties of birth-and-death and fragmentation models, however, we can show that, in these models, if dishonesty occurs along one trajectory, it must occur along any other; see Theorem 5.24. Recently, [50], such a universality of dishonesty was related to irreducibility of $BR(\lambda, A)$.

Unfortunately, much less can be said about how dishonest trajectories behave in time. One of the reasons for this is that our theory is based on the Laplace transform which gives, in some sense, time averages of solutions which provide little information about the properties which are local in time.

4 Summary of what can go wrong between building a model and their analysis

We have seen that the generator K satisfies $K_{\min} \subset K \subset K_{\max}$. The following situations are possible

1. $K_{\min} = K = K_{\max}$,
2. $K_{\min} \subsetneq K = \overline{K_{\min}} = K_{\max}$,
3. $K_{\min} = K \subsetneq K_{\max}$,
4. $K_{\min} \subsetneq K = \overline{K_{\min}} \subsetneq K_{\max}$,
5. $\overline{K_{\min}} \subsetneq K \subsetneq K_{\max}$,

and each of them has its own specific interpretation in the model.

In all cases where $K \subsetneq K_{\max}$ we don't have uniqueness; that is, there are differentiable X-valued solutions to (5.3.20) emanating from zero and therefore they are not described by the constructed dynamical system: 'there is more to life, than meets the semigroup' [14]. To achieve uniqueness here, one has to impose additional constraints on the solution.

If $\overline{K_{\min}} \subsetneq K$, then despite the fact that the model is formally conservative, (5.3.21), the solutions are not; the described quantity leaks out from the system and the mechanism of this leakage is not present in the model. In Markov processes such a case is called dishonesty of the transition function, [**?**].

Finally, the condition $u(t) \in D(A)$ for any t ensures that only a finite number of state changes can happen in the system in any finite time interval.

Therefore, strictly speaking, only problems with $K = K_{\min} = K_{\max}$ are physically realistic. However, in many applications, the case $K = \overline{K_{\min}} = K_{\max}$ is considered to be acceptable.

5 Applications to birth-and-death type problems

Recall that we consider with the system

$$\begin{aligned} u_0' &= -a_0u_0 + d_1u_1, \\ &\vdots \\ u_n' &= -a_nu_n + d_{n+1}u_{n+1} + b_{n-1}u_{n-1}, \quad n \geq 1, \\ &\vdots\ , \end{aligned} \tag{5.5.37}$$

we included a mechanism of production or annihilation with the rate $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ and $c_n = b_n + d_n - a_n$.

5.1 Existence Results

In what follows the boldface letters denote sequences, e.g. $\mathbf{u} = (u_0, u_1, \ldots)$. We assume that the sequences $\mathbf{d}$, $\mathbf{b}$, and $\mathbf{a}$ are nonnegative with $b_{-1} = d_0 = 0$.

By $\mathbb{K}$ we denote the matrix of coefficients of the right-hand side of (5.5.37) and, without causing any misunderstanding, the formal operator in the space l of all sequences, acting as

$$(\mathbb{K}\mathbf{u})_n = b_{n-1}u_{n-1} - a_nu_n + d_{n+1}u_{n+1}.$$

In the same way, we define $\mathbb{A}$ and $\mathbb{B}$ as $(\mathbb{A}\mathbf{u})_n = -a_nu_n$ and $(\mathbb{B}\mathbf{u})_n = b_{n-1}u_{n-1} + d_{n+1}u_{n+1}$, respectively.

By $\mathcal{K}$ we denote the maximal realization of $\mathcal{K}$ in l_1; that is,

$$\begin{aligned} \mathcal{K}\mathbf{u} &= \mathcal{K}\mathbf{u}, \\ D(\mathcal{K}) &= \{\mathbf{u} \in l_1;\ \mathcal{K}\mathbf{u} \in l_1\}. \end{aligned} \tag{5.5.38}$$

It is easy to check that the maximal operator $\mathcal{K}$ is closed. Next, define the operator A by restricting $\mathbb{A}$ to

$$D(A) = \{\mathbf{u} \in l_1;\ \mathbb{A}\mathbf{u} \in l_1\} = \{\mathbf{u} \in l_1;\ \sum_{n=0}^{\infty} a_n|u_n| < +\infty\}.$$

Again, it is standard that $(A, D(A))$ generates a semigroup of contractions in l_1. The situation in l_1 is completely different.

Corollary 5.21. *Assume that sequences* $\mathbf{b}$ *and* $\mathbf{d}$ *are nonnegative and*

$$a_n \geq (b_n + d_n). \tag{5.5.39}$$

Then there is an extension K of the operator $(A + B, D(A))$, where $B = \mathbb{B}|_{D(A)}$, which generates a positive semigroup of contractions in l_1.

Proof. Using the definition of $D(A)$ we see, from (5.5.39), that $0 \leq b_n \leq a_n$ and $0 \leq d_n \leq a_n$ for $n \in \mathbb{N}$. Hence, A is well defined and (5.3.23) takes the form

$$\sum_{n=0}^{\infty}((A + B)\mathbf{u})_n = -\sum_{n=0}^{\infty} a_nu_n + \sum_{n=0}^{\infty} b_nu_n + \sum_{n=0}^{\infty} d_nu_n \leq 0,$$

where we used the convention $b_{-1} = d_0 = 0$. □

As an immediate consequence of Theorem 5.18 we get

Theorem 5.22.

$$K \subset \mathcal{K}.$$

In the following two subsections we assume that the system (5.5.37) is formally conservative: $a_n = b_n + c_n$, $n \geq 0$.

5.2 Birth-and-death problem – honesty results

We now find whether the constructed semigroup is honest (conservative) or dishonest by means of the extension techniques of Subsection 3.3. In this case $\mathsf{E}_f = m$ (the set of all bounded sequences) and

$$\mathsf{L}\mathbf{u} = \left(\frac{u_n}{1+b_n+d_n}\right)_{n\in\mathbb{N}}$$

on $\mathsf{F} = \{\mathbf{u} \in m;\ \mathsf{L}\mathbf{u} \in l_1\}$, $\mathsf{A}\mathbf{u} = ((b_n+d_n)u_n)_{n\in\mathbb{N}}$ on $D(\mathsf{A}) = \mathsf{LF}$, and similarly for the other operators and spaces introduced in Subsection 3.3.

Recall that by $\mathbb{K}$ we denoted the matrix of coefficients and, at the same time, the formal operator acting on m given by multiplication by $\mathcal{K}$. It is easy to see that the maximal operator $\mathcal{K}$ (see (5.5.38)) is precisely

$$\mathcal{K} = \mathsf{K} = \mathsf{A} + \mathsf{B}. \tag{5.5.40}$$

Note too that here, for $\mathbf{u} \in D(\mathsf{K})$, the integral $\int_\Omega \mathsf{K}\mathbf{u}d\mu$ which plays an essential role in a number of theorems (e.g., Theorems 5.16 and 5.17), is given by

$$\sum_{n=0}^{\infty}(-(b_n+d_n)u_n + b_{n-1}u_{n-1} + d_{n+1}u_{n+1}) = \lim_{n\to+\infty}(-b_nu_n + d_{n+1}u_{n+1}), \tag{5.5.41}$$

where the limit exists as $\mathbf{u} \in D(\mathsf{K})$ yields the convergence of the series.

In the theorems concerning honesty and maximality we assume, to avoid technicalities, that $b_n > 0$ for $n \geq 0$ and $d_n > 0$ for $n \geq 1$.

Theorem 5.23. $K = \overline{A+B}$ *if and only if*

$$\sum_{n=0}^{\infty}\frac{1}{b_n}\left(\sum_{i=0}^{\infty}\prod_{j=1}^{i}\frac{d_{n+j}}{b_{n+j}}\right) = +\infty. \tag{5.5.42}$$

Proof. To prove honesty, we use Theorem 5.19. Thus, by (5.5.41) it suffices to prove that for any $\mathbf{u} \in D(\mathsf{K})_+$

$$\lim_{n\to+\infty}(-b_nu_n + d_{n+1}u_{n+1}) \geq 0,$$

where we know that the sequence above converges. If we assume the contrary, that for some $0 \leq \mathbf{u} \in D(\mathsf{K})$ the limit in (5.5.41) is negative, then there exists $b > 0$ such that

$$-b_nu_n + d_{n+1}u_{n+1} \leq -b \tag{5.5.43}$$

for all $n \geq n_0$ with large enough n_0. Using (5.5.43) as a recurrence we get

$$u_n \geq \frac{b}{b_n}\left(\sum_{i=0}^{\infty}\prod_{j=1}^{i}\frac{d_{n+j}}{b_{n+j}}\right)$$

and, if the assumption (5.5.42) is satisfied, we obtain $\sum_{n=0}^{\infty}u_n = +\infty$ which contradicts the assumption of the summability of $(u_n)_{n\in\mathbb{N}}$.

The proof of necessity is an application of Theorem 5.17. We define

$$u_n = \frac{b}{b_0}\prod_{i=0}^{n-1}\frac{b_i}{d_{i+1}}\left(\sum_{l=n}^{\infty}\prod_{i=1}^{l}\frac{d_i}{b_i}\right)$$

and show that $(u_n)_{n\in\mathbb{N}}$ is summable by (5.5.42) and satisfies

$$-b = -b_n u_n + d_{n+1}u_{n+1}, \quad n \geq 0$$

so that assumption (ii) of Theorem 5.17 is satisfied. By construction, $\mathcal{A}\mathbf{u} + \mathcal{B}\mathbf{u} \in l_1$, so that $\mathbf{u} \in D(\mathsf{K})$. We must show that $\mathbf{g} = \mathbf{u} - (\mathcal{A}\mathbf{u} + \mathcal{B}\mathbf{u}) \geq 0$. By direct calculations, we obtain $g_0 = u_0 + b_0 u_0 - d_1 u_1 = u_0 + b$ and for $n > 0$,

$$g_n = u_n + b_n u_n + d_n u_n - b_{n-1}u_{n-1} - d_{n+1}u_{n+1} = u_n,$$

so that $0 \leq \mathbf{g} \in l_1$. Hence assumption (i) of Theorem 5.17 is satisfied. □

5.3 Universality of Dishonesty

Theorem 5.24. *If $\{G_K(t)\}_{t\geq 0}$ is dishonest, then for each $\mathbf{u}_0 \in X_+$ there is $t_0 \geq 0$ such that $\|G_K(t)\mathbf{u}_0\| < \|\mathbf{u}_0\|$ for all $t > t_0$.*

Proof. By Theorem 5.15, $\{G_K(t)\}_{t\geq 0}$ is dishonest if and only if the functional β_λ, defined in Theorem 5.8, is not identically zero. The defect function along the trajectory originating at $\mathbf{u}_0$, which in our case is given by $\eta_{\mathbf{u}_0}(t) = \|G_K(t)\mathbf{u}_0\| - \|\mathbf{u}_0\|$, is related to β_λ by

$$\int\limits_0^\infty e^{-\lambda t}\eta_{\mathbf{u}_0}(t)dt = -\frac{1}{\lambda} <\beta_\lambda, \mathbf{u}_0> .$$

Clearly, λ is inessential. Putting $\beta_\lambda = \beta = (\beta_n)_{n\in\mathbb{N}}$ with $\beta_n \geq 0$, we see that for universality of dishonesty we must have $\beta_n > 0$ for any $n \geq 0$. On the other hand, by (5.3.31), β_λ is an eigenvector of $(BR(\lambda, A))^*$. Any eigenvector $(\phi_n)_{n\in\mathbb{N}}$ satisfies

$$\frac{b_0}{1+b_0}\phi_1 = \phi_0, \quad \ldots \quad \frac{d_n}{1+b_n+d_n}\phi_{n-1} + \frac{b_n}{1+b_n+d_n}\phi_{n+1} = \phi_n, \ldots,$$

and, because $b_0/(1+b_0) < 1$, we have $\phi_1 > \phi_0$. Rearranging the terms in the nth equation,

$$\phi_{n+1} = \left(1 + \frac{1}{b_n}\right)\phi_n + \frac{d_n}{b_n}(\phi_n - \phi_{n-1}).$$

Hence $\phi_{n+1} > \phi_n$ whenever $\phi_n \geq \phi_{n-1}$ we end the proof by induction. □

5.4 Maximality of the Generator

Let us recall that the relation between the generator K and its extensions K and $\mathcal{K}$ is given in (5.5.40). In particular, K is the maximal operator.

Lemma 5.25. *If $\{G_K(t)\}_{t\geq 0}$ is a substochastic semigroup generated by K and for some $0 \leq \mathbf{h} \in D(\mathsf{K})$,*

$$\int\limits_\Omega \mathsf{K}\mathbf{h}d\mu > 0, \tag{5.5.44}$$

then $K \neq \mathsf{K}$; that is, the generator is not maximal.

Conversely, assume that K has the property that any if $0 \neq \mathbf{u} \in l$ solves the formal solution to $\mathsf{K}\mathbf{u} = \lambda\mathbf{u}, \lambda > 0$, satisfies $\mathbf{u} \geq 0$ or $\mathbf{u} \leq 0$. If

$$\int\limits_\Omega \mathsf{K}\mathbf{h}d\mu = 0, \tag{5.5.45}$$

for any $0 \leq \mathbf{h} \in D(\mathsf{K})$, then $\mathsf{K} = K$; that is, the generator is the maximal operator.

Proof. It follows that if $\mathbf{h} \in D(K)$, then $\int_\Omega K\mathbf{h}d\mu = 0$. Because $K \subset \mathsf{K}$, (5.5.44) shows that $\mathbf{h} \notin D(K)$.

If $\mathsf{K} \neq K$ then, by Corollary 3.13, we have $N(\lambda I - \mathsf{K})_+ \neq \emptyset$. Because the problem is linear, then the assumption ascertains the existence of $0 \neq \mathbf{h} \in N(\lambda I - \mathsf{K})_+$ and for such an $\mathbf{h}$

$$\int_\Omega \mathsf{K}\mathbf{h}d\mu = \lambda \int_\Omega \mathbf{h}d\mu \neq 0, \tag{5.5.46}$$

contradicting (5.5.45). □

To be able to use this result, we have the following lemma.

Lemma 5.26. *Let $\lambda > 0$ be fixed. Any solution to*

$$\begin{aligned} \lambda u_0 &= -a_0 u_0 + d_1 u_1, \\ &\vdots \\ \lambda u_n &= -a_n u_n + b_{n-1}u_{n-1} + d_{n+1}u_{n+1},\ n \geq 1, \\ &\vdots \end{aligned} \tag{5.5.47}$$

is either nonnegative or nonpositive.

On the basis of the above lemma we obtain:

Theorem 5.27. $K \neq \mathsf{K}$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{d_n} \prod_{j=1}^{n-1} \frac{b_j}{d_j} \left(\sum_{i=0}^{n-1} \prod_{k=1}^{i} \frac{d_k}{b_k} \right) < +\infty. \tag{5.5.48}$$

Proof. By Lemma 5.26 and Proposition 5.25, $K \neq \mathsf{K}$ if and only if for each $0 \leq (u_n)_{n\in\mathbb{N}} \in l_1$, such that $(-(b_n + d_n)u_n + b_{n-1}u_{n-1} + d_{n+1}u_{n+1\,n})_{n\in\mathbb{N}} \in l_1$, we have

$$I = \sum_{n=0}^{\infty} (-(b_n + d_n)u_n + b_{n-1}u_{n-1} + d_{n+1}u_{n+1}) > 0.$$

and, similarly to the proof of Theorem 5.23 and (5.5.43), we need to investigate the behaviour of the sequence $(r_n)_{n\in\mathbb{N}}$ defined as

$$r_n = -b_n u_n + d_{n+1}u_{n+1}, \quad n \geq 0, \tag{5.5.49}$$

or, solving for u_n, for $n \geq 1$,

$$u_n = \frac{1}{d_n} \sum_{i=0}^{n-1} \left(r_i \prod_{j=1}^{n-1-i} \frac{b_{n-j}}{d_{n-j}} \right) + \frac{u_0 b_0}{d_n} \prod_{j=1}^{n-1} \frac{b_j}{d_j}. \tag{5.5.50}$$

If $K \neq \mathsf{K}$, then there is a nonnegative $(u_n)_{n\in\mathbb{N}} \in l_1$ for which $I = \lim_{n\to\infty} r_n > 0$ and, by some algebra, it is enough to consider a nonnegative sequence $(u_n)_{n\in\mathbb{N}} \in D(\mathsf{K})$ with the associated sequence $(r_n)_{n\in\mathbb{N}}$ satisfying $\inf_{n\in\mathbb{N}} r_n = r > 0$. Then it can be proved that the series in (5.5.48) is convergent.

To prove the converse, define u_n by (5.5.49) with arbitrary $(r_n)_{n\in\mathbb{N}}$ converging to $I > 0$ (e.g., we may take $r_n = r$ for all n for a constant positive r). By (5.5.48) $(u_n)_{n\in\mathbb{N}} \in l_1$, so that $(u_n)_{n\in\mathbb{N}} \in D(\mathsf{K})$ and because $I > 0$, the thesis follows by (5.5.44). □

5.5 Examples

We provide a few examples showing that all possible cases of relations between the generator and maximal and minimal operators can be realized.

Proposition 5.28. *If both sequences $(b_n^{-1})_{n\in\mathbb{N}}, (d_n^{-1})_{n\in\mathbb{N}} \notin l_1$, then $K = \overline{A+B} = \mathsf{K}$. In particular, this is true for the standard birth-and-death problem of population theory where the coefficients are affine functions of n.*

Proof. Expanding (5.5.48) we get, for a fixed n,

$$\frac{1}{d_n}\left(1 + \frac{b_{n-1}}{d_{n-1}} + \cdots + \frac{b_{n-1}\ldots b_1}{d_{n-1}\ldots d_1}\right) \geq \frac{1}{d_n}.$$

Similarly, expanding (5.5.42), we get

$$\frac{1}{b_n}\left(1 + \frac{d_{n+1}}{b_{n+1}} \cdots + \right) \geq \frac{1}{b_n}$$

which gives divergence of both series. □

The proofs of the following results are obtained in a similar way.

Proposition 5.29. *a) If $(d_n^{-1})_{n\in\mathbb{N}} \in l_1$ and $\lim_{n\to\infty} b_n/d_n = q < 1$, then $K = \overline{A+B} \neq \mathsf{K}$.*

b) If the sequence $(d_n)_{n\in\mathbb{N}}$ is of polynomial growth: $d_n = O(n^\beta)$ for some β as $n \to \infty$, $(b_n^{-1})_{n\in\mathbb{N}} \in l_1$ and $\lim_{n\to\infty} b_n/d_n = q > 1$, then $\overline{A+B} \subsetneq K = \mathsf{K}$.

c) There are sequences $(b_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ for which $\overline{A+B} \subsetneq K \subsetneq \mathsf{K}$.

To prove c) it suffices to take $b_n = 2 \cdot 3^n$ and $d_n = 3^n$.

6 Fragmentation equation

Recall that we deal with the equation

$$\partial_t u(x,t) = -a(x)u(x,t) + \int\limits_x^\infty a(y)b(x|y)u(y,t)dy \tag{5.6.51}$$

where $x \in \mathbb{R}_+ := (0,\infty)$ is the size of the particles/clusters. Here u is the density of particles of mass/size x, a is the fragmentation rate and b describes the distribution of masses x of particles spawned by a particle of mass y.

The fragmentation rate a is assumed to satisfy

$$0 \leq a \in L_{\infty,loc}(]0,\infty[), \tag{5.6.52}$$

that is, we allow a to be unbounded as $x \to \infty$ and $x \to 0$. Further, $b \geq 0$ is a measurable function satisfying $b(x|y) = 0$ for $x > y$ and (2.2.32).

The expected number of particles resulting from a fragmentation of a size y parent, denoted by $n_0(y)$, is assumed to satisfy

$$n_0(y) < +\infty \tag{5.6.53}$$

for any fixed $y \in \mathbb{R}_+$.

In fragmentation and coagulation problems, two spaces are most often used due to their physical relevance. In the space $L_1(\mathbb{R}_+, xdx)$ the norm of a nonnegative element u, given by $\int_0^\infty u(x)xdx$, represents the total mass of the system, whereas the norm of a nonnegative element u in the space $L_1(\mathbb{R}_+, dx)$, $\int_0^\infty u(x)dx$, gives the total number of particles in the system.

We use the scale of spaces with finite higher moments

$$X_m = L_1(\mathbb{R}_+, dx) \cap L_1(\mathbb{R}_+, x^m dx) = L_1(\mathbb{R}_+, (1+x^m)dx), \tag{5.6.54}$$

where $m \in \mathbb{M} := [1,\infty)$. We extend this definition to $X_0 = L_1(\mathbb{R}_+)$. We note that, due to the continuous injection $X_m \hookrightarrow X_1$, $m \geq 1$, any solution in X_m is also a solution in the basic space X_1.

Thus, we denote by $\|\cdot\|_m$ the natural norm in X_m defined in (5.6.54). To shorten notation, we define

$$w_m(x) := 1 + x^m.$$

Multiple solutions. It also has been observed in several papers that for some classes of coefficients there exist multiple solutions to (2.2.30). For instance, for

$$\partial_t u(x,t) = -xu(x,t) + 2\int\limits_x^\infty u(y,t)dy, \tag{5.6.55}$$

it is easy to see that

$$u_1(x,t) = \frac{e^t}{(1+x)^3},$$

and

$$u_2(x,t) = e^{-xt}\left(\frac{1}{(1+x)^3} + \int\limits_x^\infty \frac{1}{(1+x)^3}[2t + t^2(y-x)]dy\right)$$

are both solutions to (2.2.30) with

$$u(x,0) = (1+x)^{-3}.$$

An apparent remedy seems to be offered by an observation that the latter solution is mass-conserving, whereas the former is clearly not and therefore should be ruled out from the considerations as a non-physical one.

Dishonesty. In the context of fragmentation, dishonesty is termed 'shattering' and interpreted as accelerating fragmentation of small particles leading to the creation of a 'dust' of particles of zero mass which do not contribute to the total mass of the ensemble. For example, [17], for a binary fragmentation model with $b(x|y) = 2/y$ and the fragmentation rate $a(x) = 1/x$,

$$\partial_t u(x,t) = -x^{-1}u(x,t) + 2\int_x^\infty y^{-2}u(y,t)dt, \tag{5.6.56}$$

one can show that for mono-disperse initial condition $u(x,0) = \delta(x-l)$, $l > 0$ with unit total mass, the solution is given by

$$u_l(x,t) = e^{-t/l}\left(\delta(x-l) + \frac{2t}{l^2} - \frac{t^2}{l^2}\left(\frac{1}{l} - \frac{1}{x}\right)\right), \quad x \le l, \tag{5.6.57}$$

and $u_l(x,t) = 0$ for $x > l$. Direct calculation shows that the total mass of the ensemble at time t is given by

$$M(t) = e^{-t/l}\left(l + t + \frac{t^2}{2l}\right), \tag{5.6.58}$$

and clearly decreases monotonically in time. However, the mono-disperse initial condition is given by the Dirac measure and so is outside the scope of the general L_1 theory. Moreover, since we know that many fragmentation equations admit multiple solutions, it is not immediately clear whether the explicit solutions given by (5.6.57) is related to $\{G_K(t)\}_{t\ge 0}$. Here we can use the theory on maximality of the generator, which will be discussed below.

To remedy the first problem, we use the solution for the mono-disperse initial condition to construct a solution solutions with initial data $f \in L_1(\mathbb{R}_+, xdx)$. Since the problem is linear, formally this can be done using (5.6.57) as the source function in the integral

$$\begin{aligned} u(x,t) &= \int_x^\infty u_l(x,t)f(l)dl \\ &= e^{-t/x}f(x) + 2t\int_x^\infty \frac{e^{-t/l}}{l^2}f(l)dl + t^2\int_x^\infty \frac{e^{-t/l}}{l^2}\left(\frac{1}{x} - \frac{1}{l}\right)f(l)dl. \end{aligned} \tag{5.6.59}$$

As we shall see, if f $t \to u(\cdot,t)$ is an $L_1(\mathbb{R}_+, xdx)$-valued continuous function, which satisfies the integral version of (5.6.56):

$$u(x,t) = f(x) - x^{-1}\int_0^t u(x,s)ds + 2\int_0^t\int_x^\infty y^{-2}u(y,s)dyds \tag{5.6.60}$$

for almost all $x \in [0,\infty)$ and all $t \ge 0$, where $f \in L_1(\mathbb{R}_+, xdx)$, then $u(x,t) = [G_K(t)f](x)$.

The solution satisfies the mass equation

$$M(t) = \int_0^\infty e^{-t/l}f(l)\left(l + t + \frac{t^2}{2l}\right)dl. \tag{5.6.61}$$

or

$$\frac{dM}{dt} = -\frac{t^2}{2}\int_0^\infty f(l)\frac{e^{-t/l}}{l^2}dl,$$

which shows that for any initial condition, the mass of the ensemble is strictly decreasing for any t.

6.1 Well-posedness Results

To employ the theory introduced in Section 3.3, let $\mathcal{A}$ and $\mathcal{B}$ denote the expressions appearing on the right-hand side of the equation in (5.6.51); that is,

$$[\mathcal{A}u](x) = -a(x)u(x)$$

and

$$[\mathcal{B}u](x) = \int\limits_x^\infty a(y)b(x|y)u(y)dy,$$

defined on all measurable and finite almost everywhere functions u for which they make pointwise (almost everywhere) sense.

With these expressions we associate operators A and B in X defined by

$$[Au](x) = [\mathcal{A}u](x), \qquad [Bu](x) = [\mathcal{B}u](x)$$

and both defined on

$$D(A) = \{u \in X;\ au \in X\}.$$

Indeed, direct integration shows that $\mathcal{B}D(A) \subset X$ so that $(A + B, D(A))$ is a well-defined operator.

Thus, without any misunderstanding, the expressions $\mathcal{A}$ and $\mathcal{B}$ can be identified with the operator extensions defined in Section 3.3 where E_f is the set of all measurable and almost everywhere finite functions on $\mathbb{R}_+$.

We can now state the following theorem.

Theorem 5.30. *Under the assumptions of this section, there is an extension K of $A + B$ that generates a positive semigroup of contractions $\{G_K(t)\}_{t\geq 0}$ on X.*

Proof. It is obvious that $(A, D(A))$ generates a positive semigroup of contractions and $(B, D(B))$ is positive. Moreover, for $u \in D(A)$ we immediately have, by (2.2.32),

$$\begin{aligned}
&\int\limits_0^\infty \left(-a(x)u(x,t) + \int\limits_x^\infty a(y)b(x|y)u(y,t)dy \right) xdx \\
&\quad = -\int\limits_0^\infty a(x)u(x,t)xdx + \int\limits_0^\infty x \left(\int\limits_x^\infty a(y)b(x|y)u(y,t)dy \right) dx \\
&\quad = -\int\limits_0^\infty a(x)u(x,t)xdx + \int\limits_0^\infty a(y)u(y,t) \left(\int\limits_0^y b(x|y)xdx \right) dy \\
&\quad = -\int\limits_0^\infty a(x)u(x,t)xdx + \int\limits_0^\infty a(y)u(y,t)ydy = 0.
\end{aligned} \tag{5.6.62}$$

Thus, we see that we can use Corollary 5.12 to ascertain that there is an extension K of $A + B$ generating a substochastic semigroup $\{G_K(t)\}_{t\geq 0}$. □

6.2 Honesty

To continue, we turn to the theory of extensions (Section 3.3). First, we observe that the operator L is defined by

$$[\mathsf{L}f](x) = (1 + a(x))^{-1} f(x),$$

and therefore the operator A defined through Eq. (5.3.33) is given here by

$$[\mathsf{A}u](x) = u(x) - [\mathsf{L}^{-1}u](x) = a(x)u(x),$$

with the domain $D(\mathsf{A}) = \mathsf{X}$. Hence $\mathsf{A} \subset \mathcal{A}$. Because B is an integral operator with positive kernel, Lebesgue's monotone convergence theorem yields that $\mathsf{B} = \mathcal{B}$. Now we provide a fairly general condition for honesty of $\{G_K(t)\}_{t\geq 0}$.

Theorem 5.31. *If*

$$\limsup_{x\to 0^+} a(x) < +\infty, \tag{5.6.63}$$

then $\{G_K(t)\}_{t\geq 0}$ *is honest.*

Proof. We use Theorem 5.19. Because c is the zero functional by (5.6.62), we have to prove that for any $f \in \mathsf{F}_+$ such that $-f + \mathsf{BL}f \in X$ the following inequality holds,

$$\int_0^\infty [\mathsf{L}f](x)xdx + \int_0^\infty (-f(x) + [\mathsf{BL}f](x))\, xdx \geq 0. \tag{5.6.64}$$

We simplify (5.6.64) by defining $g(x) = [\mathsf{L}f](x) = (1+a(x))^{-1}f(x) \in X_+$ and inserting it into the inequality. Hence, we obtain that (5.6.64) holds if for any $g \in X_+$ such that $-ag + \mathsf{B}\mathsf{g} \in \mathsf{X}$ we have the inequality

$$\int_0^\infty (-a(x)g(x) + [\mathsf{B}g](x))\, xdx \geq 0. \tag{5.6.65}$$

By (5.6.63), the function ag satisfies $ag \in L_1([0,R], xdx)$ for any $0 < R < +\infty$, therefore the same is true for $\mathsf{B}g$. We observed earlier that B is given by the integral expression $\mathcal{B}$, hence

$$\begin{aligned}
\int_0^\infty (-a(x)g(x) + [\mathsf{B}g](x))\, xdx &= \lim_{R\to\infty} \int_0^R (-a(x)g(x) + [\mathsf{B}g](x))\, xdx \\
&= \lim_{R\to\infty} \left(-\int_0^R a(x)g(x)xdx + \int_0^R \left(\int_x^\infty a(y)b(x|y)g(y)dy \right) xdx \right).
\end{aligned}$$

Next, by (2.2.32),

$$\begin{aligned}
\int_0^R \left(\int_x^\infty a(y)b(x|y)g(y)dy \right) xdx &= \int_0^R \left(\int_0^y b(x|y)xdx \right) g(y)a(y)dy \\
+ \int_R^\infty \left(\int_0^R b(x|y)xdx \right) g(y)a(y)dy &= \int_0^R a(y)g(y)ydy + S_R,
\end{aligned}$$

where

$$S_R = \int_R^\infty \left(\int_0^R b(x|y)xdx \right) g(y)a(y)dy \geq 0.$$

Combining, we see that

$$\int_0^\infty (-a(x)g(x) + [\mathsf{B}g](x))\, xdx = \lim_{R\to\infty} S_R \geq 0 \tag{5.6.66}$$

so that Theorems 5.19 and 5.16 give the thesis. □

6.3 Nonuniqueness

In this section we provide a complete description of the dynamics of the fragmentation equation; that is, we give necessary and sufficient conditions for honesty and for the nonexistence of multiple solutions in the case when

$$b(x|y) = \frac{y\beta(x)}{\int\limits_0^y \beta(s)sds}.$$

We start by determining under which conditions the generator K of $\{G_K(t)\}_{t\geq 0}$ is the maximal operator. By Section 2, this is equivalent to the fact that all the X-valued solutions are given by the semigroup $\{G_K(t)\}_{t\geq 0}$ so that there are no multiple solutions.

Let us recall that the maximal operator $K_{\max}$ was defined by

$$[K_{\max}u](x) := [\mathcal{A}u](x) + [\mathcal{B}u](x) = -a(x)u(x) + \int\limits_x^\infty a(y)b(x|y)u(y)dy \tag{5.6.67}$$

on the domain

$$D_{\max} = \{u \in X;\ x \to [\mathcal{A}u](x) + [\mathcal{B}u](x) \in X\}. \tag{5.6.68}$$

Note that this definition implicitly requires $y \to a(y)b(x|y)u(y)$ to be Lebesgue integrable on $[c, \infty)$ for any $c > 0$ and almost every $x > 0$. We need the following lemma.

Lemma 5.32. *Let $\alpha > 0$ and f be integrable on compact subsets of $(0, \infty)$ and monotonic close to 0. If $\lim_{x\to 0^+} f(x) = +\infty$, then $f'e^{-\alpha f} \in L_1([0, r])$ for some $r > 0$.*

Theorem 5.33. *Let $B(x) := b(x|x)/(1 + a(x))$. Then $K \neq K_{\max}$ if and only if*

$$B(x) \in L_1([N, \infty)) \tag{5.6.69}$$

and

$$x\beta(x) \notin L_1([N, \infty)) \tag{5.6.70}$$

for some $N > 0$.

Proof. Because K is dissipative, its spectrum is contained in the negative complex half-plane. Thus, by the results of Section 2, $K \neq K_{\max}$ if and only if there are solutions in X of the eigenvalue problem

$$\lambda u(x) + a(x)u(x) - \int\limits_x^\infty a(y)b(x|y)u(y)dy = 0, \tag{5.6.71}$$

for $\lambda > 0$. Assume that there exists $u \in D_{\max}$ satisfying (5.6.71). Denoting $U(x) = u(x)/\beta(x)$ we transform (5.6.71) into

$$(\lambda + a(x))U(x) - \int\limits_x^\infty a(y)b(y|y)U(y)dy = 0. \tag{5.6.72}$$

Changing the dependent variable according to $Z(x) = \int_x^\infty a(y)b(y|y)U(y)dy$, we observe, from the definition of the maximal operator (5.6.67), that the integrand is integrable over any interval $[\epsilon, \infty)$ so that the integral is absolutely continuous at each $x > 0$ and we can thus differentiate, converting (5.6.72) into the differential equation

$$\frac{Z'}{Z} = -\frac{a(x)b(x|x)}{\lambda + a(x)}, \tag{5.6.73}$$

with the solution

$$Z(x) = C \exp\left(-\int_1^x \frac{a(s)b(s|s)}{\lambda + a(s)} ds \right), \tag{5.6.74}$$

where C is a constant, so that

$$\begin{aligned} u(x) &= C \frac{\beta(x)}{\lambda + a(x)} \exp\left(-\int_1^x \frac{a(s)b(s|s)}{\lambda + a(s)} ds \right) \\ &= C' \frac{\beta(x)}{(\lambda + a(x)) \int_0^x s\beta(s)ds} \exp\left(\lambda \int_1^x \frac{b(s|s)}{\lambda + a(s)} ds \right) \\ &= C' \frac{b(x|x)}{x(\lambda + a(x))} \exp\left(\lambda \int_1^x \frac{b(s|s)}{\lambda + a(s)} ds \right), \end{aligned} \tag{5.6.75}$$

where C' is a constant and where we used

$$\int_1^x b(s|s)ds = \int_1^x \frac{s\beta(s)}{\int_0^s t\beta(t)dt} ds = \ln \int_0^x t\beta(t)dt + C'',$$

for some constant C''.

Note that all properties of u are independent of λ as long as it is positive; thus, in what follows, we put $\lambda = 1$. Dropping unimportant constant C', we have to investigate the integrability of

$$\bar{u}(x) = xu(x) = B(x) \exp\left(\int_1^x B(s)ds \right), \tag{5.6.76}$$

close to 0 and for large x. Let us start with integrability in a neighbourhood of 0. Then we can assume that $x < 1$ so that

$$\bar{u}(x) = B(x) \exp\left(-\int_x^1 B(s)ds \right),$$

with positive integral. If $B(x) \in L_1([0,1])$, then $\bar{u} \in L_1([0,1])$ too, because the exponent is smaller than 1. Conversely, if $B(x) \notin L_1([0,1])$, then $f(x) = \int_x^1 B(s)ds$ diverges monotonically to $+\infty$ as $x \to 0^+$ so that we can apply Lemma 5.32 to see again that $\bar{u} \in L_1([0,1])$. Let us now consider integrability for large x. If $B \notin L_1([N,\infty))$, then $\bar{u} \notin L_1([N,\infty))$ too, because the exponential factor in (5.6.76) is greater than 1. Conversely, if $B \in L_1([N,\infty))$, then

$$\bar{u}(x) \leq B(x) \exp\left(\int_1^\infty B(s)ds \right) \in L_1([N,\infty)).$$

Hence, for $K \neq K_{\max}$ it is necessary that $B(x)$ be integrable at infinity.

However, at this moment we do not know whether u, defined by (5.6.75), is a solution to (5.6.71). To this end we have

$$\int_x^\infty a(y)b(x|y)u(y)dy = \beta(x)\int_x^\infty \frac{a(y)b(y|y)}{\lambda+a(y)}\exp\left(-\int_1^y \frac{a(s)b(s|s)}{\lambda+a(s)}ds\right)dy$$
$$= -\beta(x)\left[\exp\left(-\int_1^y \frac{a(s)b(s|s)}{\lambda+a(s)}ds\right)\right]_{y=x}^{y=\infty} = (\lambda+a(x))u(x)$$
$$-\beta(x)\lim_{y\to\infty}\exp\left(-\int_1^y \frac{a(s)b(s|s)}{\lambda+a(s)}ds\right). \tag{5.6.77}$$

Thus, u is a solution to (5.6.71) (with $\lambda = 1$) if and only if

$$\int_1^\infty \frac{a(s)b(s|s)}{1+a(s)}ds = \infty.$$

Transforming the integral, as in (5.6.75), we get

$$\int_1^x \frac{a(s)b(s|s)}{1+a(s)}ds = \int_1^x b(s|s)ds - \int_1^x B(s)ds$$
$$= \ln\int_0^x s\beta(s)ds - \ln\int_0^1 s\beta(s)ds - \int_1^x B(s)ds,$$

and, because the last integral tends to a finite limit as $x \to \infty$, we must have $\int_0^\infty s\beta(s)ds = \infty$. □

Example 5.34. As an example, let us consider the case of arbitrary fragmentation rate $a(x)$ satisfying the condition that both (possibly infinite) limits $\lim_{x\to\infty,0} a(x) = l_{\infty,0}$, respectively, exist and let $b(x|y) = (\nu+2)x^\nu/y^{\nu+1}$. In this case $\beta(x) = x^\nu$ with $\nu > -2$ so that $\int_N^\infty s\beta(s)ds = \infty$ and (5.6.70) is always satisfied. Moreover, because $b(x|x) = (\nu+2)/x$, we can restate Theorem 5.33 by saying that $K \neq K_{\max}$ if and only if

$$\frac{1}{xa(x)} \in L_1([N,\infty)). \tag{5.6.78}$$

Summarizing, we have the following equivalent conditions.

$$\begin{aligned} &K = K_{\max} \text{ if and only if } 1/xa(x) \notin L_1([N,\infty)), \qquad 0 < N < +\infty \\ &K = \overline{K_{min}} \text{ if and only if } 1/xa(x) \notin L_1([0,\delta]), \qquad 0 < \delta < \infty. \end{aligned} \tag{5.6.79}$$

Note that Theorem 5.31 ensures the conservativity (honesty) of $\{G_K(t)\}_{t\geq 0}$ provided $l_0 < +\infty$. However, from the above conditions, it follows that there are fragmentation rates a, infinite at $x = 0$, for which the fragmentation semigroup is still conservative. Indeed, consider $a(x) \sim -\ln x$ close to 0. Because then $1/xa(x) \notin L_1([0,\delta])$, it follows that in this case the semigroup $\{G_K(t)\}_{t\geq 0}$ is still honest.

Remark 5.35. A natural question to be asked is whether the stated result $K = \overline{A+B}$ can be improved to $K = A+B$. One can prove (see [36]) that in general the answer is negative. To simplify the presentation, we confine ourselves to the power law case: $a(x) = x^\alpha$, $b(x|y) = (\nu+2)x^\nu/y^{\nu+1}$, $\nu > -2$. From Example 5.34 we know that in this case for $\{G_K(t)\}_{t\geq 0}$ to be honest it is necessary and sufficient that $\alpha \geq 0$. If $\alpha = 0$, then the operators A and B are bounded and clearly $K = A+B$. Hence, we can assume $\alpha > 0$. In this case $1 \notin \sigma_p((BL_\lambda)^*)$, that is, $1 \notin \sigma_r(BL_\lambda)$ and because from Theorem 5.1(a) it follows that $1 \notin \sigma_p(BL_\lambda)$, we see that $1 \in \sigma_c(BL_\lambda) \cup \rho(BL_\lambda)$. Hence, by Theorem 5.1(c), to show that $K \neq A+B$ it suffices to show that $1 \notin \rho(BL_\lambda)$ for some $\lambda > 0$. Let us denote $L_1 = L$ and consider the equation

$$f_\zeta - \zeta BLf_\zeta = 0, \qquad \zeta > 0. \tag{5.6.80}$$

Denoting $u_\zeta(x) = [Lf_\zeta](x) = (1+x^\alpha)^{-1} f_\zeta(x)$, we see that u_ζ satisfies

$$u_\zeta(x) + x^\alpha u_\zeta(x) - \zeta[Bu_\zeta](x) = 0,$$

which is of the same form as Eq. (5.6.71). Hence, using the same approach and formula (5.6.75), we obtain

$$f_\zeta(x) = x^\nu \exp\left(-\zeta(\nu+2)\int\limits_1^x \frac{s^\alpha}{s(1+s^\alpha)}ds\right) = x^\nu(1+x^\alpha)^{-\zeta(\nu+2)/\alpha}$$

and $f_\zeta \in L_1(\mathbb{R}_+, xdx)$ for any $\zeta > 1$ (remember $\alpha > 0, \nu > -2$). To check whether f is the solution to (5.6.80) we evaluate

$$\begin{aligned}[BLf_\zeta](x) &= (\nu+2)x^\nu \int\limits_x^\infty \frac{y^{\alpha-1}}{(1+y^\alpha)^{1+\zeta(\nu+2)/\alpha}}dy \\ &= \frac{1}{\zeta}x^\nu\left(-\lim_{y\to\infty}\frac{1}{(1+y^\alpha)^{\zeta(\nu+2)/\alpha}} + \frac{1}{(1+x^\alpha)^{\zeta(\nu+2)/\alpha}}\right) \\ &= \frac{1}{\zeta}\frac{x^\nu}{(1+x^\alpha)^{\zeta(\nu+2)/\alpha}} = \frac{1}{\zeta}f_\zeta(x),\end{aligned}$$

as $\zeta(\nu+2) > 0$. Thus, we see that for any $\zeta > 1$, f_ζ is an eigenvector of BL corresponding to the eigenvalue $1/\zeta$ and hence $(0,1) \subset \sigma_p(BL)$. Because the spectrum of any operator is closed, we see that the value $1 \in \sigma(BL)$ and, by the previous considerations, $1 \in \sigma_c(BL)$ and $K \neq A+B$ by Theorem 5.1(c).

6.4 Uniqueness of solutions when $K \neq K_{\max}$

When $K \neq K_{\max}$ then, as we know, there are multiple solutions to (5.6.51) or, in other words, the semigroup $\{G_K(t)\}_{t\geq 0}$ does not capture all the solutions. This shows that (5.6.51), as it stands, does not determine the whole dynamics of the fragmentation process. A natural question then arises as to what additional condition should supplement (5.6.51) to ensure the uniqueness of solutions. One can show that for a large class of coefficients the solutions are unique in the class of positive and mass conserving, that is, physically reasonable, solutions. In this subsection we prove this statement for two classes of problems: when the fragmentation rate is bounded at $x = 0$ and for separable coefficient b with a such that $\{G_K(t)\}_{t\geq 0}$ is honest. It is important to note that $\{G_K(t)\}_{t\geq 0}$ is also honest in the first case: for dishonest fragmentation semigroups, mass is lost from the system at an uncontrolled rate and thus imposing a conservativity condition does not make any sense.

We also note that for a well-researched case of power law $a(x) = x^\alpha$, conditions (5.6.79) give an alternative: for $\alpha \neq 0$ the process is either honest but has multiple solutions or is dishonest with no multiple solutions. In this case the results obtained below ensure the uniqueness of solutions: unconditional for $\alpha \leq 0$ ($K = K_{\max}$) or subject to the additional requirement of positivity and conservativity of them for $\alpha > 0$.

Theorem 5.36. *Assume that the fragmentation rate satisfies (5.6.63); that is, it is bounded on bounded intervals of* $[0,\infty)$. *If* u *is a nonnegative function that is integrable on* $\mathbb{R}_+ \times [0,T]$, $T < \infty$ *with respect to the measure* $xdxdt$, *that satisfies*

$$u(x,t) = u_0(x) + \int\limits_0^t (-a(x)u(x,s) + [Bu](x,s))\,ds, \tag{5.6.81}$$

where $u_0 \in X_+$, *and*

$$\int\limits_0^\infty u(x,t)xdx = \int\limits_0^\infty u_0(x)xdx \tag{5.6.82}$$

for any $t > 0$, *then*

$$u(x,t) = [G_K(t)u_0](x) \tag{5.6.83}$$

for any $t \geq 0$ *and almost any* $x \in [0,\infty)$.

6.5 Analyticity of the fragmentation operator

Here, additionally, we assume that the fragmentation rate satisfies Further, we assume that there are $j \in (0,\infty)$, $l \in [0,\infty)$ and $a_0, b_0 \in \mathbb{R}_+$ such that for any $x \in \mathbb{R}_+$

$$a(x) \le a_0(1+x^j), \qquad n_0(x) \le b_0(1+x^l). \tag{5.6.84}$$

To formulate the main results, we have to introduce more specific assumptions and notation. First we define

$$n_m(y) := \int_0^y b(x|y)x^m dx$$

for any $m \in \mathbb{M}_0 := \{0\} \cup \mathbb{M}$ and $y \in \mathbb{R}_+$. Further, let

$$N_0(y) := n_0(y) - 1 \quad \text{ and } \quad N_m(y) := y^m - n_m(y), \quad m \ge 1.$$

It follows that

$$N_0(y) = n_0(y) - 1 \ge 0$$

and

$$N_m(y) = y^m - \int_0^y b(x|y)x^m dx \ge y^m - y^{m-1}\int_0^y b(x|y)x dx = 0 \tag{5.6.85}$$

for $m \ge 1$ with $N_1 = 0$.

Next, for $m \in \mathbb{M}$, let $(A_m u)(x) := a(x)u(x)$ on $D(A_m) = \{u \in X_m : \ au \in X_m\}$ and let B_m be the restriction to $D(A_m)$ of the integral expression

$$[\mathcal{B}u](x) = \int_x^\infty a(y)b(x|y)u(y)dy.$$

Theorem 5.37. *[21] Let a, b satisfy (2.2.32), (5.6.53) and (5.6.84), and let m be such that $m \ge j+l$ if $j+l>1$ and $m>1$ if $j+l \le 1$.*
a) The closure $(F_m, D(F_m)) = \overline{(-A_m + B_m, D(A_m))}$ generates a positive quasi-contractive semigroup, say $\{S_{F_m}(t)\}_{t\ge0}$, of the type at most $4a_0b_0$ on X_m. Furthermore, if $u \in D(F_m)_+$, then

$$N_m(x)a(x)u(x) \in X_0, \qquad m \in \mathbb{M}_0. \tag{5.6.86}$$

b) If, moreover, for some m there is $c_m > 0$ such that

$$\liminf_{x\to\infty} \frac{N_m(x)}{x^m} = c_m, \tag{5.6.87}$$

then $F_m = -A_m + B_m$ and $\{S_{F_m}(t)\}_{t\ge0}$ is an analytic semigroup on X_m.
c) If (5.6.87) holds for some m_0, then it holds for all $m \ge m_0$.

We note that (5.6.87) cannot hold for $m = 1$ as $N_1 = 0$.

Proof. We shall fix m satisfying $m \ge j+l$ if $j+l>1$ and $m>1$ otherwise; see (5.6.84). First we show that $B_m := \mathcal{B}|_{D(A_m)}$ is well defined. Next, direct integration gives for $u \in D(A_m)$

$$\int_0^\infty (-A_m + B_m)u(x)w_m(x)dx = -\phi_m(u) := \int_0^\infty (N_0(x) - N_m(x))a(x)u(x)dx. \tag{5.6.88}$$

If the term $N_0(x) > 0$ were not present, then (5.6.88) would allow a direct application of the substochastic semigroup theory. In the present case we note that for $u \in D(A_m)_+$ we have, by (5.6.85),

$$-\phi_m(u) \le \int_0^\infty N_0(y)a(y)u(y)dy \le 4a_0b_0 \int_0^\infty u(x)w_m(x)dx =: \eta\|u\|_m,$$

Then we have $\tilde{\phi}_m(u) := \phi_m(u) + \eta\int_0^\infty u(x)w_m(x)dx \ge 0$ for $0 \le u \in D(A_m)_+$ and the operator $(\tilde{A}_m, D(A_m)) := (A_m + \eta I, D(A_m))$ satisfies

$$\int_0^\infty (-\tilde{A}_m + B_m)u(x)w_m(x)dx = -\tilde{\phi}_m(u)$$

$$= -\eta\int_0^\infty u(x)w_m(x)dx + \int_0^\infty (N_0(x) - N_m(x))a(x)u(x)dx \le 0.$$

Hence an extension $\tilde{F}_m$ of $-\tilde{A}_m + B_m$ generates a substochastic semigroup $\{S_{\tilde{F}_m}(t)\}_{t\ge0}$ and thus there is an extension F_m of $(-A_m + B_m, D(A_m))$, given by $(F_m, D(F_m)) = (\tilde{F}_m + \eta I, D(\tilde{F}_m))$, generating a positive semigroup $\{S_{F_m}(t)\}_{t\ge0} = (e^{\eta t}S_{\tilde{F}_m}(t))_{t\ge0}$ on X_m.

Furthermore, $\tilde{\phi}_m$ extends to $D(F_m)$ by monotone limits of elements of $D(A_m)$. Thus, let $u \in D(F_m)_+$ with $D(A_m) \ni u_n \nearrow u$. Then, since

$$\int_0^\infty N_0(x)a(x)u(x)dx < \infty, \quad \int_0^\infty u(x)w_m(x)dx < \infty,$$

by (5.6.84), $m \ge j + l$ and $D(F_m) \subset X_m$, and the fact that $\tilde{\phi}_m(u_n)$ tends to a finite limit, we have

$$\lim_{n\to\infty} \int_0^\infty N_m(x)a(x)u_n(x)dx = \int_0^\infty N_m(x)a(x)u(x)dx < +\infty.$$

To prove part b), we begin by observing that inequality (5.6.85) implies that $0 \le N_m(x) \le x^m$. This, together with (5.6.87), yields $c_m x^m/2 \le N_m(x) \le x^m$ for large x which, by (5.6.86), establishes that if $u \in D(F_m)$, then $au \in X_m$ or, in other words, that $D(F_m) \subset D(A_m)$. Since $(F_m, D(F_m))$ is an extension of $(-A_m + B_m, D(A_m))$, we see that $D(F_m) = D(A_m)$.

It is clear that the semigroup generated by $-A_m$ is bounded. Furthermore, if $\lambda = r + is$ then $|\lambda + a(x)|^2 \ge s^2$ and therefore, for all $r > 0$

$$\|R(r + is, -A_m)f\|_m = \int_0^\infty \left|\frac{1}{r + is + a(x)}\right| |f(x)|(1 + x^m)dx \le \frac{1}{|s|}\|f\|_m.$$

The analyticity of the fragmentation semigroup then follows from the Arendt-Rhandi theorem, Theorem 5.8. □

This results plays a crucial role in the analysis of the full fragmentation-coagulation equation, [22].

Example 5.38. One of the forms of $b(x|y)$ most often used in applications is

$$b(x|y) = \frac{1}{y}h\left(\frac{x}{y}\right) \tag{5.6.89}$$

which is referred to as the homogeneous fragmentation kernel. In this case the distribution of the daughter particles does not depend directly on their relative sizes but on their ratio. In this case

$$n_m(y) = \frac{1}{y}\int\limits_0^y h\left(\frac{x}{y}\right) x^m dx = y^m \int\limits_0^1 h(z) z^m dz =: h_m y^m.$$

Since

$$y = n_1(y) = \frac{1}{y}\int\limits_0^y h\left(\frac{x}{y}\right) x dx = y \int\limits_0^1 h(z) z dz = h_1 y$$

we have $h_1 = 1$ so that $h_m < 1$ for any $m > 1$ and $N_m(y) = y^m(1 - h_m)$. Hence, (5.6.87) holds.

On the other hand, fragmentation processes in which daughter particles accumulate close both to 0 and to the parent's size may not satisfy (5.6.87), [22].

6

Semilinear problems

1 Nonhomogeneous Problems

Let us consider the problem of finding the solution to the Cauchy problem:

$$\begin{aligned} \partial_t u &= Au + f(t), \quad 0 < t < T \\ u(0) &= \overset{\circ}{u}, \end{aligned} \tag{6.1.1}$$

where $0 < T \leq \infty$, A is the generator of a semigroup and $f : (0, T) \to X$ is a known function.

If we are interested in classical solutions, then clearly f must be continuous. However, this condition proves to be insufficient. Thus we generalise the concept of the *mild solution* introduced in (3.1.13). We observe that if u is a classical solution of (6.1.1), then it must be given by

$$u(t) = G(t)\overset{\circ}{u} + \int\limits_0^t G(t-s)f(s)ds \tag{6.1.2}$$

(see, e.g., [54, Corollary 4.2.2]). The integral is well defined even if $f \in L_1([0, T], X)$ and $\overset{\circ}{u} \in X$. We call u defined by (6.1.2) the *mild solution* of (6.1.1). For an integrable f such u is continuous but not necessarily differentiable, and therefore it may be not a solution to (6.1.1).

We have the following theorem giving sufficient conditions for a mild solution to be a classical solution (see, e.g., [54, Corollary 4.2.5 and 4.2.6]).

Theorem 6.1. *Let A be the generator of a C_0-semigroup $\{G_A(t)\}_{t\geq 0}$ and $x \in D(A)$. Then (6.1.2) is a classical solution of (6.1.1) if either*

(i) $f \in C^1([0, T], X)$, or

(ii) $f \in C([0, T], X) \cap L_1([0, T], D(A))$.

The assumptions of this theorem are often too restrictive for applications. On the other hand, it is not clear exactly what the mild solutions solve. A number of weak formulations of (6.1.1) have been proposed (see e.g. [35, pp. 88–89] or [10]), all of them having (6.1.2) as their solutions. We present here a result from [34, p. 451] which is particularly suitable for applications.

Proposition 6.2. *A function $u \in C(\mathbb{R}_+, X)$ is a mild solution to (6.1.1) with $f \in L_1(\mathbb{R}_+, X)$ in the sense of (6.1.2) if and only if $\int_0^t u(s)ds \in D(A)$ and*

$$u(t) = \overset{\circ}{u} + A\int\limits_0^t u(s)ds + \int\limits_0^t f(s)ds, \qquad t \geq 0. \tag{6.1.3}$$

1.1 Semi-linear problems

Let us introduce now the simplest nonlinearity and consider the semilinear abstract Cauchy problem

$$\begin{aligned} \partial_t u &= Au + f(t,u), \quad t > 0, \\ u(0) &= \overset{\circ}{u}, \end{aligned} \tag{6.1.4}$$

where A is a generator of a C_0-semigroup $\{G_A(t)\}_{t\geq 0}$ and $f : [0,T] \times X \to X$ is a known function. Since *a priori* we know no properties of the solution u (which may even fail to exist), it is plausible to start from a weaker formulation of the problem, i.e. from the integral equation:

$$u(t) = G_A(t)\overset{\circ}{u} + \int\limits_0^t G_A(t-s)f(s,u(s))ds. \tag{6.1.5}$$

This form is typical for fixed point techniques. Here, depending on the properties of $\{G_A(t)\}_{t\geq 0}$ and f, we can use two main fixed point theorems: the Banach contraction principle and Schauder's theorem.

We shall focus on the Banach contraction principle which leads to Theorem 6.3 below. It requires a relatively strong regularity from f.

We say that $f : [0,T] \times X \to X$ is locally Lipschitz continuous in u, uniformly in t on bounded intervals, if for all $t' \in [0,T[$ and $c > 0$ there exists $L(c,t')$ such that for all $t \in [0,t']$ and $\|u\|, \|v\| \leq c$ we have

$$\|f(t,u) - f(t,v)\|_X \leq L(c,t')\|u-v\|_X.$$

Theorem 6.3. *Let $f : [0,\infty[\times X \to X$ be continuous in $t \in [0,\infty[$ and locally Lipschitz continuous in u, uniformly in t, on bounded intervals. If A is the generator of a C_0-semigroup $\{G_A(t)\}_{t\geq 0}$ on X, then for any $\overset{\circ}{u}\in X$ there exists $t_{max} > 0$ such that the problem (6.1.5) has a unique mild solution u on $[0,t_{max}[$. Moreover, if $t_{max} < +\infty$, then $\lim\limits_{t\to\infty} \|u(t)\|_X = \infty$.*

The proof is done by Picard iterations, as in the scalar case. Also, similarly to the scalar case, a sufficient condition for the existence of a global mild solution is that f be uniformly Lipschitz continuous on X. Uniform Lipschitz continuity yields at most linear growth in $\|x\|$ of $\|f(t,x)\|$. In fact, even for $f(u) = u^2$ and $A = 0$, the blow-up occurs in finite time.

There are two standard sufficient conditions ensuring that the mild solution, described in Theorem 6.3, is a strict solution. Both follow from the corresponding results for nonhomogeneous problems. They are either that $f : [0,\infty[\times X \to X$ is continuously differentiable with respect to both variables, or that $f : [0,\infty[\times D(A) \to D(A)$ is continuous. Certainly, in both cases to ensure that the solution is strict we must assume that $\overset{\circ}{u}\in D(A)$.

2 Epidemiology

This section is based on [25, 24]. To simplify the exposition we replace the SIRS model given by (2.2.13) with the SIS model describing the evolution of epidemics which does not convey any immunity. Setting $\gamma = 0$ in the system (2.2.13) and thus discarding the 'recovered' class, we have

$$\begin{aligned} \partial_t s(a,t) + \partial_a s(a,t) &= -\mu(a)s(a,t) - \Lambda(a, i(\cdot,t))s(a,t) + \delta(a)i(a,t), \\ \partial_t i(a,t) + \partial_a i(a,t) &= -\mu(a)i(a,t) + \Lambda(a, i(\cdot,t))s(a,t) - \delta(a)i(a,t), \\ s(0,t) &= \int_0^\omega \beta(a)\left\{s(a,t) + (1-q)i(a,t)\right\} da, \\ i(0,t) &= q\int_0^\omega \beta(a)i(a,t)\, da, \\ s(a,0) &= s_0(a) = \phi^s(a), \\ i(a,0) &= i_0(a) = \phi^i(a), \end{aligned} \tag{6.2.6}$$

for $0 \leq t \leq T \leq +\infty$, $0 \leq a \leq \omega \leq +\infty$.

The force of infection is defined by (2.2.14); the concrete assumptions will be introduced when needed. In both cases we deal with a semilinear problem; that is, with a nonlinear (algebraic) perturbation of a linear problem. As in Section 1.1, the decisive role is played by the semigroup generated by the linear part of the problem.

Problems like (6.2.6) have been relatively well-researched, including the cases where μ and β are nonlinear functions depending on the total population, see [30, 58] and reference therein. Our model most resembles that discussed in [58], the main difference being that in *op. cit.* the maximum age ω is infinite which makes it plausible to assume that μ is bounded. However, a biologically realistic assumption is that $\omega < +\infty$ which, however, necessitates building into the model a mechanism ensuring that no individual can live beyond ω. It follows, e.g. [40], that the probability of survival of an individual till age a is given by

$$\Pi(a) = e^{-\int_0^a \mu(s)\,ds}.$$

Thus $\Pi(\omega) = 0$ which requires

$$\int_0^\omega \mu(s)\,ds = +\infty. \tag{6.2.7}$$

Hence, μ cannot be bounded as $a \to \omega^-$. This is in contrast with the case $\omega = +\infty$, where commonly it is assumed that μ is a bounded function on $\mathbb{R}_+$, and introduces another unbounded operator in the problem. We note that this difficulty was circumvented by Inaba in [41] by introducing the maximum reproduction age $a_\dagger < \omega$ and ignoring the evolution of the post-reproductive part of the population. Also, in papers such as [30], though $\omega < +\infty$, the assumption that the population is constant removes the death coefficient from the equation. The analysis of the model without any simplification in the scalar and linear case was done in [40] by reducing it to an integral equation along characteristics. It can be proved that the solution of such a problem is given by a strongly continuous semigroup. Here we shall prove this directly by refining the argument of [41].

2.1 Notation and assumptions

We will work in the space $\mathbf{X} = L_1\big([0,\omega],\mathbb{R}^2\big)$ with norm $\|(p_1,p_2)\|_{\mathbf{X}} = \|p_1\| + \|p_2\|$, where the norm $\|\cdot\|$ refers to the norm in $L_1([0,\omega])$; the relevant norm in $\mathbb{R}^2$ will be denoted by $|\cdot|$. We also introduce necessary assumptions (cf. [40]) on the coefficients of (6.2.6) where, in what follows, for any measurable function ϕ on $[0,\omega]$ we shall use the notation

$$\bar{\phi} = \operatorname*{ess\,sup}_{a\in[0,\omega]} \phi(a), \quad \underline{\phi} = \operatorname*{ess\,inf}_{a\in[0,\omega]} \phi(a). \tag{6.2.8}$$

(H1) $0 \leq \mu \in \mathrm{L}_{\infty,loc}\big([0,\omega)\big)$, satisfying (6.2.7), with $\underline{\mu} > 0$;

(H2) $0 \leq \beta \in L_\infty\big([0,\omega]\big)$;

(H3) $0 \leq \delta \in L_\infty\big([0,\omega]\big)$;

(H4) $0 \leq K \in \mathrm{L}^\infty\big([0,\omega]^2\big)$.

Let $\mathrm{W}_1^1\big([0,\omega],\mathbb{R}^2\big)$ be the Sobolev space of vector valued functions. Further, we define $\mathbf{S} = \operatorname{diag}\{-\partial_a, -\partial_a\}$ on $D(\mathbf{S}) = \mathrm{W}_1^1\big([0,\omega],\mathbb{R}^2\big)$, $\mathbf{M}_\mu(a) = \operatorname{diag}\{-\mu(a), -\mu(a)\}$ on $D(\mathbf{M}_\mu) = \{\boldsymbol{\varphi} \in \mathbf{X} : \mu\boldsymbol{\varphi} \in \mathbf{X}\}$,

$$\mathbf{M}_\delta(a) = \begin{pmatrix} 0 & \delta(a) \\ 0 & -\delta(a) \end{pmatrix}; \tag{6.2.9}$$

$\mathbf{M}_\delta \in \mathcal{L}(\mathbf{X})$. Further, for a fixed $q \in [0,1]$,

$$\mathbf{B}(a) = \begin{pmatrix} \beta(a) & (1-q)\beta(a) \\ 0 & q\beta(a) \end{pmatrix} \tag{6.2.10}$$

with

$$\boldsymbol{\mathcal{B}}\boldsymbol{\varphi} = \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\,da;$$

the operator $\boldsymbol{\mathcal{B}}$ is bounded. Moreover, we introduce the linear operator $\mathbf{A}_\mu$ defined on the domain

$$D(\mathbf{A}_\mu) = \{\boldsymbol{\varphi} \in D(\mathbf{S}) \cap D(\mathbf{M}_\mu);\ \boldsymbol{\varphi}(0) = \boldsymbol{\mathcal{B}}\boldsymbol{\varphi}\} \tag{6.2.11}$$

by

$$\mathbf{A}_\mu = \mathbf{S} + \mathbf{M}_\mu. \tag{6.2.12}$$

Let $\boldsymbol{\mathcal{Q}}$ be the linear operator defined on the domain $D(\boldsymbol{\mathcal{Q}}) = D(\mathbf{A}_\mu)$ by $\boldsymbol{\mathcal{Q}} = \mathbf{A}_\mu + \mathbf{M}_\delta$. Using this notation, we see that (6.2.6) can be rewritten in the following compact form

$$\begin{aligned}
\partial_t \mathbf{u} &= \mathbf{A}_\mu \mathbf{u} + \mathbf{M}_\delta \mathbf{u} + \mathfrak{F}(\mathbf{u}),\\
\mathbf{u}(0,t) &= \int_0^\omega \mathbf{B}(a)\mathbf{u}(a,t)\,da,\\
\mathbf{u}(a,0) &= \mathbf{u}_0(a) = \boldsymbol{\varphi}(a),
\end{aligned} \tag{6.2.13}$$

where $\mathbf{u} = (s,i)^T$ and $\mathfrak{F}$ is a nonlinear function defined by

$$\mathfrak{F}(\mathbf{u}) = \begin{pmatrix} -\Lambda(\cdot,i)s \\ \Lambda(\cdot,i)s \end{pmatrix}$$

with Λ defined by (2.2.14).

2.2 The linear part

To prove that (6.2.6) is well-posed in $\mathbf{X}$, first we consider the linear operator $\boldsymbol{\mathcal{Q}}$ on $D(\boldsymbol{\mathcal{Q}}) = D(\mathbf{A}_\mu)$.

Theorem 6.4. *The linear operator $\boldsymbol{\mathcal{Q}}$ generates a strongly continuous positive semigroup $(\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{Q}}}(t))_{t\geq 0}$ in $\mathbf{X}$.*

To carry out the proof of Theorem 6.4, it is sufficient to prove the generation result for $\mathbf{A}_\mu$ and use Theorem 5.3 (the Bounded Perturbation Theorem) to prove the generation for $\boldsymbol{\mathcal{Q}}$; then we use some other tools to show the positivity of the combined semigroup. In this setting, first we have

Theorem 6.5. *The linear operator $\mathbf{A}_\mu$ generates a strongly continuous positive semigroup $\left(\boldsymbol{\mathcal{T}}_{\mathbf{A}_\mu}(t)\right)_{t\geq 0}$ in $\mathbf{X}$ such that*

$$\|\boldsymbol{\mathcal{T}}_{\mathbf{A}_\mu}(t)\|_{\mathcal{L}(\mathbf{X})} \leq e^{(\overline{\beta}-\underline{\mu})t}. \tag{6.2.14}$$

We prove this theorem in a sequence of lemmas in which we construct and estimate the resolvent of $\mathbf{A}_\mu$. We begin with introducing the survival rate matrix $\mathbf{L}(a)$, which represents the survival rate function in a multi-state population. $\mathbf{L}(a)$ is a solution of the matrix differential equation:

$$\frac{d\mathbf{L}}{da}(a) = \mathbf{M}_\mu(a)\mathbf{L}(a), \quad \mathbf{L}(0) = \mathbf{I}, \tag{6.2.15}$$

where $\mathbf{I}$ denotes the 2×2 identity matrix. The solution of (6.2.15) is a diagonal matrix given by

$$\mathbf{L}(a) = e^{-\int_0^a \mu(r)\,dr}\mathbf{I}. \tag{6.2.16}$$

From the above formula, we see that $\mathbf{L}(a)$ is invertible for all $a \in [0,\omega)$; its inverse is denoted $\mathbf{L}^{-1}(a)$. The inverse satisfies

$$\frac{d\mathbf{L}^{-1}}{da}(a) = -\mathbf{M}_\mu(a)\mathbf{L}^{-1}(a), \quad \mathbf{L}^{-1}(0) = \mathbf{I}. \tag{6.2.17}$$

Hence, we can define the fundamental matrix $\mathbf{L}(a,b)$ by

$$\mathbf{L}(a,b) = \mathbf{L}(a)\mathbf{L}^{-1}(b).$$

Lemma 6.6. *If* $\lambda > \overline{\beta} - \underline{\mu}$, *then* $(\lambda\mathbf{I} - \mathbf{A}_\mu)^{-1}$ *is given by*

$$\begin{aligned}\boldsymbol{\varphi} &= (\lambda\mathbf{I} - \mathbf{A}_\mu)^{-1}\boldsymbol{\psi}\\ &= e^{-\lambda a}\mathbf{L}(a)\left(\mathbf{I} - \int_0^\omega e^{-\lambda\sigma}\mathbf{B}(\sigma)\mathbf{L}(\sigma)\,d\sigma\right)^{-1}\int_0^\omega \mathbf{B}(a)\int_0^a e^{\lambda(\sigma-a)}\mathbf{L}(a,\sigma)\boldsymbol{\psi}(\sigma)d\sigma da\\ &+ e^{-\lambda a}\mathbf{L}(a)\int_0^a e^{\lambda\sigma}\mathbf{L}^{-1}(\sigma)\boldsymbol{\psi}(\sigma)\,d\sigma. \end{aligned} \tag{6.2.18}$$

Proof. Let $\lambda > \overline{\beta} - \underline{\mu}$ and $\boldsymbol{\psi} \in \mathbf{X}$. A function $\boldsymbol{\varphi} \in D(\mathbf{A}_\mu)$ if and only if

$$\begin{aligned}\lambda\boldsymbol{\varphi}(a) + \frac{d}{da}\boldsymbol{\varphi}(a) - \mathbf{M}_\mu(a)\boldsymbol{\varphi}(a) &= \boldsymbol{\psi}(a),\\ \boldsymbol{\varphi}(0) = \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\,da, \qquad \mu\boldsymbol{\varphi} &\in \mathbf{X}.\end{aligned} \tag{6.2.19}$$

By Duhamel's formula, the first equation of (6.2.19) gives

$$\begin{aligned}\boldsymbol{\varphi}(a) &= e^{-\lambda a}\mathbf{L}(a)\boldsymbol{\varphi}(0) + \int_0^a e^{-\lambda(a-s)}\mathbf{L}(a,s)\boldsymbol{\psi}(s)\,ds,\\ &= e^{-\lambda a - \int_0^a \mu(r)\,dr}\boldsymbol{\varphi}(0) + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(r)\,dr}\boldsymbol{\psi}(s)\,ds,\end{aligned} \tag{6.2.20}$$

for some unspecified as yet initial condition $\boldsymbol{\varphi}(0)$. For a fixed $\boldsymbol{\varphi}(0)$, we denote by $\boldsymbol{\mathcal{R}}_{\boldsymbol{\varphi}(0)}(\lambda)\boldsymbol{\psi}$ the operator $\boldsymbol{\psi}(s) \to \boldsymbol{\varphi}$ the operator defined above; it is easy to see that

$$(\lambda I - \mathbf{S} - \mathbf{M}_\mu)\boldsymbol{\mathcal{R}}_{\boldsymbol{\varphi}(0)}(\lambda)\boldsymbol{\psi} = \boldsymbol{\psi}, \tag{6.2.21}$$

for a.a. $a \in [0, \omega)$. The unknown $\boldsymbol{\varphi}(0)$ can be determined from the boundary condition (6.2.19) by substituting (6.2.20); we get

$$\begin{aligned}\boldsymbol{\varphi}(0) = &\int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\mathbf{B}(a)\boldsymbol{\varphi}(0)\,da\\ &+ \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\mathbf{B}(a)\left(\int_0^a e^{\lambda s + \int_0^s \mu(r)\,dr}\boldsymbol{\psi}(s)\,ds\right)\,da.\end{aligned}$$

Since

$$\left|\int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\mathbf{B}(a)\,da\right| \le \overline{\beta}\int_0^\omega e^{-(\lambda+\underline{\mu})a}\,da \le \frac{\overline{\beta}}{\lambda + \underline{\mu}} < 1 \tag{6.2.22}$$

for $\lambda > \bar{\beta} - \underline{\mu}$, $\mathbf{I} - \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\mathbf{B}(a)\,da$ is invertible with the inverse

$$\left|\left(\mathbf{I} - \int_0^\omega e^{-\lambda s - \int_0^s \mu(r)\,dr}\mathbf{B}(s)\,ds\right)^{-1}\right| \le \frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu})}. \tag{6.2.23}$$

Hence

$$\begin{aligned}\boldsymbol{\varphi}(0) = &\left(\mathbf{I} - \int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\mathbf{B}(a)\,da\right)^{-1}\\ &\int_0^\omega e^{-\lambda a - \int_0^a \mu(r)\,dr}\mathbf{B}(a)\left(\int_0^a e^{\lambda s + \int_0^s \mu(r)\,dr}\boldsymbol{\psi}(s)\,ds\right)\,da\end{aligned}$$

and we can substitute that $\boldsymbol{\varphi}(0)$ in the operator $\boldsymbol{\mathcal{R}}_{\boldsymbol{\varphi}(0)}(\lambda)$ to define

$$\begin{aligned}
&\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}(a)\\
&= e^{-\lambda a-\int_0^a \mu(r)\,dr}\left(\mathbf{I}-\int_0^\omega e^{-\lambda s-\int_0^s \mu(r)\,dr}\mathbf{B}(s)\,ds\right)^{-1}\int_0^\omega e^{-\lambda a-\int_0^a \mu(r)\,dr}\mathbf{B}(a)\\
&\quad\times\int_0^a e^{\lambda s+\int_0^s \mu(r)\,dr}\boldsymbol{\psi}(s)\,ds\,da+e^{-\lambda a-\int_0^a \mu(r)\,dr}\int_0^a e^{\lambda s+\int_0^s \mu(r)\,dr}\boldsymbol{\psi}(s)\,ds.
\end{aligned}$$

The above calculations show that $\lambda-\mathbf{A}_\mu$ is one-to-one for $\lambda>\overline{\beta}-\underline{\mu}$. Routine calculations show that

$$\left\|\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}\right\|_{\mathbf{X}}\leq\frac{1}{\lambda-(\overline{\beta}-\underline{\mu})}\left\|\boldsymbol{\psi}\right\|_{\mathbf{X}}. \tag{6.2.24}$$

Thus $\boldsymbol{\mathcal{R}}(\lambda)$ is a bounded operator in $\mathbf{X}$.

Further, by lengthy calculations we also can show that

$$\int_0^\omega\left|\mu(a)\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}(a)\right|\,da\leq\left(1+\frac{\overline{\beta}}{\lambda-(\overline{\beta}-\underline{\mu})}\right)\left\|\boldsymbol{\psi}\right\|_{\mathbf{X}}.$$

Hence $\boldsymbol{\mathcal{R}}(\lambda)\mathbf{X}\subset D(\mathbf{M}_\mu)$. Further, since for any $\boldsymbol{\psi}\in\mathbf{X}$, $\boldsymbol{\varphi}=\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}$ satisfies

$$\lambda\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}+\frac{d}{da}\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}-\mathbf{M}_\mu\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}=\boldsymbol{\psi}$$

almost everywhere, we have

$$\frac{d}{da}\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}=\boldsymbol{\psi}-\lambda\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}+\mathbf{M}_\mu\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi},$$

where, by the above estimates, all terms on the right hand side are in $\mathbf{X}$. Hence $\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}\in \mathrm{W}_1^1\big([0,\omega],\mathbb{R}^2\big)$ and, consequently, $\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}\in D(\mathbf{A}_\mu)$.

Since the boundary condition holds, using the results above we see that the operator $\boldsymbol{\mathcal{R}}(\lambda)$, given by $(\lambda I-\mathbf{A}_\mu)\boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{\psi}=\boldsymbol{\psi}$, is such that maps $\boldsymbol{\mathcal{R}}(\lambda):\mathbf{X}\to D(\mathbf{A}_\mu)$. Then $\boldsymbol{\mathcal{R}}(\lambda)$ is a right-inverse of the operator $\lambda I-\mathbf{A}_\mu$.

Summarizing, $\boldsymbol{\mathcal{R}}(\lambda)$ is the right inverse of $(\lambda I-\mathbf{A}_\mu,D(\mathbf{A}_\mu))$. To prove that it is also a left inverse, we repeat the standard argument. Assume that for some $\boldsymbol{\varphi}\in D(\mathbf{A}_\mu)$ we have

$$\boldsymbol{\mathcal{R}}(\lambda)(\lambda I-\mathbf{A}_\mu)\boldsymbol{\varphi}=\widetilde{\boldsymbol{\varphi}}\neq\boldsymbol{\varphi}.$$

Since $\boldsymbol{\mathcal{R}}(\lambda):\mathbf{X}\to D(\mathbf{A}_\mu)$, we can write

$$(\lambda I-\mathbf{A}_\mu)\boldsymbol{\varphi}=(\lambda I-\mathbf{A}_\mu)\boldsymbol{\mathcal{R}}(\lambda)(\lambda I-\mathbf{A}_\mu)\boldsymbol{\varphi}=(\lambda I-\mathbf{A}_\mu)\widetilde{\boldsymbol{\varphi}}$$

since $\boldsymbol{\mathcal{R}}(\lambda)$ is a right inverse of $\lambda I-\mathbf{A}_\mu$. But we proved that the linear operator $(\lambda I-\mathbf{A}_\mu)$ is one-to-one for $\lambda>\overline{\beta}-\underline{\mu}$, $\boldsymbol{\varphi}=\widetilde{\boldsymbol{\varphi}}$ and hence $\boldsymbol{\mathcal{R}}(\lambda)=(\lambda I-\mathbf{A}_\mu)^{-1}$. □

Lemma 6.7. $\overline{D(\mathbf{A})_+}=\mathbf{X}_+$.

Proof. A proof of this result (with some gaps) is provided in [41, p. 60]. A more comprehensive proof can be found in [58]. We present a simpler proof which, moreover, allows for the approximation of $\boldsymbol{f}\in\mathbf{X}_+$ by elements of $D(\mathbf{A}_\mu)_+$.

Fix $\boldsymbol{f}\in\mathbf{X}_+$. First we note that for any given ϵ there is $0\leq\boldsymbol{\phi}\in C_0^\infty((0,\omega),\mathbb{R}^2)$ such that $\|\boldsymbol{f}-\boldsymbol{\phi}\|_{\mathbf{X}}\leq\epsilon$. Clearly, $\boldsymbol{\phi}\in D(\mathbf{M}_\mu)$ but typically

$$\boldsymbol{\varphi}(0)\neq\int_0^\omega\mathbf{B}(a)\boldsymbol{\varphi}(a)\,da.$$

Take a function $0 \leq \eta \in C_0^\infty([0,\omega))$ with $\eta(0) = 1$ and let $\eta_\epsilon(a) = \eta(a/\epsilon)$. Further, let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ be a vector and consider

$$\boldsymbol{\psi} = \boldsymbol{\varphi} + \eta_\epsilon \boldsymbol{\alpha}.$$

Clearly, $\boldsymbol{\psi} \in W_1^1([0,\omega], \mathbb{R}^2) \cap D(\mathbf{M}_\mu)$. As far as the boundary condition is concerned we have, by the properties of the involved functions,

$$\boldsymbol{\alpha} = \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\, da + \left(\int_0^{\epsilon\omega} \begin{pmatrix} \beta(a)\eta_\epsilon(a) & (1-q)\beta(a)\eta_\epsilon(a) \\ 0 & q\beta(a)\eta_\epsilon(a) \end{pmatrix} da \right) \boldsymbol{\alpha}. \tag{6.2.25}$$

Now, since

$$0 \leq \int_0^{\epsilon\omega} \beta(a)\eta_\epsilon(a) da = \epsilon \int_0^{\omega} \beta(\epsilon s)\eta(s) ds \leq \epsilon\bar{\beta},$$

the matrix l_1-norm satisfies

$$\left\| \left(\int_0^{\epsilon\omega} \begin{pmatrix} \beta(a)\eta_\epsilon(a) & (1-q)\beta(a)\eta_\epsilon(a) \\ 0 & q\beta(a)\eta_\epsilon(a) \end{pmatrix} da \right) \right\|_{\mathcal{L}(\mathbb{R}^2)} \leq \epsilon\bar{\beta}.$$

Thus, (6.2.25) is solvable for sufficiently small ϵ. Further, by the positivity of the above matrix and the properties of the Neumann series expansion, $\boldsymbol{\alpha}$ is nonnegative and

$$|\boldsymbol{\alpha}| \leq \left| \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\, da \right| (1 - \epsilon\bar{\beta})^{-1} \leq C$$

for some constant C, which is independent of ϵ for sufficiently small ϵ (the norm of $\|\boldsymbol{\varphi}\|$, which depends on ϵ, can be bounded by e.g. $\|\boldsymbol{f}\| + 1$ for $\epsilon < 1$). Hence we have

$$\|\boldsymbol{f} - \boldsymbol{\psi}\|_{\mathbf{X}} \leq \|\boldsymbol{f} - \boldsymbol{\varphi}\|_{\mathbf{X}} + |\boldsymbol{\alpha}|\|\eta_\epsilon\| \leq (1 + C)\epsilon.$$

Proof of Theorem 6.5. Since the inverse of a bounded operator is closed, we see that $\lambda I - \mathbf{A}_\mu$, and hence $\mathbf{A}_\mu$, are closed. Thus the above lemmas with the estimate (6.2.24) show that $\mathbf{A}_\mu$ satisfies the assumptions of the Hille–Yosida theory. Hence, it generates a semigroup satisfying (6.2.14). Since the resolvent is positive, the semigroup is positive as well. □

Proof of Theorem 6.4. Since $\mathbf{M}_\delta \in \mathcal{L}(\mathbf{X})$, with $\|\mathbf{M}_\delta(a)\|_{\mathcal{L}(\mathbf{X})} \leq 2\bar{\delta}$, Theorem 5.3 (the Bounded Perturbation Theorem) is applicable and states that the linear operator $(\boldsymbol{\mathcal{Q}}, D(\mathbf{A}))$ generates a strongly continuous semigroup denoted by $\left(\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{Q}}}(t)\right)_{t\geq 0}$. Using the estimate (6.2.14) we have:

$$\|\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{Q}}}(t)\|_{\mathcal{L}(\mathbf{X})} \leq e^{t(\bar{\beta} - \underline{\mu} + 2\bar{\delta})}.$$

Using the structure of $\mathbf{M}_\delta$ we can improve this estimate and also show that the semigroup $\{\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{Q}}}(t)\}_{t\geq 0}$ is positive. Since the the variable a plays in $\mathbf{M}_\delta$ the role of a parameter, we find

$$\boldsymbol{\mathcal{T}}_{\mathbf{M}_\delta}(t) = \begin{pmatrix} 1 & 1 - e^{-t\delta(a)} \\ 0 & e^{-t\delta(a)} \end{pmatrix}$$

and so

$$\|\boldsymbol{\mathcal{T}}_{\mathbf{M}_\delta}(t)\|_{\mathcal{L}(\mathbf{X})} = 1.$$

Also, $\{\boldsymbol{\mathcal{T}}_{\mathbf{M}_\delta}(t)\}_{t\geq 0}$ is positive. Hence, by (5.1.7), we obtain

$$\|\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{Q}}}(t)\|_{\mathcal{L}(\mathbf{X})} \leq e^{t(\bar{\beta} - \underline{\mu})} \tag{6.2.26}$$

and $\{\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{Q}}}(t)\}_{t\geq 0}$ is positive. □

Remark 6.8. The estimates (6.2.14) and (6.2.26) are not optimal. In fact, for the scalar problem

$$\begin{aligned}
\partial_t n(a,t) &= -\partial_a n(a,t) - \mu(a)n(a,t), \quad t>0, a\in(0,\omega)\\
n(0,t) &= \int\limits_0^\omega \beta(a)n(a,t)da,\\
n(a,0) &= \overset{\circ}{n}(a),
\end{aligned} \tag{6.2.27}$$

it can be proved, [40], that there is a unique dominant eigenvalue λ^* of the problem, which is the solution of the renewal equation

$$1 = \int\limits_0^\omega \beta(a)e^{-\lambda a - \int\limits_0^a \mu(s)ds} da, \tag{6.2.28}$$

such that

$$\|n(t)\| \le Me^{t\lambda^*} \tag{6.2.29}$$

for some constant M. This eigenvalue is, respectively, positive, zero or negative if and only if the basic reproduction number

$$R = \int\limits_0^\omega \beta(a)e^{-\int\limits_0^a \mu(s)ds} da \tag{6.2.30}$$

is bigger, equal, or smaller, than 1.

Consider now an initial condition $(\overset{\circ}{s}, \overset{\circ}{i}) \in D(\mathbf{A}_\mu)$. Since the semigroup $\{\boldsymbol{\mathcal{T}}_{\mathfrak{Q}}(t)\}_{t\ge 0}$ is positive, the strict solution (s,i) of the linear part of (6.2.6) is nonnegative and the total population $0 \le s(a,t)+i(a,t) = n(a,t)$ satisfies (6.2.27). Using nonnegativity, we find $s(a,t) \le n(a,t)$ and $i(a,t) \le n(a,t)$ and consequently

$$\|\boldsymbol{\mathcal{T}}_{\mathfrak{Q}}(t)(\overset{\circ}{s}, \overset{\circ}{i})\|_{\mathbf{X}} \le Me^{t\lambda^*}\|(\overset{\circ}{s}, \overset{\circ}{i})\|_{\mathbf{X}}$$

for $(\overset{\circ}{s}, \overset{\circ}{i}) \in D(\mathbf{A}_\mu)_+$. However, by Lemma 6.7, the above estimate can be extended to $\mathbf{X}_+$ and, by (4.1.5), to $\mathbf{X}$.

Note that the crucial role in the above argument is played by the fact that (s,i) satisfies the differential equation (6.2.6) — if it was only a mild solution, it would be difficult to directly prove that the sum $s+i$ is the mild solution to (6.2.27).

2.3 The nonlinear problem

In the case of intercohort transmission, discussed in these lectures, individuals of any age can infect individuals of any age, though with possibly different intensity. Then

$$\Lambda(a,i) = \int_0^\omega K(a,a')i(a')\,da', \tag{6.2.31}$$

where $K(a,a')$ is a nonnegative bounded function on $[0,\omega]\times[0,\omega]$ which accounts for the age-specific probability of becoming infected through contact with infectives of a particular age.

Since the nonlinear term $\mathfrak{F}$ is quadratic, standard calculations, see [51], give

Proposition 6.9. *$\mathfrak{F}$ is continuously Fréchet differentiable with respect to $\boldsymbol{\phi} \in \mathbf{X}$ and for any $\boldsymbol{\phi} = (\phi^s, \phi^i)$, $\boldsymbol{\psi} = (\psi^s, \psi^i) \in \mathbf{X}$ the Fréchet derivative at $\boldsymbol{\phi}$, $\mathfrak{F}_{\boldsymbol{\phi}}$, is defined by*

$$\left(\mathfrak{F}_{\boldsymbol{\phi}}\boldsymbol{\psi}\right)(a) := \begin{pmatrix} -\psi^s(a)\int\limits_0^\omega K(a,a')\phi^i(a')\,da' - \phi^s(a)\int\limits_0^\omega K(a,a')\psi^i(a')\,da' \\ \\ \psi^s(a)\int\limits_0^\omega K(a,a')\phi^i(a')\,da' + \phi^s(a)\int\limits_0^\omega K(a,a')\psi^i(a')\,da' \end{pmatrix}.$$

Hence we can apply Theorem 6.3 to claim that for each $\overset{\circ}{\mathbf{u}}= (\overset{\circ}{s}, \overset{\circ}{i}) \in \mathbf{X}$, there is a $t(\overset{\circ}{\mathbf{u}})$ such that the problem (6.2.13) has a unique mild solution on $[0, t(\overset{\circ}{\mathbf{u}})[\ni t \to \mathbf{u}(t)$; this solution is strict if $\overset{\circ}{\mathbf{u}}\in D(\mathbf{A}_\mu)$.

We recall that that the proof consists of showing that the Picard iterates

$$\begin{aligned} \mathbf{u}_0 &= \overset{\circ}{\mathbf{u}} \\ \mathbf{u}_n(t) &= \mathcal{T}_{\mathcal{Q}}(t)\, \overset{\circ}{\mathbf{u}} + \int\limits_0^t \mathcal{T}_{\mathcal{Q}}(t-s)\mathfrak{F}(\mathbf{u}_{n-1}(s))ds \end{aligned} \tag{6.2.32}$$

converge in $C([0, t(\overset{\circ}{\mathbf{u}})[, B(\overset{\circ}{\mathbf{u}}, \rho))$ where $B(\overset{\circ}{\mathbf{u}}, \rho) = \{\mathbf{u} \in \mathbf{X};\ \|\mathbf{u}- \overset{\circ}{\mathbf{u}}\|_{\mathbf{X}} \leq \rho\}$ for some constant ρ. Since the nonlinearity is quadratic, it is not globally Lipschitz continuous and thus the question whether this solution can be extended to $[0, \infty[$ requires employing positivity techniques.

Since $\mathfrak{F}$ is not positive on $\mathbf{X}_+$, we cannot claim that the constructed local solution is nonnegative, as the iterates defined by (6.2.32) need not be positive even if we start with $\overset{\circ}{\mathbf{u}}\geq 0$. Hence, we re-write (6.2.13) in the equivalent form

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} = (\mathcal{Q} - \kappa I)\,\mathbf{u} + (\kappa I + \mathfrak{F})\,\big(\mathbf{u}\big), \quad t > 0, \\ \mathbf{u}(0) = \overset{\circ}{\mathbf{u}}, \end{cases} \tag{6.2.33}$$

for some $\kappa \in \mathbb{R}_+$ to be determined. Denote $\mathcal{Q}_\kappa = \mathcal{Q} - \kappa I$; then $\{\mathcal{T}_{\mathcal{Q},\kappa}(t)\}_{t\geq 0} = (e^{-\kappa t}\mathcal{T}_{\mathcal{Q}})_{t\geq 0}$ and hence $\{\mathcal{T}_{\mathcal{Q},\kappa}(t)\}_{t\geq 0}$ is positive. It is also easy to see that the following result holds.

Lemma 6.10. *For any ρ there exists κ such that $\big(\kappa\mathbf{I} + \mathfrak{F}\big)(\mathbf{X}_+ \cap B(\overset{\circ}{\mathbf{u}}, \rho)) \subset \mathbf{X}_+$.*

Then the Picard iterates corresponding to (6.2.33),

$$\mathbf{u}(t) = e^{-\kappa t}\mathcal{T}(t)\, \overset{\circ}{\mathbf{u}} + \int_0^t e^{-\kappa(t-s)}\mathcal{T}_{\mathcal{Q}}(t-s)\big(\kappa I + \mathfrak{F}\big)(\mathbf{u}(s))\, ds, \quad 0 \leq t < t(\overset{\circ}{\mathbf{u}}),$$

are nonnegative and we can repeat the standard estimates to arrive at

Theorem 6.11. *Assume that $\overset{\circ}{\mathbf{u}}\in \mathbf{X}_+$ and let $\mathbf{u} : [0, t(\overset{\circ}{\mathbf{u}})[\to \mathbf{X}$ be the unique mild solution of (6.2.13). Then this solution is nonnegative on the maximal interval of its existence. Moreover, the solutions continuously depend on the initial conditions on every compact subinterval of their joint interval of existence.*

2.4 Global existence

Since quadratic nonlinearities do not satisfy the uniform Lipschitz condition, we cannot immediately claim that the solutions to (6.2.13) are global in time. In fact, it is well known that, even for ordinary differential equations, the solution with a quadratic nonlinearity can blow up in a finite time. Here, we use positivity to show that positive solutions exist globally in time. For this, we have to show that $t \to \|\mathbf{u}(t)\|_{\mathbf{X}}$ does not blow up in finite time. We state the following result:

Theorem 6.12. *For any $\overset{\circ}{\mathbf{u}}\in D(\mathbf{A}_\mu) \cap \mathbf{X}_+$, the problem (6.2.13) has a unique strict positive solution $\mathbf{u}(t)$ defined on the whole time interval $[0, \infty[$.*

Proof. The proof uses the ideas of Remark 6.8. Under the adopted assumptions, we have a positive strict solution $\mathbf{u}(t) = \big(s(t), i(t)\big)$ to (6.2.6) in $L_1([0, \omega], \mathbb{R}^2)$ defined on its maximum interval of existence $[0, t_{max}[$. But then

$$\|\mathbf{u}(t)\|_{\mathbf{X}} = \int_0^\omega (s(a,t) + i(a,t))\, da = \int_0^\omega u(a,t)\, da, \quad t \in [0, t_{max}[,$$

where $u(a,t)$ is the solution to the McKendrick equation (6.2.27). But then, as long as $0 \leq t < t(\mathring{\mathbf{u}})$,

$$\|\mathbf{u}(t)\|_{\mathbf{X}} \leq e^{(\overline{\beta}-\underline{\mu})t}\| \mathring{\mathbf{u}} \|_{\mathbf{X}}.$$

Hence $\|\mathbf{u}(t)\|_{\mathbf{X}}$ does not blow up in finite time and the solution is global. □

Corollary 6.13. *For any* $\mathring{\mathbf{u}} \in \mathbf{X}_+$, *the problem (6.2.13) has a unique mild positive solution* $\mathbf{u}(t)$ *defined on the whole time interval* $[0, \infty[$.

The proof follows from the fact that $D(\mathbf{A}_\mu)_+$ is dense in $\mathbf{X}_+$ and the continuous dependence on initial conditions.

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