# INTRODUCTION TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS 

J. Banasiak<br>School of Mathematical Sciences<br>University of KwaZulu-Natal, Durban, South Africa

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## Chapter 1

## Origins of partial differential equations

## 1 Basic facts from Calculus

In the course we shall frequently need several facts from the integration theory. We list them here as theorems, though we shall not prove them during the lecture. Some easier proofs should be done as exercises.

Theorem 1.1 Let $f$ be a continuous function in $\bar{\Omega}$, where $\Omega \subset \mathbb{R}^{d}$ is a bounded domain. Assume that $\forall \boldsymbol{x}_{\in \bar{\Omega}} f(\boldsymbol{x}) \geq 0$ and $\int_{\Omega} f(\boldsymbol{x}) d \boldsymbol{x}=0$. Then $f \equiv 0$ in $\bar{\Omega}$.

Another "vanishing function" theorem reads as follows.

Theorem 1.2 Let $f$ be a continuous function in a domain $\Omega$ such that $\int_{\Omega_{0}} f(\boldsymbol{x}) d \boldsymbol{x}=0$ for any $\Omega_{0} \subset \Omega$. Then $f \equiv 0$ in $\Omega$.

In the proofs of both theorems the crucial role is played by the theorem on local sign preservation by continuous functions: if $f$ is continuous at $\boldsymbol{x}_{0}$ and $f\left(\boldsymbol{x}_{0}\right)>0$, then there exists a ball centered at $\boldsymbol{x}_{0}$ such $f(\boldsymbol{x})>0$ for $\boldsymbol{x}$ from this ball.

Next we shall consider perhaps the most important theorem of multidimensional integral calculus which is Green's-Gauss'-Divergence-.... Theorem. Before entering into details, however, we shall devote some time to the geometry of domains in space.
In one-dimensional space $\mathbb{R}^{1}$ typical sets with which we will be concerned are open intervals $] a, b[$, where $-\infty \leq a<b \leq+\infty$. For $-\infty<a<b<+\infty$, by $[a, b]$ we will denote the closed interval with endpoints $a, b$. In this case, we say that $] a, b[$ is the interior of the interval, $[a, b]$ is its closure and the two-point set consisting of $\{a\}$ and $\{b\}$ constitutes the boundary.
In general, for a set $\Omega$, we denote by $\partial \Omega$ its boundary.
The situation in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ is much more complicated. Let us consider first the two-dimensional situation. Then, in most cases, the boundary $\partial \Omega$ of a two-dimensional region $\Omega$ is a closed curve. The two most used analytic descriptions of curves in $\mathbb{R}^{2}$ are:
a) as a level curve of a function of two variables

$$
F\left(x_{1}, x_{2}\right)=c,
$$

b) using two functions of a single variable

$$
\begin{aligned}
& x_{1}(t)=f(t), \\
& x_{2}(t)=g(t),
\end{aligned}
$$

where $t \in\left[t_{0}, t_{1}\right]$ (parametric description). Note that since the curve is to be closed, we must have $f\left(t_{0}\right)=f\left(t_{1}\right)$ and $g\left(t_{0}\right)=g\left(t_{1}\right)$.

In many cases the boundary is composed of a number of arcs so that it is impossible to give a single analytical description applicable to the whole boundary.

Example 1.1 Let us consider the elliptical region $x_{1}^{2} / a^{2}+x_{2}^{2} / b^{2} \leq 1$. The boundary is then the ellipse

$$
\frac{x_{1}^{2}}{a^{2}}+\frac{x_{2}^{2}}{b^{2}}=1
$$

This is the description of the curve (ellipse) as a level curve of the function $F\left(x_{1}, x_{2}\right)=x_{1}^{2} / a^{2}+x_{2}^{2} / b^{2}$ (with the constant $c=1$ ). Equivalently, the boundary can be written in parametric form as

$$
x_{1}(t)=a \cos t, \quad x_{2}(t)=b \sin t
$$

with $t \in[0,2 \pi]$.

In three dimensions the boundary of a solid $\Omega$ is a two-dimensional surface. This surface can be analytically described as a level surface of a function of three variables

$$
F\left(x_{1}, x_{2}, x_{3}\right)=c,
$$

or parametrically by, this time, three functions of two variables each:

$$
\begin{aligned}
& x_{1}(t, s)=f(t, s), \\
& x_{2}(t, s)=g(t, s), \\
& x_{3}(t, s)=h(t, s), \quad t \in\left[t_{0}, t_{1}\right], s \in\left[s_{0}, s_{1}\right] .
\end{aligned}
$$

As in two dimensions, it is possible that the boundary is made up of several patches having different analytic descriptions.

Example 1.2 Consider the domain $\Omega$ bounded by the ellipsoid

$$
\frac{x_{1}^{2}}{a^{2}}+\frac{x_{2}^{2}}{b^{2}}+\frac{x_{3}^{2}}{c^{2}}=1
$$

The boundary here is directly given as the level surface of the function $F\left(x_{1}, x_{2}, x_{3}\right)=x_{1}^{2} / a^{2}+x_{2}^{2} / b^{2}+x_{3}^{2} / c^{2}$ :

$$
F\left(x_{1}, x_{2}, x_{3}\right)=1 .
$$

The same boundary can be described parametrically as $\boldsymbol{r}(t, s)=(f(t, s), g(t, s), h(t, s))$, where

$$
\begin{aligned}
f(t, s) & =a \cos t \sin s \\
g(t, s) & =b \sin t \sin s \\
h(t, s) & =c \cos s
\end{aligned}
$$

where $t \in[0,2 \pi], s \in[0, \pi]$.

One of the most important concepts in partial differential equations is that of the unit outward normal vector to the boundary of the set. For a given point $\boldsymbol{p} \in \partial \Omega$ this is the vector $\boldsymbol{n}$, normal (perpendicular) to the boundary at $p$, pointing outside $\Omega$, and having unit length.

If the boundary of (two or three dimensional) set $\Omega$ is given as a level curve of a function $F$, then the vector given by

$$
\boldsymbol{N}(\boldsymbol{p})=\left.\nabla F\right|_{\boldsymbol{p}}
$$

is normal to the boundary at $\boldsymbol{p}$. However, it is not necessarily unit, nor outward. To make it a unit vector, we divide $N$ by its length; then the unit outward normal is either $n=N /\|N\|$, or $n=-N /\|N\|$ and the proper sign must be selected by inspection.

Example 1.3 Find the unit outward normal to the ellipsoid

$$
x_{1}^{2}+\frac{x_{2}^{2}}{4}+\frac{x_{3}^{2}}{9}=1
$$

at the point $\boldsymbol{p}=(1 / \sqrt{2}, 0,3 / \sqrt{2})$.
Since $F\left(x_{1}, x_{2}, x_{3}\right)=x_{1}^{2}+\frac{x_{2}^{2}}{4}+\frac{x_{3}^{2}}{9}$, we obtain $\nabla F=\left(2 x_{1}, x_{2} / 2,2 x_{3} / 9\right)$. Therefore

$$
\boldsymbol{N}(\boldsymbol{p})=\left.\nabla F\right|_{\boldsymbol{p}}=(2 / \sqrt{2}, 0,2 / 3 \sqrt{2})
$$

Furthermore

$$
\|\boldsymbol{N}(\boldsymbol{p})\|=\sqrt{2+2 / 9}=\sqrt{20 / 9}=2 \sqrt{5} / 3
$$

thus

$$
\boldsymbol{n}(\boldsymbol{p})= \pm(3 / \sqrt{10}, 0,1 / \sqrt{10})
$$

To select the proper sign let us observe that the vector pointing outside the ellipsoid must necessarily point away from the origin. Since the coordinates of the point $\boldsymbol{p}$ are nonnegative, a vector rooted at this point and pointing away from the origin must have positive coordinates, thus finally

$$
\boldsymbol{n}(\boldsymbol{p})=(3 / \sqrt{10}, 0,1 / \sqrt{10})
$$

If the boundary is given in a parametric way, then the situation is more complicated and we have to distinguish between dimensions 2 and 3 .

Let us first consider the boundary $\partial \Omega$ of a two-dimensional domain $\Omega$, described by $\boldsymbol{r}(t)=\left(x_{1}(t), x_{2}(t)\right)=$ $(f(t), g(t))$. It is known that the derivative vector

$$
\frac{d}{d t} \boldsymbol{r}\left(t_{p}\right)=\left(f^{\prime}\left(t_{p}\right), g^{\prime}\left(t_{p}\right)\right)
$$

is tangent to $\partial \Omega$ at $\boldsymbol{p}=\left(f\left(t_{p}\right), g\left(t_{p}\right)\right)$. Since any normal vector at $\boldsymbol{p} \in \Omega$ is perpendicular to any tangent vector at this point, we immediately obtain that

$$
\begin{equation*}
\boldsymbol{N}(\boldsymbol{p})=\left(-g^{\prime}\left(t_{p}\right), f^{\prime}\left(t_{p}\right)\right) \tag{1.1.1}
\end{equation*}
$$

Therefore the unit outward normal is given by

$$
\boldsymbol{n}(p)= \pm \frac{\boldsymbol{N}(\boldsymbol{p})}{\|\boldsymbol{N}(\boldsymbol{p})\|}
$$

where the sign must be decided by inspection so that $\boldsymbol{n}$ points outside $\Omega$.
If the domain $\Omega$ is 3 -dimensional, then its boundary $\partial \Omega$ is a surface described by the 3 -dimensional vector function of 2 variables:

$$
\boldsymbol{r}(t, s)=\left(x_{1}(t, s), x_{2}(t, s), x_{3}(t, s)\right)=(f(t, s), g(t, s), h(t, s)) .
$$

Fig. 3.1 Domain $\Omega$ with boundary $\partial \Omega$ showing a surface element $d S$ with the outward normal $\mathbf{n}(\mathbf{x})$ and flux $\phi(\mathbf{x}, t)$ at point $\mathbf{x}$ and time $t$

In this case, at each point $\partial \Omega \ni \boldsymbol{p}=\boldsymbol{r}\left(t_{p}\right)$, we have two derivative vectors $\boldsymbol{r}_{s}^{\prime}\left(t_{p}\right)$ and $\boldsymbol{r}_{t}^{\prime}\left(t_{p}\right)$ which span the two dimensional tangent plane to $\partial \Omega$ at $\boldsymbol{p}$. Any normal vector must be thus perpendicular to both these vectors and the easiest way to find such a vector is to use the cross-product:

$$
\boldsymbol{N}(\boldsymbol{p})=\boldsymbol{r}_{t}^{\prime}\left(t_{p}\right) \times \boldsymbol{r}_{s}^{\prime}\left(t_{p}\right)
$$

Again, the unit outward normal is given by

$$
\boldsymbol{n}(\boldsymbol{p})= \pm \frac{\boldsymbol{N}(\boldsymbol{p})}{\|\boldsymbol{N}(\boldsymbol{p})\|}= \pm \frac{\boldsymbol{r}_{t}^{\prime}\left(t_{p}\right) \times \boldsymbol{r}_{s}^{\prime}\left(t_{p}\right)}{\left\|\boldsymbol{r}_{t}^{\prime}\left(t_{p}\right) \times \boldsymbol{r}_{s}^{\prime}\left(t_{p}\right)\right\|}
$$

where the sign must be decided by inspection.
Example 1.4 Find the outward unit normal to the ellipse $\boldsymbol{r}(t)=(\cos t, 2 \sin t)$ at the point $\boldsymbol{p}=\boldsymbol{r}(\pi / 4)$ Differentiating, we obtain $\boldsymbol{r}_{t}^{\prime}=(-\sin t, 2 \cos t)$; this is a tangent vector to ellipse at $\boldsymbol{r}(t)$. Thus,

$$
\boldsymbol{N}=(-2 \cos t,-\sin t)
$$

Next, $\|\boldsymbol{N}\|=\sqrt{4 \cos ^{2} t+\sin ^{2} t}$ and

$$
\boldsymbol{n}= \pm \frac{1}{\sqrt{4 \cos ^{2} t+\sin ^{2} t}}(2 \cos t, \sin t)
$$

At the particular point $\boldsymbol{p}$ we have $\cos \pi / 2=\sin \pi / 2=\frac{\sqrt{2}}{2}$, thus $\|\boldsymbol{N}\|=\sqrt{5 / 2}$ and

$$
\boldsymbol{n}(\boldsymbol{p})= \pm 2 / \sqrt{5}(1,1 / 2)
$$

Since the normal must point outside the ellipse, we must chose the " + " sign and finally

$$
\boldsymbol{n}(\boldsymbol{p})=2 / \sqrt{5}(1,1 / 2)
$$

Another important concept related to the normal is the normal derivative of a function. Let us recall that if $\boldsymbol{u}$ is any unit vector and $f$ is a function, then the derivative of $f$ at a point $\boldsymbol{p}$ in the direction of $\boldsymbol{u}$ is defined as

$$
f \boldsymbol{u}(\boldsymbol{p})=\lim _{t \rightarrow 0^{+}} \frac{f(\boldsymbol{p}+t \boldsymbol{u})-f(\boldsymbol{p})}{t} .
$$

Application of the Chain Rule produces the following handy formula for the directional derivative:

$$
f_{\boldsymbol{u}}(\boldsymbol{p})=\left.\nabla f\right|_{\boldsymbol{p}} \cdot \boldsymbol{u}
$$

Let now $f$ be defined in a neighbourhood of a point $\boldsymbol{p} \in \partial \Omega$. The normal derivative of $f$ at $\boldsymbol{p}$ is defined as the derivative of $f$ in the direction of $\boldsymbol{n}(\boldsymbol{p})$ :

$$
\frac{\partial f}{\partial n}(\boldsymbol{p})=f_{\boldsymbol{n}}(\boldsymbol{p})=\left.\nabla f\right|_{\boldsymbol{p}} \cdot \boldsymbol{n}(\boldsymbol{p})
$$

Example 1.5 Let us consider the spherical coordinates

$$
\begin{aligned}
& x_{1}=r \cos \theta \sin \phi, \\
& x_{2}=r \sin \theta \sin \phi, \\
& x_{3}=r \cos \phi
\end{aligned}
$$

and let $f\left(x_{1}, x_{2}, x_{3}\right)$ be a function of three variables. This function can be expressed in the spherical coordinates as the function of $(r, \theta, \phi)$

$$
F(r, \theta, \phi)=f(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi)=f\left(x_{1}, x_{2}, x_{3}\right)
$$

Using the Chain Rule we have

$$
\frac{\partial F}{\partial r}=\frac{\partial f}{\partial x_{1}} \frac{\partial x_{1}}{\partial r}+\frac{\partial f}{\partial x_{2}} \frac{\partial x_{2}}{\partial r}+\frac{\partial f}{\partial x_{3}} \frac{\partial x_{3}}{\partial r}
$$

Since, for $i=1,2,3, \partial x_{i} / \partial r=x_{i} / r$, we can write

$$
\begin{equation*}
\frac{\partial F}{\partial r}=\frac{1}{r} \nabla f \cdot \boldsymbol{r} . \tag{1.1.2}
\end{equation*}
$$

Assume now that $f$ (and thus $F$ ) be given in some neighbourhood of the sphere

$$
F\left(x_{1}, x_{2}, x_{3}\right)=x_{1}^{2}+x_{2}^{2}+x_{3}^{2}=R^{2} .
$$

To find the outward unit normal to this sphere we note that $\nabla F=\left(2 x_{1}, 2 x_{2}, 2 x_{3}\right)$ and $\|\nabla F\|=2 \sqrt{x_{1}^{2}+x_{2}^{2}+x_{3}^{2}}=$ $2 R$. Thus,

$$
\begin{equation*}
\boldsymbol{n}=\frac{1}{R}\left(x_{1}, x_{2}, x_{3}\right)=\frac{1}{R} \boldsymbol{r} . \tag{1.1.3}
\end{equation*}
$$

Geometrically, $\boldsymbol{n}$ is parallel to the radius of the sphere but has unit length.
Combining (1.1.2) with (1.1.3), we see that the normal derivative of $f$ at any point of the sphere is given by

$$
\frac{\partial f}{\partial n}=\nabla f \cdot \boldsymbol{n}=\frac{\partial F}{\partial r}
$$

In other words, the normal derivative of any function at the surface of a sphere is equal to the derivative of this function (expressed in spherical coordinates) with respect to $r$.

Next we shall discuss yet another important concept: the flux of a vector field. Let us recall that a vector field is a function $\boldsymbol{f}: \Omega \rightarrow \mathbb{R}^{d}$, where $\Omega \subset \mathbb{R}^{d}$, where $d=1,2,3 \ldots$. In other words, a vector field assigns a vector to each point of a subset of the space.

Definition 1.1 The flux of the vector field $\boldsymbol{f}$ across the boundary $\partial \Omega$ of a domain $\Omega \subset \mathbb{R}^{d}, d \geq 2$ is

$$
\int_{\partial \Omega} \boldsymbol{f} \cdot \boldsymbol{n} d \sigma
$$

Here, if $d=2$, then $\partial \Omega$ is a closed curve and the integral above is the line integral (of the second kind). The arc length element $d \sigma$ is to be calculated according to the description of $\partial \Omega$. The easiest situation occurs if $\partial \Omega$ is described in a parametric form by $\boldsymbol{r}(t)=(f(t), g(t)), t \in\left[t_{0}, t_{1}\right]$; then $d \sigma=\sqrt{\left(f^{\prime}\right)^{2}+\left(g^{\prime}\right)^{2}} d t$ and

$$
\int_{\partial \Omega} \boldsymbol{f} \cdot \boldsymbol{n} d \sigma=\int_{t_{0}}^{t_{1}} \boldsymbol{f} \cdot \boldsymbol{n}(t) \sqrt{\left(f^{\prime}\right)^{2}(t)+\left(g^{\prime}\right)^{2}(t)} d t
$$

When $d=3$, then $\partial \Omega$ is a closed surface and the integral above is the surface integral (of the second kind). The surface element $d \sigma$ is again the easiest to calculate if $\partial \Omega$ is given in a parametric form $\boldsymbol{r}(t)=$ $(f(t, s), g(t, s), h(t, s)), t \in\left[t_{0}, t_{1}\right], s \in\left[s_{0}, s_{1}\right]$. Then $d \sigma=\left|\boldsymbol{r}_{t} \times \boldsymbol{r}_{s}\right| d t d s$ and

$$
\int_{\partial \Omega} \boldsymbol{f} \cdot \boldsymbol{n} d \sigma=\int_{t_{0}}^{t_{1}} \int_{s_{0}}^{s_{1}} \boldsymbol{f} \cdot \boldsymbol{n}(t, s)\left|\boldsymbol{r}_{t} \times \boldsymbol{r}_{s}\right| d s d t
$$

Remark 1.1 With a little imagination the definition of the flux can be used also in one dimensional case. To do this, we have to understand that the integration is something like a summation of the integrand over all the points of the boundary. In one-dimensional case we have $\Omega=[a, b]$ with $\partial \Omega=\{a\} \cup\{b\}$. A vector field in one-dimension is just a scalar function. The unit outward normal at $\{a\}$ is -1 , and at $\{b\}$ is 1 . Thus $\boldsymbol{f} \boldsymbol{n}(a)=f(a)(-1)$ and $\boldsymbol{f} \boldsymbol{n}(b)=f(b)(1)$ and the flux across the boundary of $\Omega$ is

$$
\begin{equation*}
\boldsymbol{f} \cdot \boldsymbol{n}(a)+\boldsymbol{f} \cdot \boldsymbol{n}(b)=f(b)-f(a) . \tag{1.1.4}
\end{equation*}
$$

Example 1.6 To understand the meaning of the flux let us consider a fluid moving in a certain domain of space. The standard way of describing the motion of the fluid is to associate with any point $\boldsymbol{p}$ of the domain filled by the fluid its velocity $\boldsymbol{v}(\boldsymbol{p})$. In this way we have the velocity field of the fluid.

Consider first the one-dimensional case (one can think about a thin pipe). If at a certain point $x$ we have $v(x)>0$, then the fluid flows to the right, and if $v(x)<0$, then it flows to the left. Let the points $x=a$ and $x=b$ be the end-points of a section of the pipe and consider the new field $f(x)=\rho(x) v(x)$, where $\rho$ is the (linear) density of the fluid in point $x$. The flux of $f$, as defined by (1.1.4), is

$$
f(b)-f(a)=\rho(b) v(b)-\rho(a) v(a)
$$

For instance, if both $v(b)$ and $v(a)$ are positive, then at $x=b$ the fluid leaves the pipe with velocity $v(b)$ and at $x=a$ it enters the pipe with velocity $v(a)$. In a small interval of time $\Delta t$ mass of fluid which left the segment through $x=b$ is equal $\rho(b) v(b) \Delta t$ and the mass which entered the segment through $x=a$ is $\rho(a) v(a) \Delta t$, thus, the net rate at which the mass leaves (enters) the segment is equal to $\rho(b) v(b)-\rho(a) v(a)$, that is, exactly to the flux of the field $f$ across the boundary.

This idea can be generalized to more realistic case of three dimensional space. Let us consider a fluid with possibly variable density $\rho(\boldsymbol{p})$ filling a portion of space and moving with velocity $\boldsymbol{v}(\boldsymbol{p})$. We define the massvelocity field $\boldsymbol{f}=\rho \boldsymbol{v}$. Let us consider now the domain $\Omega$ with the boundary $\Omega$. Imagine a small portion $\Delta S$ of $\partial \Omega$, which could be considered flat, having the area $\Delta \sigma$. Let $\boldsymbol{n}$ be the unit normal to this surface and consider the rate at which the fluid crosses $\Delta S$. We decompose the velocity $\boldsymbol{v}$ into the component parallel to $\boldsymbol{n}$, given by $(\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{n}$ and the tangential component. It is clear that the crossing of the surface can be only due to the normal component (if in time $\Delta t$ you make two steps perpendicular to the boundary and two parallel, then you will find yourself only two steps from the boundary despite having done four steps). Therefore the mass of fluid crossing $\Delta S$ in time $\Delta t$ is given by

$$
\Delta m=\rho(\boldsymbol{v} \cdot \boldsymbol{n}) \Delta t \Delta \sigma .
$$

Thus, the rate at which the fluid is crossing the whole boundary $\partial \Omega$ (that is, the net rate at which the fluid is filling/leaving the domain $\Omega$ ) can be approximated by summing up all the contributions $\Delta m$ over all

Fig. 3.2 The fluid that flows through the patch $\Delta S$ in a short time $\Delta t$ fills a slanted cylinder whose volume is approximately equal to the base times height $-\mathbf{v} \cdot \mathbf{n} \Delta \sigma \Delta t$. The mass of the fluid in the cylinder is then $\rho(\mathbf{v} \cdot \mathbf{n}) \operatorname{Delta} \sigma \Delta t$.
patches $\Delta S$ of the boundary which, in the limit as $\Delta S$ go to zero, is nothing but the flux of $\boldsymbol{f}$ :

$$
\int_{\partial \Omega}(\rho \boldsymbol{v} \cdot \boldsymbol{n}) d \sigma
$$

Thus again we have the identity

Flux of $\rho \boldsymbol{v}$ across $\partial \Omega=$ the net rate at which the mass of fluid is leaving $\Omega$.

Let us return now to the Green-Gauss-.... Theorem. This is a theorem which relates the behaviour of a vector field on the boundary of a domain (flux) with what is happening inside the domain.
Suppose that somewhere inside the domain there is a source of the field (fluid). How can we check this? We can construct a small box around the source and measure whether there is a net outflow, inflow or whether the outflow balances the inflow. If we make this box really small, then we can be quite sure that in the first case there is a source, in the second there is a sink, and in the third case that there is neither sink nor source.
Let us then put this idea into mathematical formulae. Assume that the box $B$ with one vertex at $\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}\right)$ has the edges parallel to the axes of the coordinate system and of lengths $\Delta x_{1}, \Delta x_{2}$ and $\Delta x_{3}$.

We calculate the net rate of flow of the vector field $\boldsymbol{f}=\left(f_{1}, f_{2}, f_{3}\right)$ from the box. Following the calculations given earlier, the flow through the top side is given by

$$
\boldsymbol{f} \cdot \boldsymbol{n}\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}+\Delta x_{3}\right) \Delta x_{1} \Delta x_{2}=\boldsymbol{f} \cdot \boldsymbol{j}\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}+\Delta x_{3}\right) \Delta x_{1} \Delta x_{2}=f_{3}\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}+\Delta x_{3}\right) \Delta x_{1} \Delta x_{2},
$$

and through the bottom

$$
\boldsymbol{f} \cdot \boldsymbol{n}\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}\right) \Delta x_{1} \Delta x_{2}=\boldsymbol{f} \cdot(-\boldsymbol{j})\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}\right) \Delta x_{1} \Delta x_{2}=-f_{3}\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}\right) \Delta x_{1} \Delta x_{2},
$$

thus the net flow through the horizontal sides is

$$
\left(f_{3}\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}+\Delta x_{3}\right)-f_{3}\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}\right)\right) \Delta x_{1} \Delta x_{2} \approx \frac{\partial f_{3}}{\partial x_{3}}\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}\right) \Delta x_{1} \Delta x_{2} \Delta_{3} .
$$

Similar calculations can be done for the two remaining pairs of the sides and the total flow from the box can be approximated by

$$
\left.\left(\sum_{i=1}^{3} \frac{\partial f_{i}}{\partial x_{i}}\left(x_{1}^{0}, x_{2}^{0}, x_{3}^{0}\right)\right)\right) \Delta x_{1} \Delta x_{2} \Delta_{3} .
$$

This expression can be considered to be the net rate of the production of the field (fluid) in the box $B$. The expression

$$
\operatorname{div} \boldsymbol{f}=\sum_{i=1}^{3} \frac{\partial f_{i}}{\partial x_{i}}
$$

is called the divergence of the vector field $\boldsymbol{f}$ and can considered to be the rate of the production per unit volume (density). To obtain the total net rate of the production in the domain $\Omega$ we have to add up contributions coming from all the (small) boxes. Thus using the definition of the integral we obtain that the total net rate of the production is given by

$$
\iiint_{\Omega} \operatorname{div} \boldsymbol{f}\left(x_{1}, x_{2}, x_{3}\right) d \boldsymbol{v}
$$

Using some common sense reasoning it is easy to arrive at the identity

$$
\text { The net rate of production in } \Omega=\text { The net flow across the boundary }
$$

The Green-Gauss-... Theorem is the mathematical expression of the above law.
Theorem 1.3 Let $\Omega$ be a bounded domain in $\mathbb{R}^{d}$, $d \geq 1$, with a piecewise $C^{1}$ boundary $\partial \Omega$. Let $\boldsymbol{n}$ be the unit outward normal vector on $\partial \Omega$. Let $\boldsymbol{f}(\boldsymbol{x})=\left(f_{1}(\boldsymbol{x}), \cdots, f_{n}(\boldsymbol{x})\right)$ be any $C^{1}$ vector field on $\bar{\Omega}=\Omega \cup \partial \Omega$. Then

$$
\begin{equation*}
\int_{\Omega} \operatorname{div} \boldsymbol{f} d \boldsymbol{x}=\oint_{\partial \Omega} \boldsymbol{f} \cdot \boldsymbol{n} d \sigma \tag{1.1.5}
\end{equation*}
$$

where $d \sigma$ is the element of surface of $\partial \Omega$.

Remark 1.2 In one dimension this theorem is nothing but the fundamental theorem of calculus:

$$
\begin{equation*}
\int_{a}^{b} \frac{d f}{d x}(x) d x=f(b)-f(a) \tag{1.1.6}
\end{equation*}
$$

In fact, for a function of one variable $\operatorname{div} \boldsymbol{f}=\frac{d f}{d x}$ and the right-hand side of this equation represents the outward flux across the boundary of $\Omega=[a, b]$, as discussed in Remark 1.1.

Remark 1.3 In two dimensions the most popular form of this theorem is known as the Green Theorem which apparently differs form the one given above. To explain this, let the boundary $\partial \Omega$ of $\Omega$ be a curve given by parametric equation $\boldsymbol{r}(t)=\left(x_{1}(t), x_{2}(t)\right), t \in\left[t_{0}, t_{1}\right]$ and suppose that if $t$ runs from $t_{0}$ to $t_{1}$, then $\boldsymbol{r}(t)$ traces $\partial \Omega$ in the positive (anticlockwise) direction. Then it is easy to check that the unit outward normal vector $\boldsymbol{n}$ is given by

$$
\boldsymbol{n}(t)=\frac{1}{\boldsymbol{r}^{\prime}(t)}\left(x_{2}^{\prime}(t),-x_{1}^{\prime}(t)\right)
$$

where $\boldsymbol{r}^{\prime}(t)=\left(x_{1}(t), x_{2}(t)\right)$. Thus, if $\boldsymbol{f}=\left(f_{1}, f_{2}\right)$, then Eq. (1.1.5) takes the form

$$
\begin{equation*}
\iint_{\Omega}\left(\frac{\partial f_{1}}{\partial x_{1}}+\frac{\partial f_{2}}{\partial x_{2}}\right) d x_{1} d x_{2}=\int_{t_{0}}^{t_{1}}\left(f_{1}(t) x_{2}^{\prime}(t)-f_{1}(t) x_{1}^{\prime}(t)\right) d t \tag{1.1.7}
\end{equation*}
$$

On the other hand, the standard version of Green's theorem reads

$$
\begin{equation*}
\iint_{\Omega}\left(\frac{\partial f_{2}}{\partial x_{1}}-\frac{\partial f_{1}}{\partial x_{2}}\right) d x_{1} d x_{2}=\oint_{\partial \Omega} f_{1} d x_{1}+f_{2} d x_{2}=\int_{t_{0}}^{t_{1}}\left(f_{1}(t) x_{1}^{\prime}(t)+f_{2}(t) x_{2}^{\prime}(t)\right) d t . \tag{1.1.8}
\end{equation*}
$$

To see that these forms are really equivalent, let us define the new vector field $\boldsymbol{g}=\left(g_{1}, g_{2}\right)$ by : $g_{1}=-f_{2}, g_{2}=$ $f_{1}$. Then

$$
\operatorname{div} \boldsymbol{f}=\frac{\partial f_{1}}{\partial x_{1}}+\frac{\partial f_{2}}{\partial x_{2}}=\frac{\partial g_{2}}{\partial x_{1}}-\frac{\partial g_{1}}{\partial x_{2}}
$$

The boundary of $\partial \Omega$ is, as above, parameterized by the function $\boldsymbol{r}(t)$. Thus, if we assume that (1.1.8) holds (for an arbitrary vector field), then

$$
\begin{aligned}
\iint_{\Omega}\left(\frac{\partial f_{1}}{\partial x_{1}}+\frac{\partial f_{2}}{\partial x_{2}}\right) d x_{1} d x_{2} & =\iint_{\Omega}\left(\frac{\partial g_{2}}{\partial x_{1}}-\frac{\partial g_{1}}{\partial x_{2}}\right) d x_{1} d x_{2} \\
=\int_{t_{0}}^{t_{1}}\left(g_{1}(t) x_{1}^{\prime}(t)+g_{2}(t) x_{2}^{\prime}(t)\right) d t & =\int_{t_{0}}^{t_{1}}\left(f_{1}(t) x_{2}^{\prime}(t)-f_{2}(t) x_{1}^{\prime}(t)\right) d t,
\end{aligned}
$$

which is (1.1.7). The converse is analogous.

## 2 Conservation laws

Many of the fundamental equations that occurr in the natural and physical sciences are obtained from conservation laws. Conservation laws express the fact that some quantity is balanced throughout the process.
In thermodynamics, for example, the First Law of Thermodynamics states that the change in the internal energy in a given system is equal to, or is balanced by, the total heat added to the system plus the work done on the system. Therefore the First Law of Thermodynamics is really an energy balance, or conservation, law.

As an another example, consider a fluid flowing in some region of space that consists of chemical species undergoing chemical reaction. For a given chemical species, the rate of change in time of the total amount of that species in the region must equal the rate at which the species flows into the region, minus the rate at which the species flows out, plus the rate at which the species is created or consumed, by the chemical reactions. This is a verbal statement of a conservation law for the amount of the given chemical species.

Similar balance or conservation laws occur in all branches of science. We can recall the population equation the derivation of which was based on the observation that the rate of change of a population in a certain region must equal the birth rate, minus the death rate, plus the migration rate into or out of the region.

Such conservation laws in mathematics are translated usually into differential equations and it is surprising how many different processes of real world end up with the same mathematical formulation. These differential equations are called the governing equations of the process and dictate how the process evolves in time. Here we discuss the derivation of some governing equations starting from the first principles. We begin with a basic one-dimensional model.

### 2.1 One-dimensional conservation law

Let us consider a quantity $u=u(x, t)$ that depends on a single spatial variable $x$ in an interval $R \subset \mathbb{R}$, and time $t>0$. We assume that $u$ is a density, or concentration, measured in an amount per unit volume, where the amount may refer to the population, mass, energy or any other quantity. The fact that $u$ depends only on one spatial variable $x$ can be physically justified by assuming e.g. that we consider a flow in a tube,

## Fig. 3.3 Tube $\mathcal{I}$.

which is uniform (doesn't change in radial direction), or that the tube is very thin so that any change in the radial direction is negligible (later we shall derive the equation for the same conservation law in an arbitrary dimensional space). Note that the discussion here is related to Remark 1.1 and Example 1.6. We consider the subinterval $I=[x, x+h] \subset R$. The total amount of the quantity $u$ contained at time $t$ in the section $\mathcal{I}$ of the tube between $x$ and $x+h$ is given by the integral

$$
\text { total amount of quantity in } \mathcal{I}=A \int_{x}^{x+h} u(s, t) d s
$$

where $A$ is the area of the cross section of $\mathcal{I}$.
Assume now that there is motion of the quantity in the tube in the axial direction. We define the flux density of $u$ at time $t$ at $x$ to be the scalar function $\phi(x, t)$ which is equal to the amount of the quantity $u$ passing through the cross section at $x$ at time $t$, per unit area, per unit time. By convention, the flux density at $x$ is positive if the flow at $x$ is in the positive $x$ direction. Therefore, at time $t$ the net rate that the quantity is flowing into the section is the rate that it is flowing in at $x$ and minus the rate that it is flowing out at $x+h$, that is

Fig. 3.4 One dimensional flow through cross-sections $x$ and $x+h$.
net rate that the quantity flows into $\mathcal{I}=A(\phi(x, t)-\phi(x+h, t))$.
This equation should be compared with Example 1.6, where the flux density was easy to understand: it was the rate at which the mass of fluid flows through the boundary. Here we are considering the flux density of an arbitrary quantity (arbitrary one-dimensional vector field).

Finally, the quantity $u$ may be destroyed or created inside $\mathcal{I}$ by some internal or external source (e.g. by a chemical reaction if we consider chemical kinetics equations, or by birth/death processes in mathematical biology). We denote this source function, which is assumed to be local (acts at each $x$ and $t$ ), by $f(x, t, u)$. This function gives the rate at which $u$ is created or destroyed at $x$ at time $t$, per unit volume. Note that $f$ may depend on $u$ itself (e.g. the rate of chemical reactions is determined by concentration of the chemicals). Given $f$, we may calculate the total rate that $u$ is created/destroyed in $\mathcal{I}$ by integration:

$$
\text { rate that quantity is produced in } \mathcal{I} \text { by sources }=A \int_{x}^{x+h} f(s, t, u(s, t)) d s
$$

The fundamental conservation law can be formulated as follows: for any section $\mathcal{I}$

> the rate of change of the total amount in $\mathcal{I}$
> $\quad=$ net rate that the quantity flows into $\mathcal{I}$
> +rate that the quantity is produced in $\mathcal{I}$

Using the mathematical formulas obtained above we get, having simplified the area $A$

$$
\begin{equation*}
\frac{d}{d t} \int_{x}^{x+h} u(s, t) d s=\phi(x, t)-\phi(x+h, t)+\int_{x}^{x+h} f(s, t, u) d s \tag{1.2.1}
\end{equation*}
$$

The equation above is called a conservation law in integral form and holds even if $u, f, \phi$ are not smooth functions. This form is useful in many cases but rather difficult to handle, therefore it is convenient to reduce it to a differential equation. This requires assuming that all the functions (including the unknown $u$ ) are continuously differentiable. Using basic facts from calculus:
(i) $\int_{x}^{x+h} \phi_{s}(s, t) d s=\phi(x+h, t)-\phi(x, t)$,
(ii) $\frac{d}{d t} \int_{x}^{x+h} u(s, t) d s=\int_{x}^{x+h} u_{t}(s, t) d s$,
we can rewrite (1.2.1) in the form

$$
\begin{equation*}
\int_{I}\left(u_{t}(s, t)+\phi_{s}(s, t)-f(s, t, u)\right) d s=0 . \tag{1.2.2}
\end{equation*}
$$

Since this equation is valid for any interval $I$ we can use Theorem 1.2 to infer that the integral must vanish identically; that is, changing the independent variable back into $x$ we must have

$$
\begin{equation*}
u_{t}(x, t)+\phi_{x}(x, t)=f(x, t, u) \tag{1.2.3}
\end{equation*}
$$

for any $x \in R$ and $t>0$. Note that in (1.2.3) we have two unknown functions: $u$ and $\phi$; function $f$ is assumed to be given. Function $\phi$ is usually to be determined from empirical considerations. Equations resulting from such considerations, which specify $\phi$, are often called constitutive relations or equations of state.

Before we proceed in this direction, we shall show how the procedure described above works in higher dimensions.

### 2.2 Conservation laws in higher dimensions

It is relatively straightforward to generalize the discussion above to multidimensional space. Let $\boldsymbol{x}=$ $\left(x_{1}, x_{2}, x_{3}\right)$ denote a point in $\mathbb{R}^{3}$ and assume that $u=u(\boldsymbol{x}, t)$ is a scalar density function representing the amount per unit volume of some quantity of interest distributed throughout some domain of $\mathbb{R}^{3}$. In this domain, let $\Omega \subset \mathbb{R}^{3}$ be an arbitrary region with a smooth boundary $\partial \Omega$. As in one-dimensional case, the total amount of the quantity in $\Omega$ at time $t$ is given by the integral

$$
\int_{\Omega} u(\boldsymbol{x}, t) d \boldsymbol{x}
$$

and the rate that the quantity is produced in $\Omega$ is given by

$$
\int_{\Omega} f(\boldsymbol{x}, t, u) d \boldsymbol{x}
$$

where $f$ is the rate at which the quantity is being produced in $\Omega$.
Some changes are required however when we want to calculate the flux of the quantity in or out $\Omega$. Because now the flow can occur in any direction, the flux density is given by a vector $\boldsymbol{\Phi}$ (in Example 1.6, it was given by $\rho(\boldsymbol{x}) \boldsymbol{v}(\boldsymbol{x}))$. However, as we noted in this example, not all the flowing quantity which is close to the boundary will leave or enter the region $\Omega$ - the component of $\Phi$ which is tangential to the boundary $\partial \Omega$ will not contribute to the total loss (see also Fig. 2.1). Therefore, if we denote by $\boldsymbol{n}(\boldsymbol{x})$ the outward unit normal vector to $\partial \Omega$, then the net outward flux of the quantity $u$ through the boundary $\partial \Omega$ is given by the surface integral

$$
\int_{\partial \Omega} \boldsymbol{\Phi}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}) d \sigma
$$

where $d \sigma$ denotes a surface element on $\partial \Omega$. Finally, the balance law for $u$ is given by

$$
\begin{equation*}
\frac{d}{d t} \int_{\partial \Omega} u d \boldsymbol{x}=-\oint_{\partial \Omega} \boldsymbol{\Phi} \cdot \boldsymbol{n} d \sigma+\int_{\Omega} f d \boldsymbol{x} . \tag{1.2.4}
\end{equation*}
$$

The minus sign at the flux term occurs because the outward flux decreases the amount of $u$ in $\Omega$.
As before, the integral form can be reformulated as a local differential equation, provided $\mathbf{\Phi}$ and $u$ are sufficiently smooth in $\boldsymbol{x}$ and $t$. To this end we must use the Gauss theorem (Theorem 1.3) which gives

$$
\int_{\partial \Omega} \boldsymbol{\Phi} \cdot \boldsymbol{n} d \sigma=\int_{\Omega} \operatorname{div} \boldsymbol{\Phi} d \boldsymbol{x} .
$$

Using this formula, Equation (1.2.4) can be rewritten as

$$
\int_{\Omega}\left(u_{t}+\operatorname{div} \boldsymbol{\Phi}-f\right) d \boldsymbol{x}=0
$$

for any subregion $\Omega$. Using the vanishing theorem (Theorem 1.2) we finally obtain the differential form of the general conservation law in 3 (or higher) dimensional space

$$
\begin{equation*}
u_{t}+\operatorname{div} \boldsymbol{\Phi}=f(\boldsymbol{x}, t, u) \tag{1.2.5}
\end{equation*}
$$

In the next section we discuss a number of constitutive relations which provide examples of the flux function.

## 3 Constitutive relations and examples

In this section we describe a few constitutive relation which often appear in the applied sciences. It is important to understand that a constitutive relations appear on a different level than conservation law: the latter is is a fundamental law of nature - the mathematical expression of the fact that books should balance (at least in a decent(?) enterprise like the Universe), whereas the former is often an approximate equation having its origins in empirics.

### 3.1 Transport equation

Probably the simplest nontrivial constitutive relation is when the quantity of interest moves together with the surrounding medium with a given velocity, say, $\boldsymbol{v}$. This can be a simple model for a low density pollutant in the air moving only due to wind with velocity $\boldsymbol{v}$ or, as we discussed earlier, fluid with density $u$ moving with the velocity $\boldsymbol{v}$. Then the flux density is given by

$$
\boldsymbol{\Phi}=\boldsymbol{v} u
$$

and, if there are no sources or sinks, the transport equation is given by

$$
\begin{equation*}
u_{t}+\operatorname{div}(\boldsymbol{v} u)=0 . \tag{1.3.1}
\end{equation*}
$$

We can rewrite this equation in a more explicit form

$$
u_{t}+\boldsymbol{v} \cdot \nabla u+u \operatorname{div} \boldsymbol{v}=0
$$

or, if the velocity is constant or, more general, divergence (or source) free,

$$
\begin{equation*}
\partial_{t} u+\boldsymbol{v} \cdot \nabla u=0 \tag{1.3.2}
\end{equation*}
$$

In more general cases, $\boldsymbol{v}$ may depend on the solution $u$ itself. For example, (1.3.1) may describe a motion of a herd of animals over certain area with $u$ being (suitably rounded) density of animals per, say, square kilometer. Then the speed of the herd will depend on the density - the bigger the squeeze, the slower the animals move. This constitutes an example of a quasilinear equation in which the coefficient of the highest derivative depend on the solution:

$$
\begin{equation*}
u_{t}+\operatorname{div}(\boldsymbol{v}(u) u)=0 . \tag{1.3.3}
\end{equation*}
$$

### 3.2 McKendrick partial differential equation

The derivation of the conservation law was based on a physical intuition about a fluid flow in a physical space. Yet the argument can be used in a much broader context. Consider an age-structured population, described by the density of the population $n(a, t)$ with respect to age $n(a, t)$, and look at the population as if it was 'transported' through stages of life. Taking into account that $n(a, t) \Delta a$ is the number of individuals (females) in the age group $[a, a+\Delta a)$ at time $t$, we may write that the rate of change of this number

$$
\frac{\partial}{\partial t}[n(a, t) \Delta a]
$$

equals rate of entry at $a$ minus rate of exit at $a+\Delta a$ and minus deaths. Denoting per capita mortality rate for individuals by $\mu(a, t)$, the last term is simply $-\mu(a, t) n(a, t) \Delta t$. The first two terms require introduction of the 'flux density' of individuals $J$ describing the 'speed' of ageing. Thus, passing to the limit $\Delta a \rightarrow 0$, we get

$$
\frac{\partial n(a, t)}{\partial t}+\frac{\partial J(a, t)}{\partial a}=-\mu(a, t) n(a, t) .
$$

Let us determine the flux density in the simplest case when ageing is just the passage of time measured in the same units.

Here for consistency, we derive the full equation. If the number of individuals in the age bracket $[a, a+\Delta a)$ is $n(a, t) \Delta a$, then after $\Delta t$ we will have $n(a, t+\Delta t) \Delta a$. On the other hand, $u(a-\Delta t, t) \Delta t$ individuals moved in while $u(a+\Delta a-\Delta t, t) \Delta t$ moved out, where we assumed, for simplicity, $\Delta t<\Delta a$. Thus

$$
n(a, t+\Delta t) \Delta a-n(a, t) \Delta a=u(a-\Delta t, t) \Delta t-u(a+\Delta a-\Delta t) \Delta t-\mu(a, t) n(a, t) \Delta a \Delta t
$$

or, using the Mean Value Theorem $\left(0 \leq \theta, \theta^{\prime} \leq 1\right)$

$$
n_{t}(a, t+\theta \Delta t) \Delta a \Delta t=-n_{a}\left(a-\Delta t+\theta^{\prime} \Delta a, t\right) \Delta a \Delta t-\mu(a, t) n(a, t) \Delta a \Delta t
$$

and, passing to the limit with $\Delta t, \Delta a \rightarrow 0$, we obtain

$$
\begin{equation*}
\frac{\partial n(a, t)}{\partial t}+\frac{\partial n(a, t)}{\partial a}=-\mu(a, t) n(a, t) \tag{1.3.4}
\end{equation*}
$$

This equation is defined for $a>0$ and the flow is to the right hence we need a boundary condition. In this model the birth rate enters here: the number of neonates $(a=0)$ is the number of births across the whole age range:

$$
n(0, t)=\int_{0}^{\omega} n(a, t) m(a, t) d a
$$

where $m$ is the maternity function. Eq. (1.3.4) also must be supplemented by the initial condition

$$
n(a, 0)=n_{0}(a)
$$

describing the initial age distribution.

### 3.3 Diffusion/heat equations

Assume at first that there are no sources, and write the basic conservation law in one-dimension as

$$
\begin{equation*}
u_{t}+\phi_{x}=0 \tag{1.3.5}
\end{equation*}
$$

In many physical (and not only) problems it was observed that the substance (represented here by the density $u$ ) moves from the regions of higher concentration to regions of lower concentration (heat, population, etc. offer a good illustration of this). Moreover, the larger difference the more rapid flow is observed. At a point, the large difference can be expressed as a large gradient of the concentration $u$, so it is reasonable to assume that

$$
\phi(x, t)=-F\left(u_{x}(x, t)\right),
$$

where $F$ is an increasing function passing through $(0,0)$. The explanation of the minus sign comes from the fact that if the $x$-derivative of $u$ at $x$ is positive, that is, $u$ is increasing, then the flow occurs in the negative direction of the $x$ axis. Moreover, the larger the gradient, the larger (in magnitude) the flow. Now, the simplest increasing function passing through $(0,0)$ is a linear function with positive leading coefficient, and this assumption gives the so-called Fick law:

$$
\begin{equation*}
\phi(x, t)=-D u_{x}(x, t) \tag{1.3.6}
\end{equation*}
$$

where $D$ is a constant which is to be determined empirically. We can substitute Fick's law (1.3.6) into the conservation law (1.2.3) (assuming that the solution is twice differentiable and $D$ is a constant) and get the one dimensional diffusion equation

$$
\begin{equation*}
u_{t}-D u_{x x}=0, \tag{1.3.7}
\end{equation*}
$$

which governs conservative processes when the flux is specified by Fick's law.
To understand why this equation governs also the evolution of the temperature distribution in a body, we apply the conservation law (1.2.3) to heat (thermal energy). Then the amount of heat in a unit volume can be written as $u=c \rho \theta$, where $c$ is the specific heat of the medium, $\rho$ the density of the medium and $\theta$ is the temperature. The heat flow is governed by Fourier's law which states that the heat flux density is proportional to the gradient of the temperature, with the proportionality constant equal to $-k$, where $k$ is the thermal conductivity of the medium. Assuming $c, \rho, k$ to be constants, we arrive at (1.3.7) with $D=k / c \rho$.

Generalization of Equation (1.3.7) to multidimensional cases requires some assumptions on the medium. To simplify the discussion we assume that the medium is isotropic, that is, the process of diffusion is independent of the orientation in space. Then the Fick law states that the flux density is proportional to the gradient of $u$, that is,

$$
\boldsymbol{\Phi}=-D \nabla u
$$

and the diffusion equation, obtained from the conservation law (1.2.5) in the absence of sources, reads

$$
\begin{equation*}
u_{t}=\operatorname{div}(D \nabla u)=D \Delta u, \tag{1.3.8}
\end{equation*}
$$

where the second equality is valid if $D$ is a constant.

### 3.4 Variations of diffusion equation

In many cases the evolution is governed by more then one process. One of the most widely occuring cases is when we have simultaneously both transport and diffusion. For example, when we have a pollutant in the air or water, then it is transported with the moving medium (due to the wind or water flow) and, at the same time, it is dispersed by diffusion. Without the latter, a cloud of pollution would travel without any change - the same amount that was released from the factory would eventually reach other places with the same concentration. Diffusion, as we shall see later, has a dispersing effect, that is, the pollution will be eventually deposited but in a uniform, less concentrated way, making life (perhaps) more bearable.

Mathematical expression of the above consideration is obtained by combining equations (1.3.1) and (1.3.8):

$$
\begin{equation*}
u_{t}+\operatorname{div}(\boldsymbol{v} u)=D \Delta u, \tag{1.3.9}
\end{equation*}
$$

or, equivalently, by determining the expression for the flux taking both processes simultaneously into account. The resulting equation (1.3.9) is called the drift-diffusion equation.

When the sources are present and the constitutive relation is given by Fick's law, then the resulting equation

$$
\begin{equation*}
u_{t}-D \Delta u=f(\boldsymbol{x}, t, u) \tag{1.3.10}
\end{equation*}
$$

is called the reaction-diffusion equation. If $f$ is independent of $u$, then physically we have the situation when the sources are given a priori (like the injection performed by an observer); the equation becomes then a linear non-homogeneous equation. Sources, however, can depend on $u$. The simplest case is then when $f=c u$ with $c$ constant. Then the source term describes spontaneous decay (or creation) of the substance at the rate that is proportional to the concentration. In such a case Equation (1.3.10) takes the form

$$
\begin{equation*}
u_{t}-D \Delta u=c u \tag{1.3.11}
\end{equation*}
$$

and can be thought of as the combination of the diffusion and the law of exponential growth.

If $f$ is nonlinear in $u$, then Equation (1.3.10) has many important applications, particularly in the combustion theory and mathematical biology.

Here we describe the so called Fisher equation which appears as a combination of the one-dimensional logistic process and the diffusion. It is reasonable to assume that the population is governed by the logistic law, which is mathematically expressed as the ordinary differential equation

$$
\begin{equation*}
\frac{d u}{d t}=r u(N-u) \tag{1.3.12}
\end{equation*}
$$

where $u$ is the population in some region, $r N>0$ is the growth rate and $N>0$ is the carrying capacity of this region. If we are however interested is the spatial distribution of individuals, then we must introduce the density of the population $u(\boldsymbol{x}, t)$, that is, the amount of individuals per unit of volume (or surface) and the conservation law will take the form

$$
\begin{equation*}
u_{t}+\operatorname{div} \boldsymbol{\Phi}=r u(N-u) \tag{1.3.13}
\end{equation*}
$$

As a first approximation, it is reasonable to assume that within the region the population obeys the Fick law, that is, individuals migrate from the regions of higher density to regions of lower density. This is not always true (think about people migrating to cities) and therefore the range of validity of Fick's law is limited; in such cases $\boldsymbol{\Phi}$ has to be determined otherwise. However, if we assume that the resources are evenly distributed, then Fick's law can be used with good accuracy. In such a case, Equation (1.3.13) takes the form

$$
\begin{equation*}
u_{t}-D \Delta u=r u(N-u), \tag{1.3.14}
\end{equation*}
$$

which is called the Fisher equation. Note, that if the capacity of the region $N$ is very large, then writing the Fisher equation in the form

$$
u_{t}-D \Delta u=c u\left(1-\frac{u}{N}\right)
$$

where $c=r N$, we see that it is reasonable to neglect the small term $u / N$ and approximate the Fisher equation by the linear equation (5.5.15) which describes a combination of diffusion-type migratory processes and exponential growth.

## Chapter 2

## First order linear partial differential equations

## 1 Basic theory and examples

We start with the simplest transport equation. Assume that $u$ is a concentration of some substance (pollutant) in a fluid (amount per unit length). This substance is moving to the right with a speed $c$. Then the differential equation for $u$ has the form:

$$
\begin{equation*}
u_{t}+c u_{x}=0 . \tag{2.1.1}
\end{equation*}
$$

Let us consider more general linear first order partial differential equation (PDE) of the form:

$$
\begin{equation*}
a u_{t}+b u_{x}=0, \quad t, x \in \mathbb{R} \tag{2.1.2}
\end{equation*}
$$

where $a$ and $b$ are constants. This equation can be written as

$$
\begin{equation*}
D_{\boldsymbol{v}} u=0, \tag{2.1.3}
\end{equation*}
$$

where $\boldsymbol{v}=a \boldsymbol{j}+b \boldsymbol{i}(\boldsymbol{j}$ and $\boldsymbol{i}$ are the unit vectors in, respectively, $t$ and $x$ directions $)$, and $D \boldsymbol{v}=\nabla u \cdot \boldsymbol{v}$ denotes the directional derivative in the direction of $\boldsymbol{v}$. This means that the solution $u$ is a constant function along each line having direction $\boldsymbol{v}$, that is, along each line of equation $b t-a x=\xi$. Along each such a line the value of the parameter $\xi$ remains constant. However, the solution can change from one line to another, therefore the solution is a function of $\xi$, that is the solution to Eq. (2.1.2) is given by

$$
\begin{equation*}
u(x, t)=f(b t-a x), \tag{2.1.4}
\end{equation*}
$$

where $f$ is an arbitrary differentiable function. Such lines are called the characteristic lines of the equation.

Example 1.1 To obtain a unique solution we must specify the initial value for $u$. Hence, let us consider the initial value problem for Eq. (2.1.2): find $u$ satisfying both

$$
\begin{align*}
a u_{t}+b u_{x} & =0 \quad x \in \mathbb{R}, t>0 \\
u(x, 0) & =g(x) \quad x \in \mathbb{R} \tag{2.1.5}
\end{align*}
$$

where $g$ is an arbitrary given function. From Eq. (2.1.4) we find that

$$
\begin{equation*}
u(x, t)=g\left(-\frac{b t-a x}{a}\right) . \tag{2.1.6}
\end{equation*}
$$

We note that the initial shape propagates without any change along the characteristic lines, as seen below for the initial function $g=1-x^{2}$ for $|x|<1$ and zero elsewhere. The speed $c=b / a$ is taken to be equal to 1.


Fig. 4.1 The graph of the solution in Example 1.1

Example 1.2 Let us consider a variation of this problem and try to solve the initial- boundary value problem

$$
\begin{align*}
a u_{t}+b u_{x} & =0 \quad x \in \mathbb{R}, t>0 \\
u(x, 0) & =g(x) \quad x>0  \tag{2.1.7}\\
u(0, t) & =h(t) \quad t>0 \tag{2.1.8}
\end{align*}
$$

for $a, b>0$ From Example 1.1 we have the general solution of the equation in the form

$$
u(x, t)=f(b t-a x)
$$

Putting $t=0$ we get $f(-a x)=g(x)$ for $x>0$, hence $f(x)=g(-x / a)$ for $x<0$. Next, for $x=0$ we obtain $f(b t)=h(t)$ for $t>0$, hence $f(x)=h(x / b)$ for $x>0$. Combining these two equations we obtain

$$
u(x, t)=\left\{\begin{array}{ccc}
g\left(-\frac{b t-a x}{a}\right) & \text { for } & x>b t / a \\
h\left(\frac{b-a x}{b}\right) & \text { for } & x<b t / a
\end{array}\right.
$$

Now, let us consider what happens if $a=1>0, b=-1<0$. Then the initial condition defines $f(x)=g(-x)$ for $x<0$ and the boundary condition gives $f(x)=h(-x)$ also for $x<0$ ! Hence, we cannot specify both initial and boundary conditions in an arbitrary way as this could make the problem ill-posed.

The physical explanation of this comes from the observation that since the characteristics are given by $\xi=x+t$, the flow occurs in the negative direction and therefore the values at $x=0$ for any $t$ are uniquely determined by the initial condition. Therefore we see that to have a well-posed problem we must specify the boundary conditions at the point where the medium flows into the region.

The same principle could be used to solve equations with variable coefficients. In fact, consider the equation

$$
\begin{equation*}
a(x, t) u_{t}+b(x, t) u_{x}=f(x, t, u) . \tag{2.1.9}
\end{equation*}
$$

This equation asserts that the derivative of $u$ in the direction of the vector $(b(x, t), a(t, x))$ is equal to $f(x, t, u)$ at each point $(x, t)$ where this vector is not vanishing. We can consider a family of curves $\tau \rightarrow(x(\tau), t(\tau))$ which are tangent to these vectors at each point. In other words, we consider the family of curves determined by the system of differential equations

$$
\begin{equation*}
x_{\tau}=b(x, t), \quad t_{\tau}=a(x, t) \tag{2.1.10}
\end{equation*}
$$

subject to the initial condition $x(0)=\xi$, and $t(0)=0$, expressing the fact that the curves cross the line $t=0$ at $\tau=0$ at an unspecified for a time being point $x=\xi$. Let $I_{\xi}$ be an interval of existence of the solution
passing through $\xi$. The curves $(x(\tau, \xi), t(\tau, \xi))_{\tau \in I_{\xi}}$ are called the characteristics of the system. We note that, depending on the properties of the vector field $(b, a)$, the system (2.1.10) may not have a solution, may have non-unique solutions, the solutions may exist for a finite time interval and also may fill only a region of $\mathbb{R}^{2}$.

Let us consider a characteristic $\tau \rightarrow(x(\tau, \xi), t(\tau))$ lying in the domain $D$ of the $(x, t)$-plane in which there is a differentiable solution $u$ of (2.1.9). Then we can define

$$
v(\xi, \tau)=u(x(\tau, \xi), t(\tau, \xi))
$$

and, differentiating with respect to $\tau$ we find, by the chain rule,

$$
v_{\tau}=u_{t} t_{\tau}+u_{x} x_{\tau}=u_{t} a+u_{x} b=g
$$

where $g(\tau, \xi):=f(x(\tau, \xi), t(\tau, \xi), v(\xi, \tau))$. Thus, the partial differential equation (2.1.9) reduces to a first order ordinary equation which requires an initial condition. Since we have $u(x, 0)=u_{0}(x)$, for $v$ we can write

$$
v(\xi, 0)=u(x(0, \xi), t(0, \xi))=u(\xi, 0)=u_{0}(\xi)
$$

Conversely, if $u$ is a differentiable function of two variables such that $v(\tau, \xi)=u(x(\tau, \xi), t(\tau, \xi))$ satisfies the system

$$
\begin{align*}
v_{\tau} & =g \\
x_{\tau} & =b \\
t_{\tau} & =a \\
v(0) & =u_{0}(\xi), x(0)=\xi, t(0)=0, \tag{2.1.11}
\end{align*}
$$

then $u(x, t)$ solves (2.1.9) at any point $(x, t)$ through which a characteristic passes.
This indicates a way to solving

$$
\begin{equation*}
a(x, t) u_{t}(x, t)+b(x, t) u_{x}(x, y)=f(x, t, u(x, t)), \quad u(x, 0)=u_{0}(x), \quad x \in \mathbb{R}, t \geq 0 \tag{2.1.12}
\end{equation*}
$$

Define

$$
E=\left\{(x, t) ; x=x(\tau, \xi), t=t(\tau, \xi), \xi \in \mathbb{R}, \tau \in I_{\xi}\right\}
$$

so that $E$ is a set filled with characteristics starting from $t=0$. If the field $(b, a)$ is Lipschitz continuous, then these characteristics do not cross each other and thus present a new coordinate system. Then, for any $(x, t) \in E$ there is a unique characteristic passing through $(x, t)$; that is, there is unique pair $(\xi, \tau)$ such that $x=x(\xi, \tau), t=t(\xi, \tau)$ and then

$$
u(x, t)=v(\xi(x, t), \tau(x, t))
$$

is the solution of (2.1.12).
The above considerations can be simplified if $a \neq 0$. Then dividing the equation by $a$ we arrive at an equivalent form

$$
\begin{align*}
u_{t}+c(x, t) u_{x} & =f(x, t, u) \\
u(x, 0) & =u_{0}(x) \tag{2.1.13}
\end{align*}
$$

In this case we have $t_{\tau}=1$ and thus we can take $t=\tau$ reducing the number of equations by 1 . Then the system (2.1.11)

$$
\begin{align*}
v_{t}(\xi, t) & =g(\xi, t, v(\xi, t))=f(x(t, \xi), t, v(\xi, t)) \\
x_{t} & =c(x, t) \\
x(0) & =\xi, v(0, \xi)=u_{0}(\xi) \tag{2.1.14}
\end{align*}
$$

which, in principle, can be solved. The second equation is independent of the first so that it determines the equation of characteristics $x=x(t, \xi)$. This solution can be substituted into the first equation the solution of which gives the values of $v$ as a function of $t$ that is a parameter along a characteristic. Eliminating $\xi$ from equations for $v$ and $x$ produces the solution $u$ in terms of $x$ and $t$ only.
We illustrate the method in the examples below.

Example 1.3 Find the solution of the equation

$$
u_{t}+2 u_{x}=x, \quad x \in \mathbb{R}
$$

which satisfies the initial condition:

$$
u(x, 0)=v_{0}(x)
$$

The characteristic equation is

$$
x_{t}=2, \quad x(0)=\xi
$$

giving the characteristics

$$
x-2 t=\xi
$$

Then $v(t, \xi)=u(x(t, \xi), t), x=2 t+\xi$ and

$$
v_{t}=2 t+\xi, \quad v(\xi, 0)=v_{0}(\xi)
$$

so that we have

$$
v(t, \xi)=c+t^{2}+\xi t
$$

To determine $c$ we put $t=0$ so that

$$
v(0, \xi)=c=v_{0}(\xi)
$$

Thus

$$
v(t, \xi)=t^{2}+\xi t+v_{0}(\xi)
$$

To find the solution $u(x, t)$ we must revert to old variables. Using again the characteristic equation, we have $x-2 t=\xi$ so that

$$
u(x, t)=v(t, x-2 t)=t^{2}+(x-2 t) t+v_{0}(x-2 t)=x t-t^{2}+v_{0}(x-2 t)
$$

In the next example we consider an initial-boundary value problem. In many one-dimensional cases we encounter the so called signalling problem when the boundary condition is prescribed at one end-point of a half-line.

Example 1.4 Find the solution of the equation

$$
u_{t}+2 u_{x}=x, \quad x \in \mathbb{R},
$$

which satisfies the initial conditions: $u(x, 0)=v_{0}(x)$ for $x>0$ and $u(0, t)=\psi(t)$ for $t>0$. Let us consider the solution from the previous example $u(x, t)=x t-t^{2}+v_{0}(x-2 t)$. Since now $v_{0}(x)$ is only defined for $x>0$, the above function is defined for $x-2 t>0$ and it is a solution of our problem there. So the problem is to find the solution for $x<2 t$. We present three methods of solution.
Method I.
We must ensure that the characteristics pass through points $(x, t)=(0, \eta)$ so that from the general solution of characteristics $x-2 t=C$ we obtain $C=-2 \eta$ so that

$$
x=2 t-2 \eta
$$

and

$$
v_{t}=2 t-2 \eta
$$

subject to $\left.v(t, \eta)\right|_{t=\eta}=\psi(\eta)$. This gives

$$
v(t, \eta)=t^{2}-2 \eta t+c
$$

with $c+\eta^{2}-2 \eta^{2}=\psi(\eta)$; that is, $c=\psi(\eta)+\eta^{2} 2$. Thus

$$
v(t, \eta)=t^{2}-2 \eta t+\psi(\eta)+\eta_{2}
$$

Returning to the original variables, we find that for $x<2 t$

$$
u(x, t)=v(t, t-x / 2)=t^{2}-2 t(t-x / 2)+(t-x / 2)^{2}+\psi(t-x / 2)=\psi(t-x / 2)+x^{2} / 4
$$

Method 2.
We observe that for the boundary value part, the variables interchange their roles. Thus, we can re-write the equation as

$$
u_{x}+\frac{1}{2} u_{t}=\frac{x}{2}
$$

and proceed as before. Thus, the characteristic equation becomes

$$
t_{x}=\frac{1}{2}
$$

with the initial condition $t(0)=\eta$ giving $t-x / 2=\eta$. However, contrary to the previous method, now $x$ is the independent variable so that $u(x, t)=u(t, t(x))=v(x, \eta)$ and we obtain

$$
v_{x}=\frac{x}{2}, \quad v(0)=\psi(\eta)
$$

yielding

$$
v(x, \eta)=\frac{1}{4} x^{2}+g(\eta)
$$

or, in the old variables,

$$
u(x, t)=\frac{1}{4} x^{2}+\psi(t-x / 2), \quad x>2 t
$$

exactly as in the previous method.
Method 3.
In this method we pretend that we know the initial condition on the whole line and will try to match this solution to the particular boundary condition. Thus, consider the solution

$$
u(x, t)=x t-t^{2}+\phi(x-2 t)
$$

for some function $\phi$. Since $\phi(x)=v_{0}(x)$ for $x>0$, the above function is a solution to our problem for $x>t / 2$. On the other hand, we need $u(0, t)=\psi(t)$ so that

$$
\psi(t)=-t^{2}+\phi(-2 t)
$$

which gives $\phi(s)=s^{2} / 4+\psi(-s / 2)$ for $s<0$. Hence, for $x<t / 2$,

$$
u(x, t)=x t-t^{2}+(x-2 t)^{2} / 4+\psi(t-x / 2)=x^{2} / 4+\psi(t-x / 2)
$$

as previously.
Example 1.5 Find the solution to the following initial value problem

$$
\begin{aligned}
u_{t}+x u_{x}+u & =0, \quad t>0, x \in \mathbb{R}, \\
u(x, 0) & =u_{0}(x)
\end{aligned}
$$

where $u_{0}(x)=1-x^{2}$ for $|x|<1$, and $u_{0}(x)=0$ for $|x| \geq 1$. The characteristic system is

$$
\begin{align*}
v_{t} & =-v, \\
x_{t} & =x, \tag{2.1.15}
\end{align*}
$$

with the initial conditions $v(0, \xi)=u_{0}(\xi), x(0, \xi)=\xi$ where $\xi$ is the intercept of the characteristic and the $x$-axis. The general solution of (2.1.15) is

$$
v(t, \xi)=a e^{-t}, \quad x(t, \xi)=b e^{t}
$$



Fig 4.4. Characteristics of the equation in Example 1.5.


Fig 4.5. 3-dimensional visualization of the solution in Example 1.5.
where $a, b$ are integration constants. Using the initial conditions we get

$$
v(0, \xi)=a=u_{0}(\xi), \quad x(0, \xi)=b=\xi
$$

Thus, $x(t, \xi)=\xi e^{t}$ and, eliminating $\xi$, we get $v(t, \xi)=u_{0}(\xi) e^{-t}$. Returning to the original variables

$$
u(x, t)=u_{0}\left(x e^{-t}\right) e^{-t}
$$

To find the solution of our particular initial value problem we have

$$
u_{0}(x)=u(x, 0)=C(x)
$$

and according to the definition of $C(x)=1-x^{2}$ for $|x|<1$ and $C(x)=0$ for $|x| \geq 1$. In the solution the function $C$ appears composed with $x e^{-t}$. Accordingly, $C\left(x e^{-t}\right)=1-x^{2} e^{-2 t}$ for $\left|x e^{-t}\right|<1$ and $C\left(x e^{-t}\right)=0$ for $\left|x e^{-t}\right| \geq 1$. Thus we obtain

$$
u(x, t)=\left\{\begin{array}{ccc}
\left(1-x^{2} e^{-2 t}\right) e^{-t} & \text { for } & |t|>\ln |x| \\
0 & \text { for } & |t| \leq \ln |x|
\end{array}\right.
$$



Fig 4.6. The graph of the solution in Example 1.5 for times $t=0,0.5,1,1.5,2,2.5$.

We observe that though the flow in this example is to the right if $x>0$, the initial-boundary problem

$$
\begin{aligned}
u_{t}+x u_{x}+u & =0, \quad t>0, x>0 \\
u(x, 0) & =u_{0}(x), x>0 \quad u(0, t)=\phi(t), t>0
\end{aligned}
$$

is not well-posed. Indeed, trying to use e.g. Method III of the previous example, we would have

$$
\psi(t)=u(0, t)=u_{0}(0) e^{-t}
$$

which clearly cannot be satisfied for an arbitrary function $\psi$. In fact, this shows that the behaviour at $x=0$ is predetermined by the model - it is the exponential decay of the initial concentration at $x=0$. Roughly speaking, this is due to the fact that the speed of the flow $c=x$ equals 0 at $x=0$ so there is no flow of particles with the initial position at $x=0$ and the only change comes from decay governed by $u_{t}=-u$. Alternatively, we observe that the characteristics $x(t)=b e^{t}$ can never cross $x=0$ and therefore no data at $x=0$ can be carried to the domain $x>0$.

The described procedure is also not restricted to equations in two independent variables. Consider, for instance, the equation

$$
\begin{equation*}
a u_{t}+b u_{x}+c u_{y}=0 \tag{2.1.16}
\end{equation*}
$$

This equation expresses the fact that the directional derivative in the direction of the vector $[a, b, c]$ is equal to zero, that is, that the solution does not change along any line with parametric equation $t=t_{0}+a s, x=$ $x_{0}+b s, y=y_{0}+c s$. Such a line can be written also as the pair of equations $a x-b t=\xi, c y-b t=\eta$. For each pair $\xi, \eta$ this pair describes a single line parallel to $[a, b, c]$, that is, the solution $u$ can be a function of $\xi$ and $\eta$ only. Thus we obtain the following general solution to (2.1.16)

$$
\begin{equation*}
u(x, y, t)=f(a x-b t, a y-c t) \tag{2.1.17}
\end{equation*}
$$

where $f$ is an arbitrary differentiable function of two variables.
Example 1.6 Find the solution to the following initial value problem:

$$
u_{t}+2 u_{x}+3 u_{y}+u=0, \quad u(x, y, 0)=u_{0}(x, y)=e^{-x^{2}-y^{2}} .
$$

The characteristic equation takes the form

$$
\begin{aligned}
v_{t} & =-v, \\
x_{t} & =2, \\
y_{t} & =3
\end{aligned}
$$

with initial conditions $v(0, \xi, \eta)=u_{0}(\xi, \eta), x(0, \xi, \eta)=\xi, y(0, \xi, \eta)=\eta$. Then $x-2 t=\xi, y-3 t=\eta$ and

$$
v(t, \xi, \eta)=C(\xi, \eta) e^{-t}
$$

Thus, $C(\xi, \eta)=u_{0}(\xi, \eta)$ and

$$
u(x, y, t)=u_{0}(x-2 t, y-3 t) e^{-t}
$$

Finally, using the initial equation, we get

$$
u(x, y, t)=e^{-(x-2 t)^{2}-(y-3 t)^{2}} e^{-t}
$$

### 1.1 The McKendrick equation

Consider the simplified population dynamics problem

$$
\begin{equation*}
n_{t}(a, t)+n_{a}(a, t)=-\mu n(a, t) \tag{2.1.18}
\end{equation*}
$$

coupled with the boundary condition

$$
n(0, t)=m \int_{0}^{\infty} n(a, t) d a
$$

and the initial condition

$$
n(a, 0)=n_{0}(a)
$$

First, let us simplify the equation (2.1.18) by introducing the integrating factor

$$
\left(e^{\mu a} n(a, t)\right)_{t}=-\left(e^{\mu a} n(a, t)\right)_{a}
$$

and denote $u(a, t)=e^{\mu a} n(a, t)$. Then

$$
u(0, t)=n(0, t)=m \int_{0}^{\infty} e^{-\mu a} u(a, t) d a
$$

with $u(a, 0)=e^{\mu a} n_{0}(a)=: u_{0}(a)$. Now, if we knew $\psi(t)=u(0, t)$, then

$$
u(a, t)= \begin{cases}u_{0}(a-t), & t<a  \tag{2.1.19}\\ \psi(t-a), & a<t\end{cases}
$$

The boundary condition can be rewritten as

$$
\begin{aligned}
\psi(t) & =m \int_{0}^{\infty} e^{-\mu a} u(a, t) d a=m \int_{0}^{t} e^{-\mu a} \psi(t-a) d a+m \int_{t}^{\infty} e^{-\mu a} u_{0}(a-t) d a \\
& =m e^{-\mu t} \int_{0}^{t} e^{\mu \sigma} \psi(\sigma) d \sigma+m e^{-\mu t} \int_{0}^{\infty} e^{-\mu r} u_{0}(r) d r
\end{aligned}
$$

which, upon denoting $\phi(t)=\psi(t) e^{\mu t}$ and using the original initial value, can be written as

$$
\begin{equation*}
\phi(t)=m \int_{0}^{t} \phi(\sigma) d \sigma+m \int_{0}^{\infty} n_{0}(r) d r \tag{2.1.20}
\end{equation*}
$$

Now, if we differentiate both sides, we get

$$
\phi^{\prime}=m \phi
$$

which is just a first order linear equation. Letting $t=0$ in (2.1.20) we obtain the initial value for $\phi$ : $\phi(0)=m \int_{0}^{\infty} n_{0}(r) d r$. Then

$$
\phi(t)=m e^{m t} \int_{0}^{\infty} n_{0}(r) d r
$$

and

$$
\psi(t)=m e^{(m-\mu) t} \int_{0}^{\infty} n_{0}(r) d r
$$

Then

$$
n(a, t)=e^{-\mu a} u(a, t)=e^{-\mu t} \begin{cases}n_{0}(a-t), & t<a \\ m e^{m(t-a)} \int_{0}^{\infty} n_{0}(r) d r, & a<t\end{cases}
$$

## Chapter 3

## First order nonlinear equations

## 1 Basic theory

The linear models discussed in the previous section described propagation of signals with speed that is independent of the solution. This is the case for e.g. acoustic signals that propagate at a constant sound speed. However, signals with large amplitude propagate at a speed that is proportional to the local density of the air. Thus, it is important to consider equations in which the speed of propagation depends on the value of the solution. The qualitative picture in such cases is completely different than for linear equations.

We shall focus on the simple nonlinear value problem:

$$
\begin{align*}
u_{t}+c(u) u_{x} & =0, \quad x \in \mathbb{R}, t>0 \\
u(x, 0) & =u_{0}(x), \quad x \in \mathbb{R} . \tag{3.1.1}
\end{align*}
$$

Here, $c$ is a given smooth function of $u$. The above equation is simply the conservation law

$$
u_{t}+\phi(u)_{x}=0
$$

after differentiation with respect to $x$ so that $c(u)=\phi_{u}(u)$.
To analyse (3.1.1) we assume at first that a $C^{1}$ solution $u(x, t)$ to (3.1.1) exists for all $t>0$. Motivated by the linear approach we define characteristic curves by the differential equation

$$
\begin{equation*}
\frac{d x}{d t}=c(u(x, t)) . \tag{3.1.2}
\end{equation*}
$$

Of course, contrary to the situation for linear equations, the right hand side of this equation is not known $a$ priori. Thus, characteristics cannot be determined in advance. However, assuming that we know them, we can solve (3.1.2) getting

$$
x=x(t, \eta)
$$

where $\eta$ is an integration constant. Fixing $\eta$ we obtain, as in the linear case, that

$$
\frac{d}{d t} u(x(t, \eta), t)=u_{x} x_{t}+u_{t}=u_{x} c(u)+u_{t}=0
$$

thus the solution is constant any characteristic and depends only on the single variable $\xi$. Now, we observe that along each characteristic at each point the slope is given by $c(u(x(t, \xi), t))$ but this depends only on $\xi$ that is fixed along each characteristic. Thus, the slope of each characteristic is constant so that each characteristic is a straight line. Alternatively, we can prove it by differentiating $x=x(t, \xi)$ twice to get

$$
\frac{d^{2} x}{d t^{2}}=\frac{d}{d t} c(u(x(t, \xi), t))=c_{u}(u) \cdot\left(u_{x} x_{t}+u_{t}\right)=c_{u}(u) \cdot\left(u_{x} c(u)+u_{t}\right)=0 .
$$

Thus, from each point $(x, t)$ we draw a straight line to an unspecified (yet) point $(\xi, 0)$ on the $x$-axis so that the equation of this characteristic is given by

$$
\begin{equation*}
x-\xi=c(u(\xi, 0)) t=c\left(u_{0}(\xi)\right) t \tag{3.1.3}
\end{equation*}
$$

because the slope is constant and therefore it must be equal to the value of $c(u)$ at the initial time $t=0$. Note that since $c$ and $u_{0}$ are known functions, then (3.1.3) determines $\xi$ implicitly in terms of a given $(x, t)$ (if it can be solved). Since the solution $u$ is constant along characteristics, we have the implicit formula for it

$$
u(x, t)=u(\xi, 0)=u_{0}(\xi)
$$

where $\xi$ is given by (3.1.3). In some instances (3.1.3) can be solved explicitly.
Example 1.1 Find the solution to the initial value problem

$$
\begin{aligned}
u_{t}+u u_{x} & =0, \quad x \in \mathbb{R}, t>0 \\
u(x, 0) & =x
\end{aligned}
$$

The characteristics emanating from a point $(\xi, 0)$ on the $x$-axis have speed $c\left(u_{0}(\xi)\right)=u_{0}(\xi)$. Using the discussion above we see that the solution is given implicitly by

$$
u(x, t)=c\left(u_{0}(\xi)\right)=\xi
$$

where

$$
x-\xi=t \xi
$$

so that easily

$$
\xi=\frac{x}{1+t}
$$

and the solution is given by

$$
u(x, t)=\frac{x}{1+t} .
$$

Note that in this example the solution is defined for values of $t>0$. This is not always the case and our prior discussion based on the assumption that for any $(x, t)$ with $t>0$ we have a smooth solution can be invalid. However, we can prove the following result

Theorem 1.1 If the functions $c$ and $u_{0}$ are $C^{1}(\mathbb{R})$ and if $u_{0}$ and $c$ are either both nondecreasing or both nonincreasing on $\mathbb{R}$, then the initial value problem (3.1.1) has a unique solution defined implicitly by the parametric equations

$$
\begin{align*}
u(x, t) & =u_{0}(\xi) \\
x-\xi & =c(u(\xi, 0)) t=c\left(u_{0}(\xi)\right) t \tag{3.1.4}
\end{align*}
$$

Proof. Since we have shown that if there is a smooth solution to (3.1.1), then it must be of the form (3.1.4), all we have to do is to show that (3.1.4) is uniquely solvable and that the function $u(x, t)$ obtained from (3.1.4) is, in fact, a solution of (3.1.1). As we already mentioned, the second equation of (3.1.4) should uniquely determine $\xi=\xi(x, t)$ for any $(x, t), t>0$. This is an implicit equation of the form

$$
F(\xi, x, t)=0
$$

where $F(\xi, x, t)=\xi+c\left(u_{0}(\xi)\right) t-x$ and, from the general theory, this equation is (locally) solvable around any point at which $F_{\xi} \neq 0$. In this case,

$$
F_{\xi}(\xi, x, t)=1+c_{u}^{\prime}\left(u_{0}(\xi)\right) u_{0, \xi}^{\prime} t \neq 0
$$

and since from the assumption, $c_{0}, u_{0}$ are either both increasing, or both decreasing, the derivatives are of the same sign and the second term is always positive. Thus, under the assumptions, $F_{\xi} \neq 0$ everywhere and (3.1.4) is uniquely solvable for any choice of $(x, t)$ and $\xi(x, t)$ is differentiable. Implicit differentiation of the first equation of (3.1.4) gives

$$
u_{t}(x, t)=u_{0, \xi}^{\prime} \xi_{t}, \quad u_{x}(x, t)=u_{0, \xi}^{\prime} \xi_{x}
$$

where, by implicit differentiation of the second equation in (3.1.4),

$$
-\xi_{t}=c^{\prime}\left(u_{0}(\xi)\right) u_{0, \xi}^{\prime} \xi_{t} t+c\left(u_{0}(\xi)\right), \quad 1-\xi_{x}=c^{\prime}\left(u_{0}(\xi)\right) u_{0, \xi}^{\prime} \xi_{x} t
$$

so that

$$
u_{t}(x, t)=-\frac{c\left(u_{0}(\xi)\right) u_{0, \xi}^{\prime}}{1+c_{u}^{\prime}\left(u_{0}(\xi)\right) u_{0, \xi}^{\prime} t}, \quad u_{x}(x, t)=\frac{u_{0, \xi}^{\prime}}{1+c_{u}^{\prime}\left(u_{0}(\xi)\right) u_{0, \xi}^{\prime} t},
$$

where the denominator is always positive by the argument above. Hence,

$$
u_{t}+c(u) u_{x}=0
$$

and we have indeed a unique solution defined everywhere for $t>0$.
The general method of characteristic system, described in Remark ?? can be used also for nonlinear equations, as illustrated in the example below. The resulting system in the nonlinear case become coupled and, in many cases, rather impossible to solve.

Example 1.2 Consider the initial value problem

$$
\begin{aligned}
u_{t}+u u_{x} & =-u, \quad x \in \mathbb{R}, t>0 \\
u(x, 0) & =-\frac{x}{2}, \quad x \in \mathbb{R} .
\end{aligned}
$$

The characteristic system for $v(t, \xi)=u(x(t), t)$ is

$$
\begin{aligned}
\frac{d v}{d t} & =-v \\
\frac{d x}{d t} & =v
\end{aligned}
$$

with the initial data $v(0)=-\xi / 2, x(0)=\xi$. The general solution of the system is

$$
v(t)=a e^{-t}, \quad x(t)=b-a e^{-t}
$$

Using the initial conditions, we obtain

$$
a=-\frac{\xi}{2}, \quad b=\frac{\xi}{2},
$$

so that

$$
v=-\frac{\xi}{2} e^{-t}, \quad x=\frac{\xi}{2}\left(1+e^{-t}\right) .
$$

Dividing $v$ by $x$, we eliminate $\xi$, getting

$$
u(x, t)=-\frac{x e^{-t}}{1+e^{-t}}
$$

This is a smooth solution defined for all $x \in \mathbb{R}$ and $t>0$.

It should be stressed that the situation described in the previous two examples are far from being typical. Firstly, in most cases the implicit equation for $\xi$ in (3.1.4) cannot be solved explicitly. Secondly, a solution may cease to exist after finite time, as shown in the example below.

Example 1.3 Consider the equation

$$
u_{t}+u u_{x}=0, \quad x \in \mathbb{R}, t>0
$$

with the initial condition

$$
u_{0}(x)=\left\{\begin{array}{lll}
2 & \text { for } & x<0 \\
2-x & \text { for } & 0 \leq x \leq 1 \\
1 & \text { for } & x>1
\end{array}\right.
$$

Since $c(u)=u$, the characteristics are straight lines emanating from $(\xi, 0)$ with speed $u_{0}(\xi)$. For $\xi<0$ these lines have speed 2 ; for $\xi>1$ the lines have speed 1 . For $0 \leq \xi \leq 1$ the lines have speed $2-\xi$ and their equations are

$$
t=\frac{1}{2-\xi}(x-\xi)
$$

and we see that all these lines pass through the point $(x, t)=(2,1)$. This means that the solution $(2,1)$ cannot exist as it should be equal to the value carried by each characteristic, and each characteristic carries different value of the solution. Thus, smooth solution cannot continue beyond $t=1$. Thus, at $t=1$ breaking of the wave occurs.

To find the solution for $t<1$, we first note that $u(x, t)=2$ for $x<2 t$ and $u(x, t)=1$ for $x>t+1$. For $2 t<x<t+1$, the second equation in (3.1.4) becomes

$$
x-\xi=(2-\xi) t
$$

that gives

$$
\xi=\frac{x-2 t}{1-t}
$$

and the first equation gives then

$$
u(x, t)=u_{0}(\xi)=2-\frac{x-2 t}{1-t}=\frac{2-x}{1-t}
$$

valid for $2 t<x<t+1, t<1$. The explicit form of the solution also indicates the difficulty at the breaking time $t=1$.

The phenomenon observed in the above example is typical when the speed of propagation $c$ is increasing and $u_{0}$ is increasing, or conversely, as in the above example. To explain this, let us concentrate on the initial value problem

$$
\begin{align*}
u_{t}+c(u) u_{x} & =0 \\
u(x, 0) & =u_{0}(x) \tag{3.1.5}
\end{align*}
$$

where $c(u)>0, c^{\prime}(u)>0$, and $u_{0}$ is a differentiable function on $\mathbb{R}$. We have already seen that if $u_{0}^{\prime} \geq 0$, then a smooth solution $u(x, t)$ exists for all $t>0$ and is given implicitly by

$$
u(x, t)=u_{0}(\xi), \quad x-\xi=c\left(u_{0}(\xi)\right) t
$$

Let, on the contrary, $u_{0}$ be such that for some $\xi_{1}<\xi_{2}$ we have $u_{0}\left(\xi_{1}\right)>u_{0}\left(\xi_{2}\right)$. Then clearly also $c\left(u_{0}\left(\xi_{1}\right)\right)>$ $c\left(u_{0}\left(\xi_{2}\right)\right)$, that is the characteristic emanating from $\xi_{1}$ is faster (has greater speed) that that emanating from $\xi_{2}$. Therefore the characteristics cross at some point $(x, t)$ which is a contradiction as the value $u(x, t)$ should be uniquely determined as $u_{0}\left(\xi_{1}\right)$ (or $u_{0}\left(\xi_{2}\right)$ ?).

As we observed in the previous example, at this point the gradient $u_{x}$ becomes infinite. That is why we say that at this point a gradient catastrophe occurs. Certainly, along different characteristics the gradient catastrophe can occur in different times; along some it is possible that it will never occur. It is possible to determine the breaking time, when a gradient catastrophe occurs, even if we do not know the explicit form of the solution. To do this, let us calculate the gradient $u_{x}$ along a characteristic that has the equation

$$
x-\xi=c\left(u_{0}(\xi)\right) t
$$

We assume that $c^{\prime}>0$ and $u_{0}^{\prime}(\xi)<0$ for this $\xi$. Let $g(t)=u_{x}(x(t), t)$ denotes the gradient of the solution along this characteristic. Then

$$
\frac{d g}{d t}=u_{t x}+c(u) u_{x x}
$$

and, on the other hand, differentiating the partial differential equation in (3.1.5) with respect to $x$, we find

$$
u_{t x}+c^{\prime}(u)\left(u_{x}\right)^{2}+c(u) u_{x x}=0
$$

thus we obtain

$$
\frac{d g}{d t}=-c^{\prime}(u) g^{2}
$$

along the characteristic. Along the characteristic $c^{\prime}(u)$ is constant (as $u$ is constant) and equal to $c^{\prime}\left(u_{0}(\xi)\right)$ and therefore this equation can be solved, giving

$$
g(t)=\frac{g(0)}{1+g(0) c^{\prime}\left(u_{0}(\xi)\right) t},
$$

where $g(0)$ is the initial gradient at $t=0$. But along the characteristic the initial gradient is $u_{0}^{\prime}(\xi)$, thus

$$
u_{x}=\frac{u_{0}^{\prime}(\xi)}{1+u_{0}^{\prime}(\xi) c^{\prime}\left(u_{0}(\xi)\right) t}
$$

This is the same formula for that gradient that we obtained in the proof of Theorem 1.1 and clearly the finitness of the gradient is equivalent to the solvability of the implicit equation for $\xi$. In any case, as $u_{0}^{\prime}$ and $c^{\prime}$ have opposite sign, the product is negative and there always exist time $t$ for which

$$
t u_{0}^{\prime}(\xi) c^{\prime}\left(u_{0}(\xi)\right)=-1
$$

If the initial condition is decreasing on $\mathbb{R}$, then the gradient catastrophe will occur along any characteristic. To find when the wave brakes, we must find the characteristic along which the catastrophe occurs first. Since, for a given $\xi$ the catastrophe time along this characteristic is given by

$$
t(\xi)=-\frac{1}{u_{0}^{\prime}(\xi) c^{\prime}\left(u_{0}(\xi)\right.}
$$

and since the denominator is the derivative of

$$
F(\xi)=c\left(u_{0}(\xi)\right)
$$

we conclude that the wave first breaks along the characteristic $\xi=\xi_{b}$ for which $F^{\prime}(\xi)<0$ attains minimum. Thus, the time of (the first) breaking is

$$
t_{b}=-\frac{1}{F^{\prime}\left(\xi_{b}\right)}
$$

The positive time $t_{b}$ is called the breaking time of the wave.
We remark that is the initial function $u_{0}$ is not monotone, breaking will first occur on the characteristic $\xi=\xi_{b}$, for which $F^{\prime}\left(\xi_{b}\right)<0$ and $F^{\prime}\left(\xi_{b}\right)$ is a minimum.

Example 1.4 Consider the problem

$$
\begin{aligned}
u_{t}+u u_{x} & =0 \\
u(x, 0) & =e^{-x^{2}}
\end{aligned}
$$

To determine the breaking time, we find

$$
F(\xi)=e^{-\xi^{2}}
$$

so that

$$
F^{\prime}(\xi)=-2 \xi e^{-\xi^{2}}, \quad F^{\prime \prime}(\xi)=-\left(4 \xi^{2}-2\right) e^{-\xi^{2}}
$$

Thus, $F^{\prime}(\xi)$ attains minimum at $\xi_{b}=\frac{1}{\sqrt{2}}$ and the breaking time is

$$
t_{b}=-\frac{1}{F^{\prime}\left(\xi_{b}\right)}=\frac{e^{1 / 2}}{\sqrt{2}} \approx 1.16
$$

Thus, the breaking will occur first along the characteristic emanating from $\xi_{b}=\frac{1}{\sqrt{2}}$ at $t_{b} \approx 1.16$.

## Is there life after the break time?

We have observed that at the break time the solution should become multivalued as it is defined by many characteristics each carrying a different value. However, in most physical problems described by this theory, the solution is just the density of some medium and is inherently single-valued. Therefore when breaking occurs, then

$$
\begin{equation*}
u_{t}+c(u) u_{x}=0 \tag{3.1.6}
\end{equation*}
$$

must cease to be valid as a description of the physical problem. Even in cases such as water waves, where a multivalued solution for the height of the surface could at least be interpreted, it is still found that (3.1.6) is inadequate to describe the process. Thus the situation is that some assumption or approximate relation leading to (3.1.6) is no longer valid. In principle one must return to the physics of the problem, see what went wrong, and formulate an improved theory. However, it turns out, that fortunately the foregoing solution can be saved by allowing discontinuities into the solution: a single-valued solution with a simple jump discontinuity replaces the multivalued continuous solution. This requires some mathematical extension of what we mean by a "solution" to (3.1.6) since, strictly speaking, the derivatives of $u$ do not exist at a discontinuity.
Let us recall the modelling process that led to (3.1.6). We started with the conservation law

$$
\begin{equation*}
\frac{d}{d t} \int_{a}^{b} u(x, t) d x=\phi(a, t)-\phi(b, t) \tag{3.1.7}
\end{equation*}
$$

where $\phi$ was the flux, and under assumption that both $u$ and $\phi$ are differentiable, we arrived at (3.1.6). Now, clearly $u$ is not known beforehand and, as we have observed in several examples, it can be not differentiable. Thus, while for functions $u$ and $\phi$ with jump discontinuities the conservation law (3.1.7) may have sense, one cannot derive (3.1.6) from it. Since in the modelling process (3.1.7) is more basic than (3.1.6), we will insist on the validity of the former without necessarily having (3.1.6).
Let us find out what type of discontinuities are allowed by (3.1.7). Assume that $x=s(t)$ is a smooth curve in space-time along which $u$ suffers a simple discontinuity, that is, $u(x, t)$ is continuously differentiable for $x>s(t)$ and $x<s(t)$, and that $u$ and its derivatives have one sided limits as $x \rightarrow s(t)^{-}$and $x \rightarrow s(t)^{+}$, for any $t>0$. If we chose $a$ and $b$ such that $a<s(t)<b$ then we can write (3.1.7) in the form

$$
\begin{equation*}
\frac{d}{d t} \int_{a}^{s(t)} u(x, t) d x+\frac{d}{d t} \int_{s(t)}^{b} u(x, t) d x=\phi(a, t)-\phi(b, t) \tag{3.1.8}
\end{equation*}
$$

The Leibniz rule for differentiating an integral whose integrand and limits depend on a parameter can be applied as the integrands are differentiable. In this way we obtain

$$
\begin{equation*}
\int_{a}^{s(t)} u_{t}(x, t) d x+\int_{s(t)}^{b} u_{t}(x, t) d x+u\left(s(t)^{-}, t\right) \frac{d s}{d t}-u\left(s(t)^{+}, t\right) \frac{d s}{d t}=\phi(a, t)-\phi(b, t) \tag{3.1.9}
\end{equation*}
$$

where

$$
u\left(s^{ \pm}(t), t\right)=\lim _{x \rightarrow s^{ \pm}(t)} u(x, t)
$$

and $d s / d t$ is the speed of discontinuity $x=s(t)$. Passing with $a \rightarrow s(t)^{-}$and $b \rightarrow s(t)^{+}$, we see that the integral terms on the left-hand side become zero as the integrands are bounded and the length of integration shrinks to zero. Thus, we obtain

$$
\begin{equation*}
\frac{d s}{d t}[u]=[\phi(u)], \tag{3.1.10}
\end{equation*}
$$

where the brackets denote the jump of the quantity inside across the discontinuity. Equation (3.1.10) is called the jump condition or Rankine-Hugoniot condition, and it relates the conditions ahead of the discontinuity and behind the discontinuity to the speed of the discontinuity itself. The discontinuity in $u$ that propagates along the curve $x=s(t)$ is called a shock wave, the curve itself is called the shock path, $d s / d t$ is the shock speed, and the magnitude of the jump $[u]$ is called the shock strength. We illustrate this discussion by continuing the solution of Example 1.3.

Example 1.5 Consider the conservation law

$$
\frac{d}{d t} \int_{a}^{b} u(x, t) d x=\frac{1}{2} u^{2}(a, t)-\frac{1}{2} u^{2}(b, t)
$$

that, for smooth solutions, reduces to

$$
u_{t}+u u_{x}=0, \quad x \in \mathbb{R}, t>0
$$

with the initial condition

$$
u_{0}(x)=\left\{\begin{array}{lll}
2 & \text { for } & x<0 \\
2-x & \text { for } & 0 \leq x \leq 1 \\
1 & \text { for } & x>1
\end{array}\right.
$$

Since characteristic cannot carry values of the solution across the shock, the only possible values behind the shock are $u^{-}=2$ and in front of the shock are $u^{+}=1$, where we denoted $u^{ \pm}=u\left(s^{ \pm}(t), t\right)$. As $\phi(u)=\frac{1}{2} u^{2}$, we have

$$
\frac{d s}{d t}=\frac{3}{2}
$$

and the shock propagates as $x(t)=s(t)=\frac{3 t}{2}+\frac{1}{2}$.
It must be remembered, however, that in most cases fitting the shock cannot be done explicitly as the values of $u^{+}$and $u^{-}$are not known.

## Rarefaction waves

Another phenomenon specific to nonlinear problems is creation of rarefaction waves which, in some sense, are opposite to shock waves. To illustrate it, consider the problem

$$
u_{t}+u u_{x}=0, \quad x \in \mathbb{R}, t>0
$$

with the initial condition

$$
u_{0}(x)=\left\{\begin{array}{lll}
0 & \text { for } & x<0 \\
1 & \text { for } & x>0
\end{array}\right.
$$

The characteristic diagram is easy. Recall that the characteristic crossing $t=0$ at $x=\xi$ for this equation is given by $x-\xi=t u_{0}(\xi)$. Thus, the slope of characteristics originating from $x>0$ is 1 ; that is, the characteristics are given by $x-t=\xi, \xi \geq 0$. For $x<0$, the characteristics are vertical as $u_{0}(x)=0$ and thus their equation in $x=\xi, \xi<0$. The solution is constant along the characteristics and therefore we have

$$
u(x, t)=\left\{\begin{array}{lll}
0 & \text { for } & x<0 \\
1 & \text { for } & x>t
\end{array}\right.
$$

We note, however, that there is a region $0<x / t<1$ in which the solution is not defined. A typical argument used to construct a solution filling this void is based on the understanding of the properties of the


Figure 3.1: The void region in the creation of a rarefaction wave
solutions of this equation, namely, that the solution is constant along characteristics. However, the slopes of characteristics originating from $t=0, x=0$ are anything between 1 and $\infty-$ straight lines filling the void. In other words, we introduce the 'fan' of characteristics $x=c t, 0 \leq c \leq 1$. If the solution is to be constant along characteristics, we must have

$$
u(x, t)=f(c)=f(x / t)
$$

with $f(0)=0$ and $f(1)=0$ to make the solution continuous. We must have

$$
u_{t}+u u_{x}=-f^{\prime} \frac{x}{t^{2}}+f f^{\prime} \frac{1}{t}=0
$$

By the boundary conditions, $f$ cannot be constant, hence $f^{\prime} \neq 0$. Thus, $f(x / t)=x / t$. Thus, we obtain continuous solution

$$
u(x, t)=\left\{\begin{array}{lll}
0 & \text { for } & x<0 \\
\frac{x}{t} & \text { for } & 0 \leq x \leq t \\
1 & \text { for } & x>t
\end{array}\right.
$$

Another justification of the particular form of the rarefaction wave is given by considering an approximating problem with continuous initial data

$$
u_{0}(x)=\left\{\begin{array}{lll}
0 & \text { for } & x<0 \\
\frac{x}{a} & \text { for } & 0 \leq x \leq a \\
1 & \text { for } & x>a
\end{array}\right.
$$

for small $a$. For $0<\xi<a$ the characteristic is $x-\xi=t \xi / a$ and carries the solution $\xi / a$. Thus

$$
\xi=\frac{a t}{a+t}
$$

and the solution in $0<x<t+a$ is

$$
u_{a}(x, t)=\frac{a}{a+t} .
$$

In particular, the region $0<x<t$ is contained in the above region for any $a$ and thus we see that in this region the solutions $u_{a}$ converge to $x / t$ as $a \rightarrow 0$, in accordance with the considerations above. By using more sophisticated methods one can prove that this is a unique solution which is physically admissible in the sense that it corresponds to nondecreasing entropy. Let us consider some more complicated scenarios in which rarefaction and shocks occur.

Example 1.6 Find the solution to the following problem

$$
u_{t}+u u_{x}=0,-\infty<x<\infty, t>0
$$



Figure 3.2: Fitting the fan of characteristics


Figure 3.3: Snapshot of the solution at time $t$
with the initial condition

$$
u_{0}(x)=\left\{\begin{array}{lll}
1 & \text { for } & x<0 \\
-1 & \text { for } & 0 \leq x \leq 1 \\
0 & \text { for } & x>1
\end{array}\right.
$$

We start with considering characteristics. We have a family of characteristics with speed 1 going to the right from $x<0$ a family with speed -1 going to the left from $[0,1]$ and vertical (speed 0 ) emanating from $(1, \infty)$. Clearly, we have immediate clash of characteristics at $x=0$ where the characteristics from the left carry value 1 and from the right carrying value -1 . This will create a shock wave with speed

$$
s^{\prime}=\frac{1-1}{2}=0
$$

which will persists till $t=1$ where there are no longer characteristics coming from the right.
We observe that between the characteristic $t=-x+1$ and $x=1$ we have a void; that is, we must fit the rarefaction wave. Following the argument used earlier, we see that the characteristics fanning out from $x=1, t=0$ have the equation

$$
c=\frac{x-1}{t}
$$

with $-1<c<0$ and thus the solution is

$$
u(x, t)=\frac{x-1}{t}
$$

Hence, in the strip $0<x<1, t \geq 0$, we have values $u(x, t)=1$ coming from the left and $u(x, t)=(x-1) / t$ coming from the right. Hence, the shock speed is given by

$$
\frac{d x}{d t}=s^{\prime}=\frac{x-1+t}{2 t}
$$

which can be re-written as

$$
\frac{d x}{d t}=\frac{x}{2 t}+\frac{1}{2}-\frac{1}{2 t}
$$

which is a first order linear equation. This equation is supplemented by the initial conditions determined by the beginning of the shock $x(1)=0$. The integrating factor is $\mu=1 / \sqrt{t}$ so that, multiplying by it we get

$$
\left(\frac{x(t)}{\sqrt{t}}\right)^{\prime}=\frac{1}{2 \sqrt{t}}-\frac{1}{2 t \sqrt{t}}
$$

Integrating, we get

$$
\frac{x(t)}{\sqrt{t}}=\sqrt{t}+\frac{1}{\sqrt{t}}+C
$$

and using the initial condition $C=-2$ so that the shock is given by

$$
x(t)=t+1-2 \sqrt{t}
$$

which is valid, however, only as long as we stay in the strip $0<x<1$; that is, for $t<4$. For $x>1$, the values $u=1$ carried by the characteristics from $x<0$ meet the values $u=0$ carried by vertical characteristics from $x>1$. Thus, the shock will travel with the speed

$$
\frac{d x}{d t}=\frac{1+0}{2}=\frac{1}{2}
$$

and, starting from $t=4, x=1$, its path is given by the formula

$$
x(t)=\frac{1}{2} t-1
$$



Figure 3.4: Characteristics in Example 1.6


Figure 3.5: Development of the shock in Example 1.6

In the next two examples we consider signalling problems.
Example 1.7 Consider the following problem

$$
u_{t}+u u_{x}=0, x>0, t>0
$$

with the initial condition

$$
u_{0}(x)=\left\{\begin{array}{lll}
0 & \text { for } & 0<x \leq 1 \\
x-1 & \text { for } & 1 \leq x \leq 2 \\
1 & \text { for } & x>2
\end{array}\right.
$$

and the boundary condition $u(0, t)=2$ for $t \geq 0$.
The characteristics emanating from $x>0$ are: vertical $(x=\xi)$ for $0 \leq \xi \leq 1$,

$$
x-\xi=t(\xi-1), \quad 1 \leq \xi \leq 2
$$

and $t=x-\xi$ for $\xi>2$ and we see that, but for the boundary condition, we would have a smooth solution defined everywhere in $t>0, x>0$. However, we have characteristics entering through $x=0$ carrying value 2 of the solution. Let us find these characteristics. The fact that the characteristics are straight lines was proved without any reference to initial or boundary conditions - it is a consequence of the fact that the equation does not have any source terms and that $c$ only depends on $u$. Thus, in the same way as for the initial value problem, we find the equation to be

$$
\frac{x}{t-\eta}=c(u(\eta))
$$

where $\eta$ is the coordinate of the intercept of the characteristic passing through $(x, t)$ with the axis $x=0$. Hence $u(0, \eta)$ is the given boundary value. In our case

$$
\frac{x}{t-\eta}=2
$$



Figure 3.6: Snapshots of the solution in Example 1.6, down the columns starting from the left top


Figure 3.7: 3D picture of the solution in Example 1.6
and the equation of the characteristic is

$$
t=\frac{1}{2} x+\eta
$$

It is clear that these characteristics clash with the characteristics immediately at $t=0$ and thus the shock starts at $x=0, t=0$. Since the characteristics entering from $x=0$ carry the value 2 and the vertical characteristics from $0<x<1$ carry the value 0 , we immediately obtain the equation for the shock

$$
\frac{d x}{d t}=s^{\prime}=\frac{2+0}{2}=1
$$

and so $x(t)=t$. This shock will continue until it reaches the the line $x=1$ (at $t=1$ ) whereupon in the front of the wave we will have the value carried by the characteristics $x-\xi=t(\xi-1), 1 \leq \xi \leq 2$. To find these values, we solve for $\xi$ :

$$
\xi=\frac{t+x}{t+1}
$$

so that the solution is

$$
u(x, t)=u_{0}(\xi)=\xi-1=\frac{x-1}{t+1}
$$

Thus, at the shock we would have

$$
\frac{d x}{d t}=\frac{2+\frac{x-1}{t+1}}{2}=\frac{x}{2(t+1)}+\frac{2 t+1}{2(t+1)}
$$

which is a linear equation, which we supplement with the initial condition $x=1, t=1$ at the beginning of the shock. To simplify notation, let $\tau=t+1$ so that the equation becomes

$$
x^{\prime}=x / 2 \tau+1-1 / 2 \tau
$$

The integrating factor is $1 / \sqrt{\tau}$ so that

$$
(x / \sqrt{\tau})^{\prime}=1 / \sqrt{\tau}-1 /(2 \tau \sqrt{\tau})
$$

and solving and returning to the original variables we get

$$
x(t)=2(t+1)+1+C \sqrt{t+1}
$$

and, using the initial condition, $C=-4 / \sqrt{2}$ hence

$$
x(t)=2 t+3-4 \sqrt{\frac{t+1}{2}} .
$$

This shock continues till it reaches the characteristic $t=x-2$; that is, until the time $t$ which is the solution to

$$
2 t+3-4 \sqrt{\frac{t+1}{2}}=t+2
$$

or $t=7$. At this time the shock, still carrying the value 2 from the left, enters the region where the characteristics carry value 1 from the right and thus the shock's speed is

$$
s^{\prime}(t)=\frac{2+1}{2}=\frac{3}{2}
$$

so that the shock's path is

$$
x(t)=\frac{3}{2} t+C
$$

with $C$ to be determined by the condition $x=9, t=7: 9=21 / 2+C$; that is, $C=-3 / 2$. Hence

$$
x(t)=\frac{3}{2} t-\frac{3}{2} .
$$



Figure 3.8: Calculation of characteristics in a signalling problem


Figure 3.9: Characteristics in Example 1.7


Figure 3.10: Development of the shock in Example 1.7

Example 1.8 Consider the following problem

$$
u_{t}+u u_{x}=0, x>0, t>0
$$

with the initial condition $u_{0}(x)=1, x>0$ and the boundary condition $u(0, t)=t+1$ for $t \geq 0$.
We have characteristics $t=x-\xi, \xi>0$ while the characteristics emanating from $x=0$ are given implicitly by

$$
\frac{x}{t-\eta}=\eta+1, \quad \eta>0
$$

or

$$
\begin{equation*}
-\eta^{2}+\eta(t-1)+t-x=0 . \tag{3.1.11}
\end{equation*}
$$

The principle for determining the break time is the same as in the case of the initial value problem - for a fixed characteristics we find time $t$ for which the point $(x(t), t)$ on this characteristic does not determine $\eta$ in a unique way. In our case, the ' $\xi$-characteristics' always determine a unique $\xi$ for any given $(x, t)$. However, along the ' $\eta$-characteristics' we find that

$$
0=F(x, t, \eta)=-\eta^{2}+\eta(t-1)+t-x
$$

cannot be solved if

$$
0=F_{\eta}(x, t, \eta)=-2 \eta+t-1
$$

or $t=2 \eta+1$ (this gives the solvability region $0<x<(t+1)^{2} / 4$ ). The minimum over $(0, \infty)$ occurs at $\eta=0$ so that the breaking time is $t_{b}=1$. Thus, for $0<t<1$ we can write the solution as

$$
u(x, t)= \begin{cases}\eta(x, t) & \text { for } \quad 0<x<t, 0<t<1 \\ 1 & \text { for } \quad x>t, 0<t<1\end{cases}
$$

where $\eta(x, t)$ is given as the positive solution of (3.1.11); that is,

$$
\begin{equation*}
\eta(x, t)=\frac{t-1+\sqrt{(t-1)^{2}+4(t-x)}}{2} . \tag{3.1.12}
\end{equation*}
$$

Let us find the equation of the shock wave. The shock will have values transported by ' $\eta$-characteristics' from the left and ' $\xi$-characteristics' from the right; that is

$$
\begin{equation*}
\frac{d x}{d t}=s^{\prime}(t)=\frac{\eta(x, t)+1}{2} \tag{3.1.13}
\end{equation*}
$$

However, substituting $\eta$ found above into this expression creates a formidable equation which seems to be impossible to solve directly. Since the right hand side appears relatively simple, let us try to express $d x / d t$ in terms of $d \eta / d t$. Differentiating (3.1.11) implicitly with respect to $t$, we get

$$
-2 \eta \eta_{t}+\eta_{t}(t-1)+\eta+1-x_{t}=0
$$

or

$$
x_{t}=\eta_{t}\left(t-2 \eta_{t}-1\right)+\eta+1
$$

Substituting into (3.1.13) and simplifying, we obtain

$$
\frac{d \eta}{d t}=\frac{\eta}{2(2 \eta-(t-1))}
$$

This equation simplifies if we assume $d \eta / d t \neq 0$ (that is, $\eta \neq 0$ ). Then we can treat $\eta$ as the independent variable $t=t(\eta)$ and, substituting $\tau=t-1$, we get

$$
\begin{equation*}
\frac{d \tau}{d \eta}=4-\frac{2 \tau}{\eta} \tag{3.1.14}
\end{equation*}
$$

which is a linear ordinary differential equation. We see that the integrating factor is $\mu=\eta^{2}$ so that

$$
\left(\eta^{2} \tau\right)^{\prime}=4 \eta^{2}
$$

that is

$$
\begin{equation*}
\eta^{2} \tau=\frac{4}{3} \eta^{3}+C \tag{3.1.15}
\end{equation*}
$$

Now, the shock starts at $x=1, t=1$ or $\eta=0, \tau=0$ which is outside the range for which the equation is valid. However, if we insists that the above solution is valid as $\eta \rightarrow 0$ and $\tau \rightarrow 0$, we arrive at $C=0$. For nonzero $\eta$ we get

$$
\eta=\frac{3}{4}(t-1)
$$

and, substituting to (3.1.11), we find

$$
\begin{equation*}
x(t)=(1-\eta)(\eta-1)=\left(\frac{3}{4}(t-1)+1\right)\left(t-\frac{3}{4}(t-1)\right)=\frac{1}{16}(3 t+1)(t+3) \tag{3.1.16}
\end{equation*}
$$

We note that up to now the process has been purely formal - we are not sure that all the assumptions for the performed transformations to be correct are satisfied. However, we can validate the process of obtaining the solution by first checking that it fits into the region where $\eta_{t}$ is defined: $t<x<(t+1)^{2} / 4$ and substituting the above function defined (3.1.16) into (3.1.12) to ascertain that indeed it is a solution. The question remains whether (3.1.16) is the only solution. We note that $(1,1)$ is the singular point of (3.1.12) where the assumptions of the Picard theorem are not satisfied and thus there may be other solutions. However, we observe that in the open region $t<x<(t+1)^{2} / 4$ the equations (3.1.12) and (3.1.14) are equivalent and thus any solution in this region is given by (3.1.15). If we want this solution to converge to $(\eta, \tau)=(0,0)$ we must have $C=0$ and thus the solution $x(t)$ we obtained is the only solution in the region $t<x<(t+1)^{2} / 4$ satisfying $x(1)=1$.

## 2 Application to traffic flow

We illustrate the ideas introduced above to the classical model of one dimensional traffic flow.
Imagine a long stretch of narrow straight road with heavy traffic. Then, instead of considering separate cars we can talk about a traffic flow by introducing the traffic density $\rho$ which is the number of vehicles per unit length of the road and, instead of considering the speed of a particular car, we can consider the speed $u(x, t)$ of traffic at a particular point $x$ and time $t$. Related to it is the traffic flow $\phi(x)$ at $x$ defined, as usual, as the number of cars passing through $x$ at time $t$ per unit time. It is easy to see that, for an observer sitting at $x$, over a short time $\Delta t$ the traffic will move by the length $u \Delta t$ but in this length we have $\rho u \Delta t$ vehicles. Thus, in unit time on average we will have the flux

$$
\phi=u \rho
$$

at $x$ and $t$. Assuming that no new cars enter the road nor vanish from it, we have the conservation law

$$
\frac{d}{d t} \int_{a}^{b} \rho(x, t) d x=\phi(a, t)-\phi(b, t)
$$

or, under assumption that $u$ and $\phi$ are differentiable,

$$
\begin{equation*}
\rho_{t}+\phi_{x}=0 \tag{3.2.17}
\end{equation*}
$$

Our considerations will be carried out under simplifying assumption that the flux only depends on the density of traffic

$$
\phi=\phi(\rho)
$$



Figure 3.11: Typical shape of a flux-density dependence
which also implies that the traffic speed at any point is a function of only traffic density

$$
\phi=\phi(\rho)=\rho u(\rho) .
$$

Let us say a few words about this assumption and about reasonable functional dependance between $\rho$ and $u$. First, if there is no traffic, $\rho=0$, and the drivers can move with maximum allowed speed, say, $u_{\max }$. If there are more and more cars per kilometer; that is, the density increases the speed usually must go down. Thus

$$
u^{\prime}(\rho)<0, \quad \rho \in\left[0, \rho_{j}\right]
$$

and finally there is the density $\rho_{j}$ at which the traffic becomes jammed: $u\left(\rho_{j}\right)=0$. For the flux these assumptions yield:

$$
\begin{array}{lll}
\phi(\rho)>0 & \text { for } \quad \rho \in\left(0, \rho_{j}\right), \\
\phi(\rho)=0 & \text { for } & \rho \notin\left(0, \rho_{j}\right) . \tag{3.2.18}
\end{array}
$$

Further,

$$
\phi^{\prime}(\rho)=u(\rho)+\rho u^{\prime}(\rho)
$$

with $\phi^{\prime}(0)=u(0)=u_{\max }>0$ and $\phi^{\prime}\left(\rho_{j}\right)=u\left(\rho_{j}\right)+\rho_{j} u^{\prime}\left(\rho_{j}\right)=\rho_{j} u^{\prime}\left(\rho_{j}\right)<0$. Next,

$$
\phi^{\prime \prime}(\rho)=2 u^{\prime}(\rho)+u^{\prime \prime}(\rho) .
$$

Typically it is assumed that $u$ is such that $\phi$ is concave down; that is, $\phi^{\prime \prime}<0$. Under this assumption there is a unique $\rho_{\max } \in\left(0, \rho_{j}\right)$ such that $\phi^{\prime}\left(\rho_{\max }\right)$ and $\phi$ attains maximum there. This maximum flow is often called by traffic engineers the capacity of the road - note that it occurs at a speed which is lower than the maximum speed - here we have a source of a conflict between traffic authorities who want to maintain the maximum traffic flow and reduce speed to $u\left(\rho_{\max }\right)$ and individual drivers who want to get from A to B in minimum time opting for $u_{\max }$.

We say that the traffic is light if $0 \leq \rho \leq \rho_{\max }$ and heavy if $\rho_{\max }<\rho \leq \rho_{j}$.
To make another observation which does not require solving the traffic flow equation we write the equation (3.2.17) in a more familiar form

$$
\begin{equation*}
\rho_{t}+\phi_{x}(\rho)=\rho_{t}+\phi^{\prime}(\rho) \rho_{x}=0 \tag{3.2.19}
\end{equation*}
$$

where $\phi^{\prime}(\rho)=c(\rho)=u(\rho)+\rho u^{\prime}(\rho)$ is the characteristic speed. We remember from the theory of the quasilinear equation that the characteristic equation is $d x / d t=c(\rho)$ and we see that the density wave moves through traffic at this speed that is, since the equation is homogeneous, the density stays the same along each characteristic. In particular, if traffic is light, the density wave moves forward and if the traffic is heavy, it moves backward.

It is important to realize here that we have two speeds involved in the model - the traffic speed and the speed of the density wave (the speed with which the signal propagates through traffic). Now

$$
c(\rho)=u(\rho)+\rho u^{\prime}(\rho)<u(\rho)
$$

on account of $u^{\prime}$ being a decreasing function, thus the density wave travels slower than the traffic.
In the following examples we shall take a simple linear speed-density dependance

$$
u(\rho)=u_{\max }\left(1-\frac{\rho}{\rho_{j}}\right)
$$

yielding

$$
\phi(\rho)=u_{\max } \rho\left(1-\frac{\rho}{\rho_{j}}\right)
$$

It is easy to see that the characteristic speed is

$$
c(\rho)=\phi^{\prime}(\rho)=u_{\max }\left(1-\frac{2 \rho}{\rho_{j}}\right) .
$$

In other words, we will be working with the equation

$$
\begin{equation*}
\rho_{t}+u_{\max }\left(1-\frac{2 \rho}{\rho_{j}}\right) \rho_{x}=0 \tag{3.2.20}
\end{equation*}
$$

From the expression for $\phi^{\prime}$ we see that the maximum flow occurs at $\rho_{\max }=\rho_{j} / 2$ corresponding to the optimal speed $u_{\max } / 2$.
We can simplify the equation by introducing a dimensionless density $g=\rho / \rho_{j}$. We also shorten notation by denoting $u_{\max }=v$. Then the equation turns into

$$
\begin{equation*}
g_{t}+v(1-2 g) g_{x}=0 \tag{3.2.21}
\end{equation*}
$$

with $0 \leq g \leq 1$.

Example 2.1 Consider the following problem. It is observed that at a red light at an intersection a typical queue of cars is of length $x_{0}$ with no traffic behind the back of the queue and no traffic in front of the traffic lights. What should be the green phase to allow all cars in the queue pass the intersection.

We assume that the traffic obeys Eq. (3.2.23) and that the traffic lights are at $x=0$. The fact that the cars do not move in front of the red light is modelled by the initial condition

$$
g_{0}(x)=\left\{\begin{array}{lll}
0 & \text { for } & x<-x_{0} \\
1 & \text { for } & -x_{0}<x<0 \\
0 & \text { for } & x>0
\end{array}\right.
$$

(recall that $g=1$ corresponds to the maximum density.) To solve the problem, first we find characteristics. Observe that contrary to the previous examples, here we have $c(g)=v(1-2 g)$ and thus the characteristic speed is $v$ for $\xi<-x_{0}$ and $\xi>0$ and $-v$ for $-x_{0}<\xi<0$. Hence, we have the following families of characteristics $x-\xi=v t$ for $\xi<-x_{0}$ and $x>0$ and $x-\xi=-v t$ for $-x_{0}<\xi<0$. Therefore we have a rarefaction wave originating at $x=0, t=0$ and a shock wave originating at $x=-x_{0}$ and $t=0$. In this model the interpretation of these waves is simply that in the front of the queue cars immediately start moving with the maximum velocity $v$ (as there is no traffic in front and the shock wave corresponds to the discontinuity between the back of the queue and the no traffic zone behind the queue). Mathematically, the fan of characteristics is given by $x / t=c(\rho)$ with $-v<c(\rho)<v$ but here $c$ is given by

$$
c(g)=v(1-2 g)
$$

so that the solution is

$$
\begin{equation*}
g(x, t)=\frac{1}{2}\left(1-\frac{x}{v t}\right), \quad-v<\frac{x}{t}<v \tag{3.2.22}
\end{equation*}
$$

as long as the fan is in the region not affected by characteristics coming from $x<-x_{0}$. We note that a car in the queue at position $x$ can move only when the rarefaction wave reaches $x$; that is, at time $t=-x / v$. Thus, the last car at $x=-x_{0}$ starts moving only at $t=x_{0} / v$ after the light turned green.

At the other end we have immediate clash of characteristics generating a shock wave of speed

$$
s^{\prime}(t)=\frac{\phi^{+}-\phi^{-}}{g^{+}-g^{-}}=\frac{0-v 1(1-1)}{0-1}=0
$$

where we used $\phi(g)=v g(1-g)$ which is 0 at both $g=0$ and $g=1$. Thus the shock initially does not move which corresponds to the situation when the rarefaction wave has not yet reached the end of the queue; that is, till $t=x_{0} / v$. When $t>x_{0} / v$ the characteristics $x-v t=\xi$, carrying value 0 , will meet characteristics from the fan. The resulting shock is determined by the following equation

$$
\frac{d x}{d t}=s^{\prime}(t)=\frac{\phi^{+}-\phi^{-}}{g^{+}-g^{-}}=\frac{0-v g(1-g)}{0-g}=v(1-g)=v\left(\frac{1}{2}-\frac{x}{2 v t}\right)
$$

where $g$ was determined from (3.2.22). This is a linear first order equation

$$
x^{\prime}=\frac{x}{2 t}+\frac{v}{2}
$$

with integrating factor $1 / \sqrt{t}$. Thus

$$
\left(\frac{x}{\sqrt{t}}\right)^{\prime}=\frac{v}{2 \sqrt{t}}
$$

solution of which is given by

$$
x(t)=v t+C \sqrt{t}
$$

with $C$ determined from the initial condition $x=-x_{0}$ at $t=x_{0} / v$, that is,

$$
-x_{0}=x_{0}+C \sqrt{\frac{x_{0}}{v}}
$$

or $C=-2 \sqrt{v x_{0}}$, giving

$$
x(t)=v t-2 \sqrt{v x_{0} t} .
$$

Since the shock wave traces the last car, the queue disappears when the shock reaches $x=0$ so at time

$$
t=4 \frac{x_{0}}{v}
$$

which is the minimum time for the green light to be on to clear the congestion of a queue of length $x_{0}$.
It is instructive to look at this process from the point of view of a particular car in the queue. Suppose the car is located at $-x_{0}<\xi<0$. As we know from the above considerations, the car will remain stationary for $t=\xi / v$ from the moment the light changes to green. We know that the speed of the traffic at point $x$ is given by $u(x, t)=v(1-g(x, t))$ but this is also the speed of any car which happens to be at this point at time $t$. Thus, the speed of the car is given implicitly as

$$
\frac{d x_{c}}{d t}=v\left(1-g\left(x_{c}, t\right)\right)
$$

where $g$ is the normalized density of the traffic and $x_{c}$ the position of the car. We observe that as soon as the car starts moving, it moves in the rarefaction wave with density determined by (3.2.22). That is

$$
\frac{d x_{c}}{d t}=v\left(1-\frac{1}{2}\left(1-\frac{x_{c}}{v t}\right)\right)=\frac{v}{2}+\frac{x_{c}}{2 t}
$$



Figure 3.12: Rarefaction wave when light turns green


Figure 3.13: Path of a car when light turns green
which is exactly the differential equation of the shock wave (which we interpreted as the trajectory of the last car in the queue), only with different initial conditions. This gives the trajectory of the car

$$
x_{c}(t)=v t-2 \sqrt{-v \xi t}
$$

(remember $\xi<0$ ) and the time when the car passes the lights is $t=-4 \xi / v$.
Next we consider what happens when the light turns red.

Example 2.2 Consider a uniform traffic moving to the right with (normalized) density $g_{0}<1$. Assume that at $t=0$ the traffic light at $x=0$ turns red. To model this we note that when the lights are red, the moving cars must stop and cars do not move only when $g=1$ ( $\left.\rho=\rho_{\text {max }}\right)$. Thus, the traffic behind the lights can be modelled by the signalling problem

$$
\begin{equation*}
g_{t}+v(1-2 g) g_{x}=0, \quad x<0, t>0 \tag{3.2.23}
\end{equation*}
$$

and

$$
g(x, 0)=g_{0}, x<0, \quad g(0, t)=1,0<t<t_{0}
$$

where $t_{0}$ is the length of the red phase. Then the characteristics emanating from $x<0$ are given by

$$
x-\xi=t c\left(g_{0}\right)=t v\left(1-2 g_{0}\right)
$$

whereas from $x=0$ by

$$
x=(t-\eta) c(1)=v(t-\eta) .
$$

Depending on whether the traffic is heavy $\left(g_{0}>1 / 2\right)$ or light $\left(g_{0}<1 / 2\right)$, the characteristic speed of ' $\xi$ 'characteristics is, respectively, negative or positive, but in both cases $v(1-2 g)>-v$ which means that the ' $\xi$ '-characteristics are steeper than the ' $\eta$ '-characteristics and therefore we observe creation of a shock originating from $t=0$ and $x=0$. In this case the position of the shock indicated the point to the right of which the traffic is already jammed and to the left it is still moving. The shock speed is given as before by

$$
\frac{d x}{d t}=s^{\prime}(t)=\frac{\phi^{+}-\phi^{-}}{g^{+}-g^{-}}=\frac{0-g_{0} u\left(g_{0}\right)}{1-g_{0}}=\frac{0-v g_{0}\left(1-g_{0}\right)}{1-g_{0}}=-v g_{0}
$$

and the shock path is given by

$$
\begin{equation*}
x(t)=-v g_{0} t \tag{3.2.24}
\end{equation*}
$$

As before, we can look at this problem from the point of view of an individual car which is at a distance $\xi$ from the origin when the light turns red. Here it is better to use the real density $\rho_{0}$ and not the normalized one. Since the density of traffic is uniform, we have approximately $N=\xi \rho_{0}$ cars between our car and the traffic light. The car will stop at the point $-\xi_{1}$ such that the density on $\left[-\xi_{1}, 0\right]$ is $\rho_{j}$. Thus, we must have $\rho_{j}=N / \xi_{1}$. Solving for $N$ we obtain

$$
\xi_{1}=\xi \rho_{0} / \rho_{j} .
$$

On the other hand, since the density, and hence the traffic speed, is uniform, the path between $-\xi$ and $-\xi_{1}$ is covered in $t=\left(\xi-\xi_{1}\right) / u\left(\rho_{0}\right)$. Denoting $-\xi_{1}=x(t)$ and eliminating $\xi$ we have

$$
x(t)=\frac{\rho_{0}}{\rho_{j}}\left(u\left(\rho_{0}\right) t+x(t)\right)
$$

which, due to $u\left(\rho_{0}\right)=v\left(1-\rho_{0} / \rho_{j}\right)$, yields

$$
x(t)=-\frac{v \rho_{0}}{\rho_{j}} t=-v g_{0} t
$$

in perfect agreement with (3.2.24).



Figure 3.14: Characteristics for heavy traffic (left) and light traffic (right)


Figure 3.15: Shock path when light turns red

## 3 An exactly solvable McKendrick nonlinear model

In Subsection 1.1 we introduced the age structured McKendrick population model

$$
\begin{aligned}
& \frac{\partial n(a, t)}{\partial t}+\frac{\partial n(a, t)}{\partial a}=-\mu(a, t) n(a, t), \quad t, a>0 \\
& n(0, t)=\int_{0}^{\infty} n(a, t) m(a, t) d a, \quad t>0 \\
& n(a, 0)=n_{0}(a), \quad a>0
\end{aligned}
$$

where $\mu$ is the death rate, $m$ is the maternity rate and $\infty$ is the maximum age of individuals in the population. In many cases the assumption that $\mu$ and $m$ do not depend on the population is an oversimplification. A more realistic system

$$
\begin{align*}
& \frac{\partial n(a, t)}{\partial t}+\frac{\partial n(a, t)}{\partial a}=-\mu(a, t, N) n(a, t), \quad t, a>0 \\
& n(0, t)=\int_{0}^{\infty} n(a, t) m(a, t, N) d a, \quad t>0 \\
& n(a, 0)=n_{0}(a), \quad a>0 \tag{3.3.25}
\end{align*}
$$

where

$$
N(t)=\int_{0}^{\infty} n(a, t) d a
$$

is the total population at time $t$. This makes (3.3.25) a (badly) nonlinear equation which only can be solved in very special cases. We shall describe one such case. Consider

$$
\begin{align*}
& \frac{\partial n(a, t)}{\partial t}+\frac{\partial n(a, t)}{\partial a}=-\mu_{0} N(t) n(a, t), \quad t, a>0 \\
& n(0, t)=\int_{0}^{\infty} n(a, t) m_{0} N(t) d a, \quad t>0, \\
& n(a, 0)=n_{0}(a), \quad a>0 \tag{3.3.26}
\end{align*}
$$

where $\mu_{0}, m_{0}$ are positive constants. Let us integrate the first and second equation with respect to $a$. Let us start with observing that

$$
\int_{0}^{\infty} n_{a}(a, t) d a=\lim _{a \rightarrow \infty} n(a, t)-n(0, t)=-m_{0} N \int_{0}^{\infty} n(a, t) d a-0=-m_{0} N^{2}(t),
$$

where we used $\lim _{a \rightarrow \infty} n(a, t)=0$ which reflects the the assumption that the number of individuals surviving till an old age is decreasing to 0 . Thus, integration of the first and the last equation in (3.3.28) yields

$$
\begin{equation*}
N_{t}=N^{2}\left(m_{0}-\mu_{0}\right), \quad N(0)=N_{0}=\int_{0}^{\infty} n_{0}(a) d a \tag{3.3.27}
\end{equation*}
$$

There are two cases to consider.
a) $\mu_{0}=m_{0}$.

In this case, $N(t)=N_{0}$ for all $t \geq 0$ and (3.3.28) turns into

$$
\begin{align*}
& \frac{\partial n(a, t)}{\partial t}+\frac{\partial n(a, t)}{\partial a}=-\mu_{0} N_{0} n(a, t), \quad t, a>0 \\
& n(0, t)=\int_{0}^{\infty} n(a, t) m_{0} N_{0} d a, \quad t>0 \\
& n(a, 0)=n_{0}(a), \quad a>0 \tag{3.3.28}
\end{align*}
$$

which is of the form (2.1.18). We could use the formula which we derived there but here we can use the fact that

$$
n(0, t)=\int_{0}^{\infty} n(a, t) m_{0} N_{0} d a=m_{0} N_{0} \int_{0}^{\infty} n(a, t) d a=m_{0} N_{0} N(t)=m_{0} N_{0}^{2}
$$

and we deal with a straightforward initial boundary value problem

$$
\begin{align*}
& \frac{\partial n(a, t)}{\partial t}+\frac{\partial n(a, t)}{\partial a}=-\mu_{0} N_{0} n(a, t), \quad t, a>0 \\
& n(0, t)=m_{0} N_{0}^{2}, \quad t>0 \\
& n(a, 0)=n_{0}(a), \quad a>0 \tag{3.3.29}
\end{align*}
$$

the solution of which is given by (2.1.19)

$$
\begin{align*}
n(a, t)=e^{-m_{0} N_{0} a} u(a, t) & =e^{-m_{0} N_{0} a} \begin{cases}e^{m_{0} N_{0}(a-t)} n_{0}(a-t), & t<a \\
m_{0} N_{0}^{2}, & a<t\end{cases} \\
& = \begin{cases}e^{-m_{0} N_{0} t} n_{0}(a-t), & t<a \\
e^{-m_{0} N_{0} a} m_{0} N_{0}^{2}, & a<t\end{cases} \tag{3.3.30}
\end{align*}
$$

b) $\mu_{0} \neq m_{0}$.

In this case the solution of (3.3.27) is given by

$$
\frac{1}{N_{0}}-\frac{1}{N(t)}=\left(m_{0}-\mu_{0}\right) t
$$

or

$$
\begin{equation*}
N(t)=\frac{N_{0}}{1-\left(m_{0}-\mu_{0}\right) N_{0} t} . \tag{3.3.31}
\end{equation*}
$$

At this moment we note that the evolution of the population is determined by the sign of $r_{0}=m_{0}-\mu_{0}$ which plays the role of the net growth rate: if $r_{0}>0$ (that is, if the maternity coefficient exceeds the mortality rate), the population will explode at $t=1 /\left(r_{0}\right) N_{0}$. On the other hand, if $r_{0}<0$ (that is, if the maternity coefficient is lower than the mortality rate), the population will exists for all time gradually decaying to 0 . However, in both cases we can give explicit formulae for $n(a, t)$. Indeed, similarly to a) we arrive at the straightforward initial boundary value problem

$$
\begin{align*}
& \frac{\partial n(a, t)}{\partial t}+\frac{\partial n(a, t)}{\partial a}=-\mu_{0} N(t) n(a, t), \quad t, a>0 \\
& n(0, t)=m_{0} N(t)^{2}, \quad t>0 \\
& n(a, 0)=n_{0}(a), \quad a>0 \tag{3.3.32}
\end{align*}
$$

where $N(t)$ is given (3.3.31). However, contrary to (3.3.29) (and (2.1.18)), the coefficient $\mu$ is dependent on time and thus the reduction must be made with more care. We begin by noting that

$$
\frac{d}{d t} e^{\mu_{0} \int_{0}^{t} N(s) d s}=m_{0} N(t) e^{\mu_{0} \int_{0}^{t} N(s) d s}
$$

and thus the first equation of (3.3.32) can be written as

$$
\frac{\partial}{\partial t}\left(n(a, t) e^{\mu_{0} \int_{0}^{t} N(s) d s}\right)+\frac{\partial}{\partial a}\left(n(a, t) e^{\mu_{0} \int_{0}^{t} N(s) d s}\right)=0
$$

If we denote

$$
u(a, t)=n(a, t) e^{\mu_{0} \int_{0}^{t} N(s) d s}
$$

then

$$
u(a, 0)=n(a, 0) e^{\mu_{0} \int_{0}^{0} N(s) d s}=n_{0}(a)
$$

and

$$
u(0, t)=n(0, t) e^{\mu_{0} \int_{0}^{t} N(s) d s}=m_{0} N(t)^{2} e^{\mu_{0} \int_{0}^{t} N(s) d s}
$$

so that, using again (2.1.19), we obtain

$$
\begin{align*}
n(a, t)=e^{-\mu_{0} \int_{0}^{t} N(s) d s} u(a, t) & =e^{-\mu_{0} \int_{0}^{t} N(s) d s} \begin{cases}n_{0}(a-t), & t<a, \\
m_{0} N(t-a)^{2} e^{\mu_{0}} \int_{0}^{t-a} N(s) d s\end{cases} \\
& = \begin{cases}e^{-\mu_{0} \int_{0}^{t} N(s) d s} n_{0}(a-t), & t<a, \\
m_{0} N(t-a)^{2} e^{-\mu_{0} \int_{t-a}^{t} N(s) d s}, & a<t\end{cases} \tag{3.3.33}
\end{align*}
$$

as long as $N$ is defined. For our particular case we have

$$
\int N(s) d s=N_{0} \int \frac{d s}{1-r_{0} N_{0} t}=\ln \left(\frac{1}{1-N_{0} r_{0} t}\right)^{\frac{1}{r_{0}}}+C
$$

for $0 \leq t<1 / r_{0} N_{0}$ if $r_{0}>0$ and for $0 \leq t<\infty$ otherwise. Hence, for such $t$

$$
e^{-\mu_{0} \int_{0}^{t} N(s) d s}=\left(1-r_{0} N_{0} t\right)^{\frac{\mu_{0}}{r_{0}}}
$$

and

$$
e^{-\mu_{0} \int_{t-a}^{t} N(s) d s}=\left(\frac{1-r_{0} N_{0} t}{1-r_{0} N_{0}(t-a)}\right)^{\frac{\mu_{0}}{r_{0}}}
$$

Thus

$$
n(a, t)= \begin{cases}\left(1-r_{0} N_{0} t\right)^{\frac{\mu_{0}}{r_{0}}} n_{0}(a-t), & t<a, \\ \frac{m_{0} N_{0}^{2}}{\left(1-r_{0} N_{0}(t-a)\right)^{2}}\left(\frac{1-r_{0} N_{0} t}{1-r_{0} N_{0}(t-a)}\right)^{\frac{\mu_{0}}{r_{0}}}, & a<t .\end{cases}
$$

## Chapter 4

## Similarity methods for linear and non-linear diffusion

## 1 Similarity method

The method described in this section can be applied to equations of arbitrary order. However, having in mind concrete application we shall focus on the general second order partial differential equation in two independent variables in the form

$$
\begin{equation*}
G\left(t, x, u, u_{x}, u_{t}, u_{x x}, u_{x t}, u_{t t}\right)=0 \tag{4.1.1}
\end{equation*}
$$

To shorten notation we introduce notation

$$
p=u_{x}, \quad q=u_{t}, \quad r=u_{x x}, \quad s=u_{x t}, \quad v=u_{t t} .
$$

We introduce the one-parameter family of stretching transformations, denoted by $T_{\epsilon}$, by

$$
\begin{equation*}
\bar{x}=\epsilon^{a} x, \quad \bar{t}=\epsilon^{b} t, \quad \bar{u}=\epsilon^{c} u \tag{4.1.2}
\end{equation*}
$$

where $a, b, c$ are real constants and $\epsilon$ is a real parameter restricted to some open interval $I$ containing $\epsilon=1$. We note that (4.1.2) induces a transformation of the derivatives in the following way:

$$
\begin{equation*}
\bar{p}=\frac{\partial \bar{u}}{\partial \bar{x}}=\epsilon^{c} \frac{\partial u}{\partial x} \frac{d x}{d \bar{x}}=\epsilon^{c-a} p \tag{4.1.3}
\end{equation*}
$$

and similarly for other derivatives

$$
\begin{equation*}
\bar{q}=\epsilon^{c-b} q, \quad \bar{r}=\epsilon^{c-2 a} r, \quad \bar{s}=\epsilon^{c-a-b} s, \quad \bar{v}=\epsilon^{c-2 b} v . \tag{4.1.4}
\end{equation*}
$$

Further, we say that PDE (4.1.1) is invariant under the one parameter family $T_{\epsilon}$ of stretching transformations if there exists a smooth function $f(\epsilon)$ such that

$$
\begin{equation*}
G(\bar{t}, \bar{x}, \bar{u}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{v})=f(\epsilon) G(t, x, u, p, q, r, s, v) \tag{4.1.5}
\end{equation*}
$$

for all $\epsilon \in I$, with $f(1)=1$. If $f(\epsilon) \equiv 1$ for all $\epsilon \in I$, then the $\operatorname{PDE}(4.1 .1)$ is said to be absolutely invariant. We shall formulate and prove the fundamental theorem.

Theorem 1.1 If the equation (4.1.1) is invariant under the family $T_{\epsilon}$ defined by (4.1.2), then the transformation

$$
\begin{equation*}
u=t^{c / b} y(z), \quad z=\frac{x}{t^{a / b}} \tag{4.1.6}
\end{equation*}
$$

reduces (4.1.1) to a second order ordinary differential equation in $y(z)$.

Proof. By invariance, we know that (4.1.5) holds for all $\epsilon$ in some open interval containing 1 thus we can differentiate (4.1.5) and set $\epsilon=1$ after differentiation getting

$$
a x G_{x}+b t G_{t}+c u G_{u}+(c-a) p G_{p}+(c-b) q G_{q}+(c-2 a) r G_{r}+(c-a-b) s G_{s}+(c-2 b) v G_{v}=f^{\prime}(1) G
$$

where we used formulae like

$$
\left.\frac{d \bar{x}}{d \epsilon}\right|_{\epsilon=1}=\left.a \epsilon^{a-1} x\right|_{\epsilon=1}=a x
$$

etc. The above equation is a first order equation so that we can integrate it using $t$ as the parameter along characteristics. The characteristic system will be then

$$
\begin{aligned}
\frac{d G}{d t} & =\frac{f^{\prime}(1) G}{b t} \\
\frac{d x}{d t} & =\frac{a x}{b t} \\
\frac{d u}{d t} & =\frac{c u}{b t} \\
\frac{d p}{d t} & =\frac{(c-a) p}{b t} \\
\frac{d q}{d t} & =\frac{(c-b) q}{b t} \\
\frac{d r}{d t} & =\frac{(c-2 a) r}{b t} \\
\frac{d s}{d t} & =\frac{(c-a-b) s}{b t} \\
\frac{d v}{d t} & =\frac{(c-2 b) v}{b t}
\end{aligned}
$$

Thus, we obtain characteristics defined by

$$
\begin{aligned}
x t^{-a / b} & =z, \\
u t^{-c / b} & =\xi_{1}, \quad p t^{-(c-a) / b}=\xi_{2}, \quad q t^{-(c-b) / b}=\xi_{3} \\
r t^{-(c-2 a) / b} & =\xi_{4}, \quad s t^{-(c-a-b) / b}=\xi_{5}, \quad v t^{-(c-2 b) / b}=\xi_{6}
\end{aligned}
$$

and

$$
\begin{equation*}
G=t^{f^{\prime}(1)} b \Psi\left(z, \xi_{1}, \xi_{2}, \xi_{3}, \xi_{4}, \xi_{5}, \xi_{6}\right) \tag{4.1.7}
\end{equation*}
$$

where $\Psi$ is an arbitrary function. Now, we have $y=u t^{-c / b}=\xi_{1}, p=u_{x}=t^{c / b} y_{z}^{\prime} z^{\prime} x=y_{z}^{\prime} t^{(c-a) / b}$, hence $\xi_{2}=y_{z}^{\prime}$. Further,

$$
q=u_{t}=\frac{c}{b} t^{-1+c / b} y-\frac{a}{b} t^{c / b-a / b-1} x y_{z}^{\prime}
$$

thus

$$
\xi_{3}=q t^{1-c / b}=\frac{c}{b} y-\frac{a}{b} z y_{z}^{\prime}
$$

Further,

$$
r=u_{x x}=p_{x}=y_{z z}^{\prime \prime} t^{(c-a) / b} z_{x}^{\prime}=y_{z z}^{\prime \prime} t^{(c-2 a) / b}
$$

hence $\xi_{4}=y_{z z}^{\prime \prime}$. Similarly,

$$
s=u_{t x}=q_{x}=\frac{c}{b} t^{-1+c / b} y_{z}^{\prime} z_{x}^{\prime}-\frac{a}{b} t^{c / b-a / b-1} y_{z}^{\prime}-\frac{a}{b} t^{c / b-a / b-1} y_{z z}^{\prime \prime} z_{x}^{\prime}=t^{c / b-a / b-1}\left(\frac{c-a}{b} y_{z}^{\prime}-\frac{a}{b} y_{z z}^{\prime \prime} z\right)
$$

giving $\xi_{5}=\frac{c-a}{b} y_{z}^{\prime}-\frac{a}{b} y_{z z}^{\prime \prime} z$ and finally

$$
\begin{aligned}
v= & q_{t}=\frac{c}{b}\left(\frac{c}{b}-1\right) t^{-2+c / b} y+\frac{c}{b} t^{-1+c / b} y_{z}^{\prime} z_{t}^{\prime} \\
& -\frac{a}{b}\left(\left(\frac{c}{b}-1\right) t^{-2+c / b} z y_{z}^{\prime}+t^{-1+c / b} z_{t}^{\prime} y_{z}^{\prime}+t^{-1+c / b} z y_{z z}^{\prime \prime} z_{t}^{\prime}\right) \\
= & t^{-2+c / b}\left(\frac{c}{b}\left(\left(\frac{c}{b}-1\right) y-\frac{a}{b} z y_{z}^{\prime}\right)-\frac{a}{b}\left(\left(\frac{c}{b}-1\right) z y_{z}^{\prime}-\frac{a}{b} z y_{z}^{\prime}-\frac{a}{b} z^{2} y_{z z}^{\prime \prime}\right)\right) \\
= & t^{-2+c / b}\left(\frac{c}{b}\left(\frac{c}{b}-1\right) y-2 \frac{a c}{b^{2}} z y_{z}^{\prime}+\frac{a}{b} z y_{z}^{\prime}+\frac{a^{2}}{b^{2}} z y_{z}^{\prime}+\frac{a^{2}}{b^{2}} z^{2} y_{z z}^{\prime \prime}\right)
\end{aligned}
$$

so that

$$
\xi_{6}=\frac{c}{b}\left(\frac{c}{b}-1\right) y-\frac{a}{b}\left(2 \frac{c}{b}-1-\frac{a}{b}\right) z y_{z}^{\prime}+\frac{a^{2}}{b^{2}} z^{2} y_{z z}^{\prime \prime}
$$

Hence, combining (4.1.7) with the original equation (4.1.1) we obtain

$$
\Psi\left(z, y, y^{\prime}, \frac{c}{b} y-\frac{a}{b} z y_{z}^{\prime}, y_{z z}^{\prime \prime}, \frac{c-a}{b} y_{z}^{\prime}-\frac{a}{b} y_{z z}^{\prime \prime} z, \frac{c}{b}\left(\frac{c}{b}-1\right) y-\frac{a}{b}\left(2 \frac{c}{b}-1-\frac{a}{b}\right) z y_{z}^{\prime}+\frac{a^{2}}{b^{2}} z^{2} y_{z z}^{\prime \prime}\right)=0
$$

which is a second order ordinary differential equation in $z$.

## 2 Linear diffusion equation

Consider the diffusion equation

$$
\begin{equation*}
u_{t}-D u_{x x} \tag{4.2.1}
\end{equation*}
$$

We shall try to find a stretching transformation under which this equation is invariant. Using our simplified notation for derivatives we have

$$
\bar{q}-D \bar{r}=\epsilon^{c-b} q-D \epsilon^{c-2 a} r .
$$

We achieve invariance if $\epsilon$ is risen to the same power. Thus, we must have

$$
b=2 a
$$

with $c$ and $a$ at this moment arbitrary. Thus, (4.2.1) is invariant under the stretching transformation

$$
\begin{equation*}
\bar{x}=\epsilon^{a}, \quad \bar{t}=\epsilon^{2 a} t, \quad \bar{u}=\epsilon^{c} u \tag{4.2.2}
\end{equation*}
$$

and the similarity transformation is given by

$$
\begin{equation*}
u=t^{c / 2 a} y(z), \quad z=\frac{x}{\sqrt{t}} \tag{4.2.3}
\end{equation*}
$$

We have $z_{x}^{\prime}=\frac{1}{\sqrt{t}}, z_{t}^{\prime}=-\frac{1}{2} x t^{-3 / 2}=-\frac{1}{2} z t^{-1}$, hence

$$
u_{t}=-\frac{c}{2 a} t^{-1+c / 2 a} y+t^{c / 2 a} y_{z}^{\prime} z_{t}^{\prime}=-t^{-1+c / 2 a}\left(\frac{c}{2 a} y-\frac{z}{2} y_{z}^{\prime}\right)
$$

and

$$
u_{x}=t^{c / 2 a-1 / 2} y_{z}^{\prime}, \quad u_{x x}=t^{c / 2 a-1} y_{z z}^{\prime \prime} .
$$

Substituting the above relations into the diffusion equation yields

$$
\begin{equation*}
D y^{\prime \prime}+\frac{z}{2} y^{\prime}-\frac{c}{2 a} y=0 \tag{4.2.4}
\end{equation*}
$$

Constants $c$ and $a$ are in general arbitrary.
An important comment is in place here. Though the diffusion equation has been reduced to an ordinary differential equation, one should not think that any problem related to the diffusion equation is reducible to
an ODE problem so the similarity approach by no means solves all PDE problems. In fact, an inherent part of a diffusion problem are initial and boundary conditions and these, in general, cannot be translated into side conditions for (4.2.4). For instance, consider the initial value problem for the diffusion equation

$$
\begin{align*}
u_{t} & =D u_{x x}, \quad t>0,-\infty<x<\infty \\
u(x, 0) & =u_{0}(x) \tag{4.2.5}
\end{align*}
$$

Using the similarity transformation, we can convert the equation into an ODE for $f$ defined as $u(x, t)=$ $t^{c / 2 a} y(x / \sqrt{t})$ but then putting $t=0$ in the preceding formula in general does not make any sense as, at best, we would have something like

$$
\begin{align*}
& y(\infty)=\lim _{t \rightarrow 0^{+}} t^{-c / 2 a} u(x, t), \\
& y(-\infty)=\lim _{t \rightarrow 0^{+}} t^{-c / 2 a} u(x, t),  \tag{4.2.6}\\
& x<0
\end{align*}
$$

with the right hand side equal to 0 if $c / 2 a<0, \infty$ if $c / 2 a>0$ or $u_{0}(x)$ if $c=0$. Now, in the first two cases all the information coming from the initial condition is lost and the last one imposes a strict condition on $u_{0}$ : $u_{0}$ must be constant on each semi-axis. Note that such a condition is also invariant under the transformation $z=x / \sqrt{t}: z \lessgtr 0$ if and only if $x \lessgtr 0$. In general, the similarity method provides a full solution to the initial-boundary value problems only if the side conditions are also invariant under the same transformation or, in other words, can be expressed in terms of the similarity variable. Otherwise, this method can serve as a first step in building more general solution, as we shall see below.

Consider the initial value problem

$$
\begin{align*}
u_{t} & =D u_{x x}, \quad t>0,-\infty<x<\infty \\
u(x, 0) & =H(x) \tag{4.2.7}
\end{align*}
$$

where $H$ is the Heaviside function: $H(x)=1$ for $x \geq 0$ and $H(x)=0$ for $x<0$. According to the discussion above, this initial condition yields to the similarity method provided $c=0$; in this case $a$ is irrelevant and we put it equal to 1 . Thus, the initial value problem (4.2.7) is transformed into

$$
\begin{array}{cl}
y^{\prime \prime}+\frac{z}{2 D} y^{\prime}= & 0, \\
y(-\infty)=0, & y(\infty)=1 . \tag{4.2.8}
\end{array}
$$

Denoting $y^{\prime}=h$, we reduce the equation to the first order equation

$$
h^{\prime}+\frac{z}{2 D} h=0
$$

which can be integrated to $y^{\prime}=h=c_{1} \exp \left(-z^{2} / 4 D\right)$. Integrating this once again, we obtain

$$
\begin{equation*}
y(z)=c_{1} \int_{0}^{z} e^{-\frac{\eta^{2}}{4 D}} d \eta+c_{2} \tag{4.2.9}
\end{equation*}
$$

and the constants $c_{1}$ and $c_{2}$ can be obtained from the initial conditions

$$
\begin{aligned}
& 1=\lim _{z \rightarrow+\infty} y(z)=c_{1} \int_{0}^{\infty} e^{-\frac{\eta^{2}}{4 D}} d \eta+c_{2}=\sqrt{4 D} c_{1} \int_{0}^{\infty} e^{-s^{2}} d s+c_{2}=c_{1} \frac{\sqrt{4 D \pi}}{2}+c_{2} \\
& 0=\lim _{z \rightarrow-\infty} y(z)=c_{1} \int_{0}^{-\infty} e^{-\frac{\eta^{2}}{4 D}} d \eta+c_{2}=\sqrt{4 D} c_{1} \int_{0}^{-\infty} e^{-s^{2}} d s+c_{2}=-c_{1} \frac{\sqrt{4 D \pi}}{2}+c_{2}
\end{aligned}
$$

where we used $\int_{0}^{\infty} e^{-s^{2}} d s=\sqrt{\pi} / 2$, so that

$$
c_{2}=\frac{1}{2}, \quad c_{1}=\frac{1}{\sqrt{4 \pi D}}
$$

Hence

$$
\begin{equation*}
u(x, t)=y\left(\frac{x}{\sqrt{t}}\right)=\frac{1}{2}+\frac{1}{\sqrt{4 \pi D}} \int_{0}^{\frac{x}{\sqrt{t}}} e^{-\frac{\eta^{2}}{4 D}} d \eta \tag{4.2.10}
\end{equation*}
$$

The fundamental rôle in the theory of the diffusion equation is played by the derivative of $u$ with respect to $x$ :

$$
\begin{equation*}
S(x, t)=\frac{\partial u}{\partial x}(x, t)=\frac{1}{\sqrt{4 \pi D t}} e^{-\frac{x^{2}}{4 D t}} \tag{4.2.11}
\end{equation*}
$$

that is called source function or fundamental solution of the diffusion equation.
The reason for the importance of the fundamental solution is that it describes diffusion of a unit quantity of the medium concentrated at the origin and thus provides the solution to the initial value problem for the diffusion equation of the form

$$
\begin{aligned}
u_{t} & =D u_{x x} \\
u(x, 0) & =\delta(x)
\end{aligned}
$$

where $\delta(x)$ is Dirac's delta "function". One can check that $S(x, t)$ has the following properties:

$$
\begin{align*}
\lim _{t \rightarrow 0^{+}} S(x, t) & =0, \quad x \neq 0  \tag{4.2.12}\\
\lim _{t \rightarrow 0^{+}} S(x, t) & =\infty, \quad x=0  \tag{4.2.13}\\
\lim _{x \rightarrow \pm \infty} S(x, t) & =0, \quad x>0  \tag{4.2.14}\\
\int_{-\infty}^{\infty} S(x, t) d x & =1, \quad t>0 \tag{4.2.15}
\end{align*}
$$

In general, it can be proved that the initial value problem (4.2.5) has a unique solution given by

$$
\begin{equation*}
u(x, t)=\int_{-\infty}^{\infty} S(x-\xi) u_{0}(\xi) d \xi \tag{4.2.16}
\end{equation*}
$$

provided $u_{0}$ is sufficiently regular (e.g. bounded and continuous).

## 3 Nonlinear diffusion models

The linear diffusion equation, though extremely useful and important in applications, has at least one major drawback - it is physically incorrect as it admits infinite speed of signal transmission. To alleviate this problem a number of nonlinear versions of this equation has been presented and we shall discuss some of them.

### 3.1 Models

Let us recall that the starting point for the derivation of the diffusion equation was the conservation law

$$
\begin{equation*}
u_{t}+\phi_{x}=0 \tag{4.3.1}
\end{equation*}
$$

and the linear equation was obtained by postulating the flux in the form of Fick's law

$$
\begin{equation*}
\phi=-D u_{x} \tag{4.3.2}
\end{equation*}
$$

In population models usually there is the increase in diffusion due to the population pressure and it is then reasonable to assume that the coefficient $D$ itself should depend on the density $u$, that is,

$$
\begin{equation*}
\phi=-D(u) u_{x} \tag{4.3.3}
\end{equation*}
$$

and substituting this to the conservation law (4.3.1) yields the nonlinear equation

$$
\begin{equation*}
u_{t}=\left(D(u) u_{x}\right)_{x} \tag{4.3.4}
\end{equation*}
$$

Another place where the nonlinearity can occur is at the time derivative. For instance, deriving the heat equation we start with the energy conservation law

$$
\begin{equation*}
(\rho C T)_{t}+\phi_{x}=0 \tag{4.3.5}
\end{equation*}
$$

where $T$ is the temperature, $\rho$ is the density and $C$ is the specific heat of the medium (so that the first terms describes the rate of change of the amount of energy contained in a unit volume). The flux $\phi$ is given by the
Fourier law

$$
\phi=-K T_{x},
$$

where $K$ is thermal conductivity of the medium. In many applications, where the temperature range is limited, the specific heat and the density may be regarded as constants. However, over a wide temperature ranges, they are not constants but rather depend on the temperature and in this case (4.3.5) combined with Fourier's law give the equation

$$
\begin{equation*}
(\rho(T) C(T) T)_{t}=K T_{x x} \tag{4.3.6}
\end{equation*}
$$

which is a nonlinear diffusion equation for the temperature $T$.
These two types of nonlinearities can, of course, appear in a single equation. Consider, for instance, the porous medium equation, where we wish to describe a fluid (e.g. water) seeping downward through the soil. Let $\rho(x, t)$ be the density of the fluid, with positive $x$ measured downward. In a given volume of soil only a fraction of the space is available to the fluid, the remaining being reserved for the soil itself. If the fraction of the volume that is available to the fluid is denoted by $\kappa$, then mass balance for the fluid can be written as

$$
\begin{equation*}
\kappa \rho_{t}+(\rho v)_{x}=0 \tag{4.3.7}
\end{equation*}
$$

where we used the formula $\phi=v \rho$ with $v$ being the volumetric flow rate (velocity of the flow). The conservation law (4.3.7) contains two unknowns: the density and the velocity of the flow so we need another constitutive equation relating them. Without going through too much of physics we state that in many cases the constitutive relation

$$
v=a \rho_{x}^{\gamma}
$$

for some constants $\gamma>1, a>0$. Thus, the porous media equation can be written as

$$
\begin{equation*}
\kappa \rho_{t}=a\left(\rho \rho_{x}^{\gamma}\right)_{x}=a \gamma\left(\rho^{\gamma} \rho_{x}\right)_{x}=\alpha \rho_{x x}^{m} \tag{4.3.8}
\end{equation*}
$$

where $\alpha=a \gamma /(\gamma+1)$ and $m=\gamma+1>2$. Since, in general, $\kappa$ can be $\rho$-dependent, the porous media equation combines nonlinearities of (4.3.6) and (4.3.4) (with $D(\rho)=a \gamma \rho^{\gamma}$.)

### 3.2 Some solutions

Let us consider a special case of Eq. (4.3.4) with $D(u)=u$, that is,

$$
\begin{equation*}
u_{t}-\left(u u_{x}\right)_{x}=0 \tag{4.3.9}
\end{equation*}
$$

To compare the nonlinear equation with the linear diffusion, we shall try to solve it on $\mathbb{R}$ subject to the initial condition of a unit point source applied at $x=0$, that is,

$$
u(x, 0)=\delta(x)
$$

To simplify considerations we shall change this condition into two, more amenable to the similarity method, so that we shall look for solutions satisfying

$$
\begin{equation*}
\int_{-\infty}^{\infty} u(x, t) d x=1, \quad t>0 \tag{4.3.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{x \rightarrow \pm \infty}=0 \tag{4.3.11}
\end{equation*}
$$

As follows from Eqs. (4.2.12)-(4.2.15), in the linear case the solution satisfying these conditions is the fundamental solution $S(x, t)$ to the diffusion equation.

Let us introduce the stretching transformation

$$
\begin{equation*}
\bar{x}=\epsilon^{a} x, \quad \bar{t}=\epsilon^{b} t, \bar{u}=\epsilon^{c} u \tag{4.3.12}
\end{equation*}
$$

and substitute it into our nonlinear diffusion equation

$$
u_{t}-\left(u u_{x}\right)_{x}=u_{t}-u_{x}^{2}-u u_{x x}=p-q^{2}-u r=\epsilon^{b-c} \bar{p}-\epsilon^{2(a-c)} \bar{q}^{2}-\epsilon^{-2 c+2 a} \bar{u} \bar{r}
$$

from where we see that $b=2 a-c$ so that

$$
\begin{equation*}
\bar{x}=\epsilon^{a} x, \quad \bar{t}=\epsilon^{b} t, \bar{u}=\epsilon^{2 b-a} u \tag{4.3.13}
\end{equation*}
$$

and the similarity transformation is given by

$$
\begin{equation*}
u(x, t)=t^{(2 a-b) / b} y(z), \quad z=\frac{x}{t^{a / b}} . \tag{4.3.14}
\end{equation*}
$$

Let us first specialize the parameters $a$ and $b$ by imposing the condition (4.3.10). We have

$$
1=\int_{-\infty}^{\infty} u(x, t) d x=t^{(2 a-b) / b} \int_{-\infty}^{\infty} y\left(\frac{x}{t^{a / b}}\right) d x=t^{\frac{3 a-b}{b}} \int_{-\infty}^{\infty} y(z) d z
$$

so that $3 a-b=0$. Hence

$$
\begin{equation*}
u(x, t)=t^{-1 / 3} y(z), \quad z=x t^{-1 / 3} . \tag{4.3.15}
\end{equation*}
$$

and substituting these equations we obtain

$$
\begin{aligned}
u_{t} & =-\frac{1}{3} t^{-4 / 3}\left(y+y^{\prime} z\right) \\
u_{x} & =t^{-2 / 3} y^{\prime} \\
\left(u u_{x}\right)_{x} & =t^{-1}\left(y y^{\prime}\right)_{x}=t^{-1}\left(y y^{\prime}\right)^{\prime} z_{x}=t^{-4 / 3}\left(y y^{\prime}\right)^{\prime}
\end{aligned}
$$

and consequently (4.3.11) turns into

$$
\begin{equation*}
3\left(y y^{\prime}\right)^{\prime}+y+z y^{\prime}=0 . \tag{4.3.16}
\end{equation*}
$$

As $y+z y^{\prime}=(z y)^{\prime}$, this equation can be integrated at once giving

$$
\begin{equation*}
3 y y^{\prime}+z y=\text { constant } . \tag{4.3.17}
\end{equation*}
$$

Now, the equation is invariant under the change of variable $x \rightarrow-x$ and the side conditions also are not altered when we make this change so that it is reasonable to expect the solution to have the same property, that is $u(x)=u(-x)$. In other ways, we expect the solution to be even and in such a case we derive another condition, namely $u_{x}(0, t)=0$ for any $t>0$. Since $y^{\prime}(z)=t^{2 / 3} u_{x}(x, t)$ we see that $y^{\prime}(0)=0$. Putting $z=0$ in (4.3.17) we see that the constant must be zero and hence we obtain a first order separable equation

$$
3 y y^{\prime}+z y=0
$$

that can be immediately integrated giving either $y=0$ or $y=\frac{-z^{2}+A^{2}}{6}$, where $A$ is a constant of integration. Separately, neither solution makes sense as the former does not satisfy the integral condition and the second becomes negative for $|z|>A$. Let us then patch these two solutions together and consider

$$
y(z)= \begin{cases}\frac{-z^{2}+A^{2}}{6} & \text { if }|z|<A  \tag{4.3.18}\\ 0 & \text { if }|z| \geq A\end{cases}
$$

At first glance this seems to be a ridiculous idea as such a function is merely continuous and we are looking for a solution to the second order differential equation. However, a closer look at (4.3.16) shows that the equation does not involve the second derivative of $y$ by itself but rather requires differentiability of $y y^{\prime}$ that is less stringent as $y$ is zero at the discontinuity of $y^{\prime}$. We note thus that

$$
y^{\prime}(z)= \begin{cases}\frac{-z}{3} & \text { if }|z|<A,  \tag{4.3.19}\\ 0 & \text { if }|z|>A,\end{cases}
$$

with one-sided derivatives at $z= \pm A: y_{-}^{\prime}(-A)=y_{+}^{\prime}(A)=0, y_{+}^{\prime}(-A)=A / 3$ and $y_{-}^{\prime}(A)=-A / 3$. Thus

$$
y y^{\prime}(z)= \begin{cases}\frac{-z\left(A^{2}-z^{2}\right)}{18} & \text { if }|z|<A  \tag{4.3.20}\\ 0 & \text { if }|z| \geq A\end{cases}
$$

so that

$$
\left(y y^{\prime}\right)^{\prime}(z)= \begin{cases}\frac{-A^{2}+3 z^{2}}{18} & \text { if }|z|<A  \tag{4.3.21}\\ 0 & \text { if }|z|>A\end{cases}
$$

with one-sided derivatives at $\pm A:\left(y y^{\prime}\right)_{-}^{\prime}(-A)=\left(y y^{\prime}\right)_{+}^{\prime}(A)=0,\left(y y^{\prime}\right)_{+}^{\prime}(-A)=\left(y y^{\prime}\right)_{-}^{\prime}(A)=A^{2} / 9$. Clearly, the equation is satisfied on open intervals $|z|<A$ and $|z|>A$. Taking one-sided values from the left at $-A$ and from the right at $A$ we have zeros; from the right at $-A$ we obtain $3 A^{2} / 9+0+(-A) A / 3=0$ and similarly from the left at $A: 3 A^{2} / 9+0+A(-A / 3)=0$. Hence, the equation is satisfied everywhere if we allow interpretation of derivatives as one-sided ones. Such solutions are called piecewise smooth.

Having accepted (4.3.18) as a solution to our problem, we can specify $A$ by requiring (4.3.10) to be satisfied. Thus

$$
1=\int_{-\infty}^{\infty} y(z) d z=\int_{-A}^{A} y(z) d z=\frac{2}{9} A^{3}
$$

that gives $A=\sqrt[3]{2 / 9}$.
Therefore we have constructed a piecewise smooth solution

$$
u(x, t)= \begin{cases}\frac{1}{6} t^{-2 / 3}\left((9 t / 2)^{2 / 3}-x^{2}\right) & \text { if }|x|<(9 t / 2)^{1 / 3}  \tag{4.3.22}\\ 0 & \text { if }|x| \geq(9 t / 2)^{1 / 3}\end{cases}
$$

It is easy to see that as $t \rightarrow 0^{+}, u(x, t)$ converges to $\delta(x)$ in the sense of (4.2.12)-(4.2.13). Snapshots of the solution are shown on Fig. 6.1.

Thus, we see that solution (4.3.22) is fundamentally different from the smooth and everywhere positive solution $u(x, t)=(4 \pi)^{-1 / 2} \exp \left(-x^{2} / 4 t\right)$ of the corresponding linear problem. In fact, (4.3.22) represents a type of a sharp wavefront $x_{f}=(9 t / 2)^{1 / 3}$ propagating into the medium with speed

$$
\frac{d x_{f}}{d t}=\sqrt[3]{t / 6}
$$

The last example we are going to discuss is the nonlinear heat equation

$$
\begin{equation*}
u u_{t}-u_{x x}=0, \quad x>0, t>0 \tag{4.3.23}
\end{equation*}
$$



Fig. 6.1 Snapshots of (4.3.22).
subject to side conditions

$$
\begin{align*}
u(x, 0) & =0, \quad x>0  \tag{4.3.24}\\
u(\infty, t) & =0, \quad t>0  \tag{4.3.25}\\
u_{x}(0, t) & =-1, \quad t>0 \tag{4.3.26}
\end{align*}
$$

This problem is a simplified version of the nonlinear heat equation (4.3.6) with temperature zero initially and as $x \rightarrow \infty$. The flux condition describes the heat flowing into the medium at the constant rate -1 (gradient of the temperature is negative and heat flows from regions of higher temperature to regions of lower temperature).

As before, we introduce the stretching transformation

$$
\begin{equation*}
\bar{x}=\epsilon^{a} x, \quad \bar{t}=\epsilon^{b} t, \bar{u}=\epsilon^{c} u \tag{4.3.27}
\end{equation*}
$$

and substitute it to our nonlinear diffusion equation

$$
u u_{t}-u_{x x}=u p-r=\epsilon^{b-2 c} \bar{p}-\epsilon^{2 a-c} \bar{r}
$$

from where we see that we have $c=b-2 a$ so that

$$
\begin{equation*}
\bar{x}=\epsilon^{a} x, \quad \bar{t}=\epsilon^{b} t, \bar{u}=\epsilon^{b-2 a} u \tag{4.3.28}
\end{equation*}
$$

and the similarity transformation is given by

$$
\begin{equation*}
u(x, t)=t^{(b-2 a) / b} y(z), \quad z=\frac{x}{t^{a / b}} \tag{4.3.29}
\end{equation*}
$$

Restrictions on the constants $a$ and $b$ can be determined by the initial and boundary conditions. Evaluating $u_{x}(x, t)$ gives

$$
\begin{equation*}
u_{x}(x, t)=t^{(b-3 a) / b} y^{\prime}(z) \tag{4.3.30}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
u_{x}(0, t)=t^{(b-3 a) / b} y^{\prime}(0)=-1 \tag{4.3.31}
\end{equation*}
$$

which could be possible only if $b=3 a$. Consequently, the similarity transformation is given by

$$
\begin{equation*}
u(x, t)=t^{1 / 3} y(z), \quad z=\frac{x}{\sqrt{3} t^{1 / 3}} \tag{4.3.32}
\end{equation*}
$$

where the factor $\sqrt{3}$ was introduced to simplify further calculations. Since in the initial condition (4.3.26) $t>0$ is arbitrary, the flux condition will become

$$
\begin{equation*}
y^{\prime}(0)=-\sqrt{3}, \tag{4.3.33}
\end{equation*}
$$

where the factor $\sqrt{3}$ was introduced due to the modified expression for $z$. Condition (4.3.25) implies

$$
\begin{equation*}
y(\infty)=0 \tag{4.3.34}
\end{equation*}
$$

and the initial condition can be written as

$$
\lim _{t \rightarrow 0^{+}} u(x, t)=\lim _{z \rightarrow 0} t^{1 / 3} y(z)=\lim _{z \rightarrow 0} \frac{x}{\sqrt{3} z} y(z)=0, \quad x>0
$$

that is even stronger condition than (4.3.34). Thus, conditions (4.3.24) and (4.3.26) have coalesced into the single condition (4.3.34). Once again we emphasize here that the fact that the original boundary conditions can be written as conditions in similarity variables follows from their special form - in general such a transformation is impossible.

Let us now transform the equation. We have

$$
\begin{aligned}
z_{t} & =-\frac{z}{3 \sqrt{3} t} \\
u_{t} & =\frac{1}{3} t^{-2 / 3}\left(y-y^{\prime} z\right) \\
u_{x} & =\frac{y^{\prime}}{\sqrt{3}} \\
u_{x x} & =t^{-1 / 3} \frac{y^{\prime \prime}}{3} \\
u u_{t} & =\frac{1}{3} t^{-1 / 3}\left(y^{2}-\frac{y y^{\prime} z}{\sqrt{3}}\right)
\end{aligned}
$$

and consequently (4.3.23) turns into

$$
\begin{equation*}
f^{\prime \prime}-f\left(f-z f^{\prime}\right)=0 . \tag{4.3.35}
\end{equation*}
$$

Unfortunately, contrary to the previous cases, this is second order non-autonomous equation and cannot be easily solved. However, we can still simplify it using once again the similarity variables. We shall formulate and prove the following lemma.

Lemma 3.1 Assume that the equation

$$
\begin{equation*}
f^{\prime \prime}-G\left(z, f, f^{\prime}\right)=0 \tag{4.3.36}
\end{equation*}
$$

is invariant under the transformation

$$
\begin{equation*}
s=\epsilon z, \quad g=\epsilon^{b} f, \quad \epsilon \in I \tag{4.3.37}
\end{equation*}
$$

where $I$ is an open interval containing 1 and $b$ is a constant, that is,

$$
\frac{d^{2} g}{d s^{2}}-G\left(s, g, \frac{d g}{d s}\right)=A(\epsilon)\left(\frac{d^{2} f}{d z^{2}}-G\left(z, f, \frac{d f}{d z}\right)\right)
$$

Then (4.3.36) can be reduced to a first order ordinary differential equation of the form

$$
\begin{equation*}
\frac{d \eta}{d \xi}=\frac{H(\xi, \eta)-(b-1) \eta}{\eta-b \xi} \tag{4.3.38}
\end{equation*}
$$

where $\xi=\phi(z, f)$ and $\eta=\psi(z, f, p)$ are solutions of the characteristic system

$$
\begin{align*}
\frac{d f}{d z} & =\frac{b f}{z} \\
\frac{d p}{d z} & =\frac{(b-1) p}{z} \tag{4.3.39}
\end{align*}
$$

and $H$ is some function depending on $G$.

Proof. The proof is similar to the proof of Theorem 1.1. Denoting $p=f^{\prime}$, we obtain from invariance

$$
\frac{d^{2} g}{d s^{2}}-G\left(s, g, \frac{d g}{d s}\right)=\epsilon^{b-2} f^{\prime \prime}-G\left(\epsilon z, \epsilon^{b} f, \epsilon^{b-1} p\right)=A(\epsilon)\left(\frac{d^{2} f}{d z^{2}}-G(z, f, p)\right)
$$

so that

$$
\epsilon^{b-2} G\left(\epsilon z, \epsilon^{b} f, \epsilon^{b-1} p\right)=G(z, f, p)
$$

for all $\epsilon \in I$. Differentiating with respect to $\epsilon$ and putting $\epsilon=1$ we obtain

$$
z G_{z}+b f G_{f}+(b-1) p G_{p}=(b-2) G
$$

The characteristic system is

$$
\begin{aligned}
\frac{d G}{d z} & =(b-2) \frac{G}{z} \\
\frac{d f}{d z} & =\frac{b f}{z} \\
\frac{d p}{d z} & =(b-1) \frac{p}{z}
\end{aligned}
$$

The solutions of the characteristic system are

$$
\xi=f z^{-b}, \quad \eta=p z^{1-b}
$$

so that $G=z^{b-2} H(\xi, \eta)$ for some function $H$. To arrive at (4.3.38) we observe that, since

$$
\frac{d \xi}{d z}=f_{z}^{\prime} z^{-b}-b f z^{-b-1}=\eta z^{-1}-b \xi z^{-1}
$$

we obtain

$$
\begin{aligned}
f^{\prime \prime} & =\frac{d p}{d z}=z^{1-b} \eta_{z}^{\prime}+\eta(b-1) z^{b-2}=z^{1-b} \frac{d \eta}{d \xi} \frac{d \xi}{d z}+\eta(b-1) z^{b-2} \\
& =z^{1-b} \frac{d \eta}{d \xi}\left(\eta z^{-1}-b \xi z^{-1}\right)+\eta(b-1) z^{b-2}=z^{b-2}\left(\frac{d \eta}{d \xi}(\eta-b \xi)+(b-1) \eta\right)
\end{aligned}
$$

so that

$$
\frac{d \eta}{d \xi}=\frac{H(\xi, \eta)-(b-1) \eta}{\eta-b \xi}
$$

that is exactly (4.3.38).
Returning to our problem, we take the stretching transformation $s=\epsilon x, \quad g=\epsilon^{b} y$ and substitute it to (4.3.35). We get

$$
g_{s s}^{\prime \prime}-g\left(g-s g^{\prime}\right)=\epsilon^{b-2} y_{z z}^{\prime \prime}-\epsilon^{2 b} y\left(y-z y_{z}^{\prime}\right)
$$

that gives invariance if $b=-2$. Hence, introducing new variables

$$
\begin{equation*}
\xi=z^{2} y, \quad \eta=z^{3} y^{\prime} \tag{4.3.40}
\end{equation*}
$$

we obtain directly

$$
\frac{d^{2} y}{d z^{2}}=\eta_{z} z^{-3}-3 \eta z^{-4}=\frac{d \eta}{d \xi} \frac{d \xi}{d z}-3 \eta z^{-4}=z^{-4}\left(\frac{d \eta}{d \xi}(\eta+2 \xi)-3 \eta\right)
$$

so that

$$
\begin{equation*}
\frac{d \eta}{d \xi}=\frac{\xi^{2}-\xi \eta+3 \eta}{\eta+2 \xi} \tag{4.3.41}
\end{equation*}
$$

This is first order non-autonomous equation that still cannot be solved. However, we can obtain some information on the behaviour of the solution by re-writing it as an autonomous system and performing
phase plane analysis. To do this, in some sense we perform the procedure leading to (??) in the opposite direction. To identify the proper parameter, we write

$$
\frac{d \xi}{d z}=z^{-1}(\eta+2 \xi)
$$

and, using (4.3.41)

$$
\frac{d \eta}{d z}=3 z^{2} y^{\prime}+z^{3} y^{\prime \prime}=3 z^{-1} \eta+z^{3}\left(y^{2}-z y y^{\prime}\right)=z^{-1}\left(3 \eta+\xi^{2}-\xi \eta\right)
$$

Thus, introducing a new variable via $d \tau / d z=1 / z$, that is, $\tau=\ln z$, we have

$$
\begin{align*}
\xi_{\tau}^{\prime} & =\eta+2 \xi \\
\eta_{\tau}^{\prime} & =\xi^{2}-\xi \eta+3 \eta \tag{4.3.42}
\end{align*}
$$

Moreover, we see that as $z \rightarrow 0^{+}, \tau \rightarrow-\infty$ and as $z \rightarrow+\infty, \tau \rightarrow+\infty$, thus the boundary conditions (4.3.33) and (4.3.34) can be translated into conditions at $\pm \infty$. Firstly, we observe that (4.3.33) implicitly imposes the condition of boundedness on $f$ at 0 so that $\xi=\eta=0$ at $s=0$ that is at $\tau=-\infty$.
If we look at the equilibrium points of $(4.3 .42)$, then we obtain $(0,0)$ and $(2,-4)$. The Jacobi matrix of the right-hand side is

$$
J(\xi, \eta)=\left(\begin{array}{cc}
2 & 1 \\
2 \xi-\eta & 3-\xi
\end{array}\right)
$$

Eigenvalues at $(0,0)$ are calculated from

$$
\left|\begin{array}{cc}
2-\lambda & 1 \\
0 & 3-\lambda
\end{array}\right|=(2-\lambda)(3-\lambda)=0
$$

hence $\lambda_{ \pm}=2,3$ and $(0,0)$ is an unstable node (source) which is consistent with the requirement $(\xi(\tau), \eta(\tau)) \rightarrow$ $(0,0)$ as $\tau \rightarrow \infty$.
Eigenvalues at $(2,-4)$ are calculated from

$$
\left|\begin{array}{cc}
2-\lambda & 1 \\
8 & 1-\lambda
\end{array}\right|=\lambda^{2}-3 \lambda-6=0
$$

hence $\lambda_{ \pm}=\frac{2 \pm \sqrt{33}}{2}$ and since $\lambda_{ \pm}$are real with opposite sign, $(2,-4)$ is a saddle. It is important to note that the tangent of the stable manifold at $(2,-4)$ is $\lambda_{+}-2$.
The isoclines are $\eta=-2 \xi$ and $\eta=\xi^{2} / \xi-3$ and since tangents of these isoclines at $(2,-4)$ are -2 and -8 respectively $\left(\eta^{\prime}=\left(\xi^{2}-6 \xi\right) /(\xi-3)^{2}\right.$ at $\left.\xi=3\right)$, we see that the stable manifold is between these two tangents, as seen in Fig. 6.2 Now, we observe that in the bounded region $R I$ between isoclines we have $\xi^{\prime}>0, \eta^{\prime}<0$, in the region $R I I$ immediately above $\xi^{\prime}>0, \eta^{\prime}>0$ and in RIII immediately below $\xi^{\prime}<0, \eta^{\prime}<0$.
For the phase-plane analysis, it may be easier to change the direction of $\tau$ defining $\bar{\tau}=-\tau$ and consequently $\bar{\xi}(\bar{\tau})=\xi(\tau)$ and $\bar{\eta}(\bar{\tau})=\eta(\tau)$ so that the stable manifold at $(2,-4)$ becomes the unstable manifold and the source at $(0,0)$ becomes a sink. Then in $R I$ we have $\bar{\xi}^{\prime}<0, \bar{\eta}^{\prime}>0$, in $R I I \bar{\xi}^{\prime}<0$ and $\bar{\eta}^{\prime}<0$, and in RIII $\bar{\xi}^{\prime}>0$ and $\bar{\eta}^{\prime}<0$. In these new variables, there is a unique trajectory starting at $(2,-4)$ at $\bar{\tau}=-\infty$ and entering $R I$. If this trajectory was to leave $R I$ through the isocline $\bar{\eta}^{\prime}=0$, that is, into $R I I$, then the intercept would be a local maximum of $\bar{\eta}$ with $\bar{\xi}$ still moving to the left. This is, however, impossible, as the isocline is a decreasing function. Similarly, if the trajectory was to leave through the isocline $\bar{\xi}^{\prime}=0$, then at the intercept $\bar{\xi}$ would have local minimum, with $\bar{\eta}$ still moving up. This is again impossible as the isocline is a decreasing function. Thus, the trajectory must stay in $R I$ and since $\bar{\xi}$ and $\bar{\eta}$ are monotonic functions, it must converge to $(0,0)$.
Returning to the old variable, we have proved the existence of a solution $(\xi, \eta)$ such that $(\xi(z), \eta(z)) \rightarrow(0,0)$ as $z \rightarrow 0$ and $(\xi(z), \eta(z)) \rightarrow(2,-4)$ as $z \rightarrow \infty$. In particular,

$$
\xi(z) \sim \frac{2}{z^{2}}, \quad z \rightarrow \infty
$$



Fig. 6.2 Phase plane for the system (4.3.42).
and going back to the original problem (4.3.23) we have determined the asymptotic behaviour of the solution $u(x, t)$

$$
\begin{equation*}
u(x, t)=t^{1 / 3} y\left(\frac{x}{\sqrt{3} t^{1 / 3}}\right) \sim \frac{6 t}{x^{2}} \tag{4.3.43}
\end{equation*}
$$

for large $x / t^{1 / 3}$.

## Chapter 5

## Travelling waves

## 1 Introduction

One of the cornerstones in the study of both linear and nonlinear PDEs is the wave propagation. A wave is a recognizable signal which is transferred from one part of the medium to another part with a recognizable speed of propagation. Energy is often transferred as the wave propagates, but matter may not be. We mention here a few areas where wave propagation is of fundamental importance.

Fluid mechanics (water waves, aerodynamics)
Acoustics (sound waves in air and liquids)
Elasticity (stress waves, earthquakes)
Electromagnetic theory (optics, electromagnetic waves)
Biology (epizootic waves)
Chemistry (combustion and detonation waves)
The simplest form of a mathematical wave is a function of the form

$$
\begin{equation*}
u(x, t)=f(x-c t) \tag{5.1.1}
\end{equation*}
$$

We adopt the convention that $c>0$. We have already seen such a wave as a solution of the constant coefficient transport equation

$$
u_{t}+c u_{x}=0
$$

it can, however, appear in many other contexts.
At $t=0$ the wave has the form $f(x)$ which is the initial wave profile. Then $f(x-c t)$ represents the profile at time $t$, that is just the initial profile translated to the right by ct spatial units. Thus the constant $c$ represents the speed of the wave and thus, evidently, (5.1.1) represents a wave travelling to the right with speed $c>0$. Similarly,

$$
u(x, t)=f(x+c t)
$$

represents a wave travelling to the right with speed $c$. These waves propagate undistorted along the lines $x \pm c t=$ const .

One of the fundamental questions in the theory of nonlinear PDEs is whether a given PDE admit such a travelling wave as a solution. This question is generally asked without regard to initial conditions so that the wave is assumed to have existed for all times. However, boundary conditions of the form

$$
\begin{equation*}
u(-\infty, t)=\text { constant }, \quad u(+\infty, t)=\text { constant } \tag{5.1.2}
\end{equation*}
$$

are usually imposed. A wavefront type-solution to a PDE is a solution of the form $u(x, t)=f(x \pm c t)$ subject to the condition (5.1.2) of being constant at $\pm \infty$ (this constant are not necessarily the same); the function $f$ is assumed to have the requisite degree of smoothness defined by the PDE. If $u$ approaches the same constant at both $\pm \infty$, then the wavefront is called a pulse.

## 2 Basic examples

We have already seen that the transport equation with constant $c$ admits the travelling wave solution and it is the only solution this equation can have. To illustrate the technique of looking for travelling wave solutions on a simple example first, let us consider the wave equation.

Example 2.1 Find travelling wave solutions to the wave equation

$$
u_{t t}-a^{2} u_{x x}=0
$$

According to definition, travelling wave solution is of the form $u(x, t)=f(x-c t)$. Inserting this into the equation, we find

$$
u_{t t}-a^{2} u_{x x}=c^{2} f^{\prime \prime}-a^{2} f^{\prime \prime}=f^{\prime \prime} \cdot\left(c^{2}-a^{2}\right)=0
$$

so that either $f(s)=A+B s$ for some constants $A, B$ or $c= \pm a$ and $f$ arbitrary. In the first case we would have

$$
u(x, t)=A+B(x \pm c t)
$$

but the boundary conditions (5.1.2) cannot be satisfied unless $B=0$. Thus, the only travelling wave solution in this case is constant. For the other case, we see that clearly for any twice differentiable function $f$ such that

$$
\lim _{s \rightarrow \pm \infty}=d_{ \pm \infty}
$$

the solution

$$
u(x, t)=f(x \pm a t)
$$

is a travelling wave solution (a pulse if $d_{+\infty}=d_{-\infty}$ ).
In general, it follows that any solution to the wave equation can be obtained as a superposition of two travelling waves: one to the right and one to the left

$$
u(x, t)=f(x-a t)+g(x+a t)
$$

Not all equations admit travelling wave solutions, as demonstrated below.
Example 2.2 Consider the diffusion equation

$$
u_{t}=D u_{x x}
$$

Substituting the travelling wave formula $u(x, t)=f(x-c t)$, we obtain

$$
-c f^{\prime}-D f^{\prime \prime}=0
$$

that has the general solution

$$
f(s)=a+b \exp \left(-\frac{c s}{D}\right) .
$$

It is clear that for $f$ to be constant at both plus and minus infinity it is necessary that $b=0$. Thus, there are no nonconstant travelling wave solutions to the diffusion equation.

We have already seen that the non-viscid Burger's equation

$$
u_{t}+u u_{x}=0
$$

does not admit a travelling wave solution: any profile will either smooth out or form a shock wave (which can be considered as a generalized travelling wave - it is not continuous!). However, some amount of dissipation represented by a diffusion term allows to avoid shocks.

## 3 The Burger equation

Consider Burger's equation with viscosity

$$
\begin{equation*}
u_{t}+u u_{x}-\nu u_{x x}=0, \quad \nu>0 \tag{5.3.3}
\end{equation*}
$$

The term $u u_{x}$ will have a shocking up effect that will cause waves to break and the term $\nu u_{x x}$ is a diffusion like term.

### 3.1 Travelling wave solutions

We attempt to find a travelling wave solution of (5.3.3) of the form

$$
u(x, t)=f(x-c t) .
$$

Substituting this to (5.3.3) we obtain

$$
-c f^{\prime}(s)+f(s) f^{\prime}(s)-\nu f^{\prime \prime}(s)=0,
$$

where $s=x-c t$. Noting that $f f^{\prime}=\frac{1}{2}\left(f^{2}\right)^{\prime}$ we re-write the above as

$$
-c f^{\prime}+\frac{1}{2}\left(f^{2}\right)^{\prime}-\nu f^{\prime \prime}=0
$$

hence we can integrate getting

$$
-c f+\frac{1}{2} f^{2}-\nu f^{\prime}=B
$$

where $B$ is a constant of integration. Hence

$$
\begin{equation*}
\frac{d f}{d s}=\frac{1}{2 \nu}\left(f^{2}-2 c f-2 B\right) . \tag{5.3.4}
\end{equation*}
$$

Let us consider the case when the quadratic polynomial above factorizes into real linear factors, that is

$$
\left(f^{2}-2 c f-2 B\right)=\left(f-f_{1}\right)\left(f-f_{2}\right)
$$

where

$$
\begin{equation*}
f_{1}=c-\sqrt{c^{2}+2 B}, \quad f_{2}=c+\sqrt{c^{2}+2 B} \tag{5.3.5}
\end{equation*}
$$

This requires $c^{2}>2 B$ and yields, in particular, $f_{2}>f_{1}$. Eq. (5.3.4) can be easily solved by separating variables. First note that $f_{1}$ and $f_{2}$ are the only equilibrium points and $\left(f-f_{1}\right)\left(f-f_{2}\right)<0$ for $f_{1}<f<f_{2}$ so that any solution starting between $f_{1}$ and $f_{2}$ will stay there tending to $f_{1}$ as $s \rightarrow+\infty$ and to $f_{2}$ as $s \rightarrow-\infty$. Any solution starting above $f_{2}$ will tend to $\infty$ as $s \rightarrow+\infty$ and any one starting below $f_{1}$ will tend to $-\infty$ as $s \rightarrow-\infty$. Thus, the only non constant travelling wave solutions are possible for $f_{1}<f<f_{2}$. For such $f$ integration if (4.3.26) yields

$$
\begin{aligned}
\frac{s-s_{0}}{2 \nu} & =\int \frac{d f}{\left(f-f_{1}\right)\left(f-f_{2}\right)}=-\frac{1}{f_{2}-f_{1}} \int\left(\frac{1}{f-f_{1}}+\frac{1}{f_{2}-f}\right) d f \\
& =\frac{1}{f_{2}-f_{1}} \ln \frac{f_{2}-f}{f-f_{1}}
\end{aligned}
$$

Solving for $f$ yields

$$
\begin{equation*}
f(s)=\frac{f_{2}+f_{1} e^{K\left(s-s_{0}\right)}}{1+e^{K\left(s-s_{0}\right)}} \tag{5.3.6}
\end{equation*}
$$

where $K=\frac{1}{2 \nu}\left(f_{2}-f_{1}\right)>0$. We see that for large positive $s f(s) \sim f_{1}$ whereas for negative values of $s$ we obtain asymptotically $f(s) \sim f_{2}$. It is clear that the initial value $s_{0}$ is not essential so that we shall suppress it in the sequel. The derivative of $f$ is

$$
f^{\prime}(s)=K \frac{e^{K s}\left(f_{1}-f_{2}\right)}{\left(1+e^{K s}\right)^{2}}<0 .
$$

It is easy to see that for large $|s|$ the derivative $f(s)$ is close to zero so that $f$ is almost flat. Moreover, the larger $\nu$ (so that the smaller $K$ ), the more flat is $f$ as the derivative is closer to zero. Hence, for small $\nu$ we obtain a very steep wave front that is consistent with the fact for $\nu=0$ we obtain inviscid Burger's equation that admits only discontinuous travelling waves.
The formula for travelling wave solution to (5.3.3) is then

$$
\begin{equation*}
u(x, t)=\frac{f_{2}+f_{1} e^{K(x-c t)}}{1+e^{K(x-c t)}} \tag{5.3.7}
\end{equation*}
$$

where the speed of the wave is determined from (5.3.5) as

$$
c=\frac{1}{2}\left(f_{1}+f_{2}\right) .
$$

Graphically the travelling wave solution is the profile $f$ moving to the right at speed $c$. This solution, because it resembles the actual profile of a shock wave, is called the shock structure solution; it joints the asymptotic states $f_{1}$ and $f_{2}$. Without the term $\nu u_{x x}$ the solutions of (5.3.3) would shock up and tend to break. The presence of the diffusion term prevents this breaking effect by countering the nonlinearity. The result is competition and balance between the nonlinear term $u u_{x}$ and the diffusion term $-\nu u_{x x}$, much the same as occurs in a real shock wave in the narrow region where the gradient is steep. In this context the $-\nu u_{x x}$ term could be interpreted as a viscosity term.

### 3.2 The Cole-Hopf transformation and analytic solution to the Burgers equation

Let us consider the simplified version of the Burgers equation

$$
\begin{equation*}
u_{t}+u u_{x}-u_{x x}=0 \tag{5.3.8}
\end{equation*}
$$

It can be proved that the Burgers equation is equivalent to the linear diffusion equation. First we introduce a new function by

$$
u=w_{x}
$$

Then (5.3.8) takes the form

$$
w_{t x}+w_{x} w_{x x}-w_{x x x}=w_{t x}+\frac{1}{2}\left(w_{x}\right)_{x}^{2}-w_{x x x} 0
$$

which can be integrated with respect to $x$ to give

$$
w_{t}+\frac{1}{2} w_{x}^{2}-w_{x x}=c(t)
$$

where $x(t)$ is a function of time $t$. Next we introduce a new function $v$ by the formula

$$
w=-2 \ln v .
$$

Differentiation gives

$$
w_{t}=-2 \frac{v_{t}}{v}, \quad w_{x}=-2 \frac{v_{x}}{v}, \quad w_{x x}=-2 \frac{v_{x x}}{v}+2 \frac{v_{x}^{2}}{v^{2}} .
$$

Hence

$$
w_{t}+\frac{1}{2} w_{x}^{2}-w_{x x}=-2 \frac{v_{t}}{v}+2 \frac{v_{x}^{2}}{v^{2}}+2 \frac{v_{x x}}{v}-2 \frac{v_{x}^{2}}{v^{2}}=-2 \frac{v_{t}}{v}+2 \frac{v_{x x}}{v}=c(t)
$$

which is the diffusion equation (with a growth/decay term).

$$
v_{t}-v_{x x}=-2 c(t) v
$$

If we denote $C(t)=\int c(t) d t$ then, using the integrating factor, we can write the above as the standard diffusion equation

$$
\left(v e^{C(t)}\right)_{t}-\left(v e^{C(t)}\right)_{x x}=0
$$

Summarizing, if we can find a solution $V(t, x)$ of the diffusion equation, then

$$
\begin{equation*}
u(x, t)=-2 \frac{\partial}{\partial x} \ln \left(v(x, t) e^{C(t)}\right)=-2 \frac{v_{x}(x, t)}{v(x, t)} \tag{5.3.9}
\end{equation*}
$$

is a solution to the Burgers equation. We see, that the arbitrary function $c(t)$ does not enter into the solution $u$ and thus we can focus on solving just

$$
v_{t}-v_{x x}=0
$$

So, any nonzero solution of the diffusion equation generates a unique solution of the Burgers equation. Conversely, if $u$ is a solution to the Burgers equation, then the formula (5.3.9) determines $v$ according to

$$
v(x, t)=\phi(t) e^{-\frac{1}{2} \int_{0}^{x} u(s, t) d s}
$$

where $\phi$ is an arbitrary function of $t$. Differentiating, we obtain

$$
v_{t}(x, t)=\phi^{\prime}(t) e^{-\frac{1}{2} \int_{0}^{x} u(s, t) d s}-\frac{1}{2} \phi(t) \int_{0}^{x} u_{t}(s, t) d s e^{-\int_{0}^{x} u(s, t) d s}=\frac{\phi^{\prime}(t)}{\phi(t)} v(x, t)-\frac{1}{2} v(x, t) \int_{0}^{x} u_{t}(s, t) d s
$$

and, similarly,

$$
v_{x}=-\frac{1}{2} u v, \quad v_{x x}=-\frac{1}{2} u_{x} v+\frac{1}{4} u^{2} v .
$$

Consequently,

$$
v_{t}-v_{x x}=\frac{1}{2} \frac{\phi^{\prime}}{\phi} v-\frac{1}{2} v\left(\int_{0}^{x} u_{t}(s, t) d s-u_{x}+\frac{1}{2} u^{2}\right)
$$

where the expression in brackets is the integral with respect to $x$ of the left hand side of (5.3.8), which is an arbitrary function of $t$. Hence, any solution to the Burgers equation generates a solution to the diffusion equation

$$
v_{t}-v_{x x}=\phi(t) v
$$

for some function $\phi(t)$.
As our main interest is finding the solution to the Burgers equation, we write down the analytical formula for the solution to the initial value problem

$$
\begin{equation*}
u_{t}+u u_{x}-u_{x x}=0, \quad u(x, 0)=\stackrel{\circ}{u}(x) . \tag{5.3.10}
\end{equation*}
$$

Using (5.3.9), the initial condition (5.3.10) on $u$ can be obtained from the initial condition on $v$ given by

$$
\stackrel{\circ}{v}(x)=e^{-\frac{1}{2} \int_{0}^{x} \dot{u}(s) d s}
$$

The solution of the initial value problem for the diffusion equation is given by (4.2.16)

$$
v(x, t)=\frac{1}{2 \sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^{2}}{4 t}} \stackrel{\circ}{v}(\xi) d \xi
$$

Therefore

$$
v_{x}(x, t)=-\frac{1}{2 \sqrt{\pi t}} \int_{-\infty}^{\infty} \frac{(x-\xi)}{2 t} e^{-\frac{(x-\xi)^{2}}{4 t}} \stackrel{\circ}{v}(\xi) d \xi
$$

and

$$
u(x, t)=-2 \frac{v_{x}(x, t)}{v(x, t)}=-\frac{\int_{-\infty}^{\infty} \frac{(x-\xi)}{t} e^{-\frac{(x-\xi)^{2}}{4 t}} \stackrel{\circ}{v}(\xi) d \xi}{\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^{2}}{4 t}} \stackrel{\circ}{v}(\xi) d \xi}
$$

or, introducing the kernel

$$
G(x, \xi, t)=-\frac{1}{2} \int_{0}^{\xi} \stackrel{\circ}{u}(s) d s-\frac{(x-\xi)^{2}}{4 t}
$$

we obtain the formula explicitly involving $\stackrel{\circ}{u}$ as

$$
u(x, t)=-\frac{\int_{-\infty}^{\infty} \frac{(x-\xi)}{t} e^{G(x, \xi, t)} d \xi}{\int_{-\infty}^{\infty} e^{G(x, \xi, t)} d \xi}
$$

Remark 3.1 The considerations above have been done for a spacial case of the Burgers equation with $\nu=1$. If we have the general form of the equation

$$
u_{t}+a u u_{x}=\nu u_{x x}
$$

then the substitution $u(x, t)=\nu a^{-1} U(x, \nu t)$ gives the simplified equation

$$
U_{t}+U U_{x}=U_{x x}
$$

Example 3.1 Find the solution to the following initial value problem

$$
u_{t}+u u_{x}=2 u_{x x}, \quad u(x, 0)=2 x .
$$

Introducing a new unknown function $u(x, t)=2 U(x, 2 t)$ we see that $u_{t}=4 U_{\tau}, u_{x}=2 U_{x}, u_{x x}=2 U_{x x}$ and thus have

$$
0=u_{t}+u u_{x}=2 u_{x x}=4 U_{\tau}+4 U U_{x}-4 U_{x x}
$$

so that $U$ satisfies the simplified Burgers equation with the initial condition $U(x, 0)=x$. We can then apply the transformation

$$
U=-2 \frac{V_{x}}{V}
$$

to get the diffusion equation

$$
V_{t}-V_{x x}=0
$$

subject to the initial condition

$$
\frac{x}{2}=-2 \frac{V_{x}(x, 0)}{V(x, 0)}
$$

that is

$$
V(x, 0)=e^{-\frac{1}{4} x^{2}}
$$

To solve the diffusion equation with this initial condition we can use several methods. We present the solution by matching and by using the integral formula (4.2.16).
Method 1.
We recognize that $V(x, 0)$ has the form similar to the fundamental solution of the diffusion equation (4.2.11)

$$
S(x, t)=\frac{1}{\sqrt{4 \pi t}} e^{-\frac{x^{2}}{4 t}}
$$

Since a solution of the diffusion equation multiplied by a constant is again a solution, we look for time $t$ and constant $c$ such that

$$
c S(x, t)=e^{-\frac{1}{4} x^{2}}
$$

which gives $t=1$ and $c=2 \sqrt{\pi}$. Thus $2 \sqrt{\pi} S(x, t)$ takes the value $V(x, 0)$ at $t=1$. Since the diffusion equation is invariant with respect to the shift of time, we obtain that

$$
V(x, t)=2 \sqrt{\pi} S(x, t+1)=\frac{e^{-\frac{x^{2}}{4(t+1)}}}{\sqrt{t+1}}
$$

is the sought solution. Consequently

$$
U(x, t)=-2 \frac{V_{x}}{V}=\frac{x}{t+1}
$$

and

$$
u(x, t)=2 U(x, 2 t)=\frac{2 x}{2 t+1}
$$

## Method 2.

We have

$$
V(x, t)=\frac{1}{2 \sqrt{\pi} t} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^{2}}{4 t}} e^{-\frac{y^{2}}{4}} d y
$$

Completing the quare in the exponent we get

$$
\frac{x^{2}-2 x y+y^{2}}{4 t}+\frac{y^{2}}{4}=\frac{x^{2}-2 x y+y^{2}+t y^{2}}{4 t}=\frac{(x-y(1+t))^{2}}{4 t(1+t)}+\frac{x^{2}}{4(1+t)}
$$

hence

$$
V(x, t)=\frac{e^{-\frac{x^{2}}{4(t+1)}}}{2 \sqrt{\pi} t} \int_{-\infty}^{\infty} e^{-\frac{(x-y(1+t))^{2}}{4 t(1+t)}} d y
$$

Introducing the change of variable $z=(x-y(1+t)) / 2 \sqrt{t(1+t)}$, so that $d y=-2 \sqrt{t} d z / \sqrt{1+t}$, we obtain

$$
V(x, t)=\frac{e^{-\frac{x^{2}}{4(t+1)}}}{\sqrt{\pi}(1+t)} \int_{-\infty}^{\infty} e^{-z^{2}} d z=\frac{e^{-\frac{x^{2}}{4(t+1)}}}{\sqrt{(1+t)}}
$$

The last step follows as before.

## 4 The Korteweg-de Vries equation and solitons

The last example related to travelling waves is concerned with the so called solitons that appear in solutions of numerous important partial differential equations. The simples equation producing solitons is the KortewegdeVries equation that governs long waves in shallow water. Find travelling wave solutions of the KdV equation

$$
\begin{equation*}
u_{t}+u u_{x}+k u_{x x x}=0 \tag{5.4.11}
\end{equation*}
$$

where $k>0$ is a constant. As before, we are looking for a solution of the form

$$
u(x, t)=f(s), \quad s=x-s t,
$$

where the waveform $f$ and the wave speed $c$ are to be determined. Substituting $f$ into (5.4.11) we get

$$
-c f^{\prime}+\frac{1}{2}\left(f^{2}\right)^{\prime}+k f^{\prime \prime \prime}=0
$$

integration of which gives

$$
-c f+\frac{1}{2} f^{2}+k f^{\prime \prime}=a
$$

for some constant $a$. This is a second order equation that does not contain the independent variable, hence we can use substitution $f^{\prime}=F(f)$. Hence, $f_{s}^{\prime \prime}=F_{f}^{\prime} f_{s}^{\prime}=F_{f}^{\prime} F$ and we can write

$$
-c f+\frac{1}{2} f^{2}+k F_{f}^{\prime} F=-c f+\frac{1}{2} f^{2}+\frac{k}{2}\left(F_{f}^{2}\right)^{\prime}=a .
$$

Integrating, we get

$$
F^{2}=\frac{1}{k}\left(c f^{2}-\frac{1}{3} f^{3}+2 a f+2 b\right)
$$

where $b$ is a constant. Thus,

$$
\begin{equation*}
f^{\prime}= \pm \sqrt{\frac{1}{3 k}}\left(-f^{3}+3 c f^{2}+6 a f+6 b\right)^{1 / 2} \tag{5.4.12}
\end{equation*}
$$

To fix attention we shall take the "+" sign. Denote by $\phi(f)$ the cubic polynomial on the right-hand side. We have the following possibilities:
(i) $\phi$ has one real root $\alpha$;
(ii) $\phi$ has three distinct real roots $\gamma<\beta<\alpha$;
(iii) $\phi$ has three real roots satisfying $\gamma=\beta<\alpha$;
(iv) $\phi$ has three real roots satisfying $\gamma<\beta=\alpha$;
(v) $\phi$ has a triple root $\gamma$.

Since we are looking for travelling wave solutions that should be bounded at $\pm \infty$ and nonnegative, we can rule out most of the cases by qualitative analysis describedNote first that the right-hand side of (5.4.12) is defined only where $\phi(f)>0$. Then, in the case (i), $\alpha$ is a unique equilibrium point, $\phi>0$ only for $f<\alpha$ and hence any solution converges to $\alpha$ as $t \rightarrow \infty$ and diverges to $+\infty$ as $t \rightarrow-\infty$. Similar argument rules out case (v).
If we have three distinct roots, then by the same argument, the only bounded solutions can exist in the cases (iii) and (ii) (in the case (iv) the bounded solutions could only appear where $\phi<0$.) The case (ii) leads to the so-called cnoidal waves expressible through special functions called cn-functions and hence the name. We shall concentrate on case (iii) so that

$$
\phi(f)=-f^{3}+3 c f^{2}+6 a f+6 b=(\gamma-f)^{2}(\beta-f)
$$

and, since $f>\gamma$

$$
\sqrt{\phi(f)}=(f-\gamma) \sqrt{\alpha-f}
$$

Thus, the differential equation (5.4.12) can be written

$$
\begin{equation*}
\frac{s}{\sqrt{3 k}}=\int \frac{d f}{(f-\gamma) \sqrt{\alpha-f}} \tag{5.4.13}
\end{equation*}
$$

To integrate, we first denote $v=f-\gamma$ and $B=\alpha-\gamma$ (with $0<v<b$ ), getting

$$
\frac{s}{\sqrt{3 k}}=\int \frac{d v}{v \sqrt{B-v}}
$$

Next, we substitute $w=\sqrt{B-v}$, hence $v=B-w^{2}$ and $d w=-d s / 2 \sqrt{B-v}$ so that the above will be transformed to

$$
\frac{s}{\sqrt{3 k}}=2 \int \frac{d w}{w^{2}-B}
$$

This integral can be evaluated by partial fractions, so that, using $0<w<\sqrt{B}$, we get

$$
\frac{s}{\sqrt{3 k}}=\frac{1}{\sqrt{B}} \int\left(\frac{d w}{w-\sqrt{B}}-\frac{d w}{w+\sqrt{B}}\right) d w=\frac{1}{\sqrt{B}} \ln \frac{\sqrt{B}-w}{\sqrt{B}+w} .
$$

Solving with respect to $w$ we obtain

$$
\frac{\sqrt{B}-w}{\sqrt{B}+w}=\exp s \sqrt{\frac{B}{3 k}}
$$

and thus

$$
w=\sqrt{B} \frac{1-\exp s \sqrt{\frac{B}{3 k}}}{1+\exp s \sqrt{\frac{B}{3 k}}}=-\sqrt{B} \frac{\sinh s \sqrt{\frac{B}{12 k}}}{\cosh s \sqrt{\frac{B}{12 k}}} .
$$

Returning to the original variables $w^{2}=B-v=\alpha-\gamma-f+\gamma=\alpha-f$ and using the hyperbolic identity $\cosh ^{2} \theta-\sinh ^{2} \theta=1$, we get

$$
\begin{aligned}
f(s) & =\alpha-w^{2}(s)=\alpha-(\alpha-\gamma) \frac{\sinh ^{2} s \sqrt{\frac{\alpha-\gamma}{12 k}}}{\cosh ^{2} s \sqrt{\frac{\alpha-\gamma}{12 k}}} \\
& =\gamma+(\alpha-\gamma) \operatorname{sech}^{2}\left(s \sqrt{\frac{\alpha-\gamma}{12 k}}\right)
\end{aligned}
$$

Clearly, $f(s) \rightarrow \gamma$ as $s \rightarrow \pm \infty$ so that the travelling wave here is a pulse. It is instructive to write the roots $\alpha$ and $\gamma$ in terms of the original parameters. To identify them we observe

$$
\begin{aligned}
\phi(f) & =-f^{3}+3 c f^{2}+6 a f+6 b=(\gamma-f)^{2}(\beta-f) \\
& =-f^{3}+f^{2}(\alpha+2 \gamma)+f\left(-\gamma^{2}-2 \alpha \gamma\right)+\gamma^{2} \alpha .
\end{aligned}
$$

Thus, the wave speed is given by

$$
c=\frac{\alpha+2 \gamma}{3}=\frac{\alpha-\gamma}{3}+\gamma .
$$

Since $\gamma$ is just the level of the wave at $\pm \infty$, by moving the coordinate system we can make it equal to zero. In this case we can write the travelling wave solution to the $K d V$ equation as

$$
\begin{equation*}
u(x, t)=3 \operatorname{csech}^{2}\left(\sqrt{\frac{\sqrt{c}}{4 k}}(x-c t)\right) \tag{5.4.14}
\end{equation*}
$$

It is important to note that the velocity of this wave is proportional to its amplitude which makes it different from linear waves governed by, say the wave equation, where the wave velocity is the property of the medium rather than of the wave itself.

## 5 The Fisher equation

Many natural processes involve mechanisms of both diffusion and reaction, and such problems are often modelled by so-called reaction-diffusion equations of the form

$$
\begin{equation*}
u_{t}-D u_{x x}=f(u), \tag{5.5.15}
\end{equation*}
$$

where $f$ is a given, usually nonlinear function of $u$. We introduced earlier the Fisher equation

$$
\begin{equation*}
u_{t}-D u_{x x}=r u\left(1-\frac{u}{K}\right) \tag{5.5.16}
\end{equation*}
$$

to model the diffusion of a species (e.g. insect population) when the reaction (or, at this instance) growth term is given by the logistic law. Here $D$ is the diffusion constant, and $r$ and $K$ are the growth rate and carrying capacity, respectively.

We shall examine the Fisher equation and, in particular, we shall address the question of existence of a travelling wave solution.

Let us consider the Fisher equation in dimensionless form

$$
\begin{equation*}
u_{t}-u_{x x}=u(1-u) \tag{5.5.17}
\end{equation*}
$$

and, as before, we shall look for solutions of the form

$$
\begin{equation*}
u(x, t)=U(s), \quad s=x-c t \tag{5.5.18}
\end{equation*}
$$

where $c$ is a positive constant and $U$ has the property that it approaches constant values at $s \rightarrow \pm \infty$. The function $U$ to be determined should be twice differentiable. The wave speed $c$ is a priori unknown and must be determined as a part of the solution of the problem. Substituting (5.5.18) into (5.5.17) yields a second order ordinary differential equation for $U$ :

$$
\begin{equation*}
-c U^{\prime}-U^{\prime \prime}=U(1-U) \tag{5.5.19}
\end{equation*}
$$

Contrary to the previous cases, this equation cannot be solved in a closed form and the best approach to analyze it is to perform the phase plane analysis. In a standard way we write (5.5.19) as a simultaneous system of first order equations by defining $V=U^{\prime}$. In this way we obtain

$$
\begin{align*}
U^{\prime} & =V \\
V^{\prime} & =-c V-U(1-U) \tag{5.5.20}
\end{align*}
$$

We find equilibrium points of this system solving

$$
\begin{aligned}
& 0=V \\
& 0=-c V-U(1-U)
\end{aligned}
$$

which gives two: $(0,0)$ and $(0,1)$. The Jacobi matrix of the system is

$$
J(U, V)=\left(\begin{array}{cc}
0 & 1 \\
2 U-1 & -c
\end{array}\right)
$$

so that

$$
J(0,0)=\left(\begin{array}{cc}
0 & 1 \\
-1 & -c
\end{array}\right)
$$

with eigenvalues

$$
\lambda_{ \pm}^{0,0}=\frac{-c \pm \sqrt{c^{2}-4}}{2}
$$

and

$$
J(1,0)=\left(\begin{array}{cc}
0 & 1 \\
1 & -c
\end{array}\right)
$$

with eigenvalues

$$
\lambda_{ \pm}^{1,0}=\frac{-c \pm \sqrt{c^{2}+4}}{2} .
$$

It is easily seen that for any $c, \lambda_{ \pm}^{1,0}$ are real and of opposite sign and therefore $(1,0)$ is a saddle. On the other hand, $\lambda_{ \pm}^{0,0}$ are both real and negative if $c \geq 2$ and in this case $(0,0)$ is a stable node (for the linearized system), and are complex with negative real part if $0<c<2$ in which case $(0,0)$ is a stable focus.
Since the wave profile $U(s)$ must have finite limits as $s \rightarrow \pm \infty$ and since we know that the only limit points of solutions of autonomous systems are equilibrium points, search for travelling wave solutions of (5.5.19) is


Fig. 5.1 Phase portrait of system (5.5.20).
equivalent to looking for orbits of (5.5.20) joining equilibria, that is approaching them as $s \rightarrow \pm \infty$. Such orbits are called heteroclinic if they join different equilibrium points and homoclinic if the orbit returns to the same equilibrium point from which it started.

We shall use the Stable Manifold Theorem. According to it, there are two orbits giving rise, together with the equilibrium point $(1,0)$, to the unstable manifold defined at least in some neighbourhood of the saddle point $(1,0)$, such that each orbit $\phi(s)=(U(s), V(s))$ satisfies: $\phi(s)$ to $(1,0)$ as $s \rightarrow-\infty$. Our aim is then to show that at least one of these orbits can be continued till $s \rightarrow \infty$ and reaches then $(0,0)$ in a monotonic way. Let us consider the orbit that moves into the fourth quadrant $U>0, V<0$. This quadrant is divided into to regions by the isocline $0=V^{\prime}=-c V-U(1-U)$ : region I with $\frac{1}{c}\left(U^{2}-U\right)<V<0$ and region II where $V<\frac{1}{c}\left(U^{2}-U\right)$, see Fig. 5. 1. In region I we have $V^{\prime}<0, U^{\prime}<0$ and in region II we have $V^{\prime}>0, U^{\prime}<0$. From the earlier theory we know that if a solution is not defined for all values of the argument $s$ then it must blow up as $s \rightarrow s^{\prime}$ where $\left(-\infty, s^{\prime}\right)$ is the maximal interval of existence of the solution. We note first that the selected orbit on our unstable manifold enters Region I so that $(U(s), V(s)) \in$ Region I for $-\infty<s \leq s_{0}$ for some $s_{0}$. In fact, the tangent of the isocline at $(1,0)$ is $1 / c$ and the tangent of the unstable manifold is $\lambda_{+}^{1,0}=\frac{\sqrt{c^{2}+4}-c}{2}$. Denoting

$$
\psi(c)=\frac{c\left(\sqrt{c^{2}+4}-c\right)}{2}
$$

we see that $\psi(0)=0, \lim _{c \rightarrow+\infty} \psi(s)=1$ and

$$
\psi^{\prime}(c)=\frac{4}{\sqrt{c^{2}+4}\left(\sqrt{c^{2}+4}+c\right)^{2}}>0
$$

thus $0 \leq \psi(c) \leq 1$ for all $c \geq 0$ and hence

$$
\frac{1}{c} \geq \frac{\sqrt{c^{2}+4}-c}{2}
$$

so that the slope of the isocline is larger than that of the orbit and the orbit must enter Region 1. Then, since $V^{\prime}<0$ there, $V(s)<V\left(s_{0}\right)$ as long as $V(s)$ stays in Region I. Hence, the orbit must leave Region I in finite time as there is no equilibrium point with strictly negative $V$ coordinate. Of course, there cannot be any blow up as long as the orbit stays in Region I. However, as at the crossing point the sign of $V^{\prime}$ changes, this point is a local minimum for $V$ so that the orbit starts moving up and continues to move left.

The slope of $V=\frac{1}{c} U(1-U)$ at the origin is $-1 / c$ so for $c \geq 1$ (and in particular for $c \geq 2$ ) the parabola $V=\frac{1}{c} U(1-U)$ stays above the line $V=-U$ so that any orbit entering Region II from Region I must stay


Fig. 5.2. The orbit in Region II.
for some time above $V=-U$, that is, $V / U>-1$. Consider any point $(U,-U), U>0$, and estimate the slope of the vector field on the line, see Fig. 5.2, We have

$$
\tan \phi=\frac{-c V-U(1-U)}{-V}=c+\frac{U}{V}(1-U)=c-1+U>c-1
$$

Considering the direction, we see that if $c \geq 2$, then the vector field points to the left from the line $U=-V$. Hence, the whole trajectory must stay between $U=-V$ and $V=\frac{1}{c} U(1-U)$. Hence, the orbit is bounded and therefore exists for all $s$ and enters $(0,0)$ in a monotonic way, that is $U$ is decreasing monotonically from 1 at $s=-\infty$ to 0 at $s=+\infty$ while $U^{\prime}=V$ is non-positive and goes from zero at $s=-\infty$ through minimum back to 0 at $s=+\infty$.

Thus, summarizing, the orbit $(U(s), V(s))$ is globally defined for $-\infty<s<+\infty$ joining the equilibrium points $(1,0)$ and $(0,0)$. Thus, $U(s) \rightarrow 1$ as $s \rightarrow-\infty$ and $U(s) \rightarrow 0$ as $s \rightarrow \infty$. Moreover, as $0>U^{\prime}(s)=$ $V(s) \rightarrow 0$ as $s \rightarrow \pm \infty, U$ is monotonically decreasing and becomes flat at both "infinities" giving a travelling wavefront solution.

We note that for $c<2$ the orbit no longer enters $(0,0)$ monotonically but, as suggested by linearization, spirals into the equilibrium point with $U$ passing through positive and negative values. Thus, if we are interested only in positive values of $U$ in order to have a physically realistic solution (e.g. if $U$ is a population density), we should reject this case.

### 5.1 An explicit travelling wave solution to the Fisher equation

In many cases by postulating a special form of the travelling wave, one can obtain explicit solutions having this particular form. An important example in the Fisher equation case are solutions of the form

$$
\begin{equation*}
U_{d}(z)=\frac{1}{\left(1+a e^{b z}\right)^{d}}, \quad z=x-c t \tag{5.5.21}
\end{equation*}
$$

where constants $a, b, d$ are to be determined. To determine these constants, we substitute (5.5.21) to (5.5.19)

$$
-c U^{\prime}-U^{\prime \prime}=U(1-U)
$$

First let us evaluate the necessary derivatives.

$$
\begin{aligned}
U_{d}^{\prime} & =-d b a e^{b z}\left(1+a e^{b z}\right)^{-d-1}=-d b\left(1+a e^{b z}-1\right)\left(1+a e^{b z}\right)^{-d-1} \\
& =-d b\left(\left(1+a e^{b z}\right)^{-d}-\left(1+a e^{b z}\right)^{-d-1}\right)=-d b\left(U_{d}-U_{d+1}\right)
\end{aligned}
$$

and similarly

$$
U_{d}^{\prime \prime}=d b^{2}\left(d U_{d}-(2 d+1) U_{d+1}+(d+1) U_{d+2}\right)
$$

On the other hand

$$
U(1-U)=U_{d}-U_{2 d}
$$

so that

$$
c d b\left(U_{d}-U_{d+1}\right)-d b^{2}\left(d U_{d}-(2 d+1) U_{d+1}+(d+1) U_{d+2}\right)=U_{d}-U_{2 d}
$$

Since $U_{d}=\left(U_{1}\right)^{d}$ and $y=U_{1}(z)$ is a one-to-one mapping of $(-\infty, \infty)$ onto $(0,1)$, the above equation is equivalent to the polynomial equation

$$
y^{2 d}-d(d+1) b^{2} y^{d+2}+\left(d(2 d+1) b^{2}-c d b\right) y^{d+1}+\left(c d b-(d b)^{2}-1\right) y^{d}=0, \quad y \in(0,1)
$$

Since a polynomial is identically equal zero on an open interval if and only if all coefficients are 0 , we obtain three possible cases: $2 d=d+2,2 d=d+1$ or $2 d=d$. The last one gives $d=0$ and thus a constant solution. The second results in $d=1$ and thus $d(d+1) b=0$, yielding $b=0$ and again we obtain a constant solution. Let us consider the first case. Then $d=2$ and thus, equating to zero coefficients of like powers

$$
\begin{aligned}
& 1-6 b^{2}=0 \\
& 2 b(5 b-c)=0 \\
& 2 c b-4 b^{2}-1=0
\end{aligned}
$$

we immediately obtain $b= \pm 1 / \sqrt{6}, c= \pm 5 / \sqrt{6}$ and it is easy to see that the last equation is automatically satisfied. Hence, we obtained a family of travelling wave of the Fisher equation

$$
u(x, t)=\frac{1}{\left(1+a e^{ \pm \frac{\sqrt{6} x \mp 5 t}{6}}\right)^{2}}
$$

The arbitrary parameter $a>0$ determines how steep is the wave.

## 6 The Nagumo equation

The Fisher equation describes spreading of a population which locally reproduces according to the logistic law. More sophisticated reproduction laws include the so-called Allee effect describing the situation in which small populations die out. Mathematically the Allee effect is represented by a term with three equilibria $0<L<K$, where $K$ is the carrying capacity of the environment and $L$ is the threshold below which the population perishes. In the normalized case we consider the equation

$$
\begin{equation*}
u_{t}=u_{x x}+u(u-a)(1-u) \tag{5.6.22}
\end{equation*}
$$

where $0<a<1$. $u$ is normalized density of the population and we assume that $0<u<1$. This equation is called the Nagumo equation. We look for some explicit travelling wave solutions to this equation. As before, we introduce $u(x, t)=U(x-c t), 0<U<1$, which converts (5.6.22) to

$$
-c U_{z}^{\prime}=U_{z z}^{\prime \prime}+U(U-a)(1-U), \quad z=x-c t .
$$

As long as $U_{z}^{\prime} \neq 0$, we can reduce this equation to the first order equation

$$
\left(\Psi^{2}\right)_{U}^{\prime}=-2 c \Psi-2 U^{3}-2(1+a) U^{2}+2 a U
$$

where $\Psi(U)=U_{z}^{\prime}$ and $U_{z z}^{\prime \prime}=\Psi_{U}^{\prime} \Psi$. If we try to find polynomial (in $U$ ) solutions to the above equation, then we see that the lowest order polynomial which could provide a solution is quadratic. Consider thus

$$
\Psi(U)=\gamma+\beta U+\alpha U^{2}
$$

Then

$$
\left(\Psi^{2}(U)\right)^{\prime}=2 \gamma \beta+2\left(2 \gamma \alpha+\beta^{2}\right) U+6 \alpha \beta U^{3}+4 \alpha^{2} U^{3}
$$

and, comparing like powers of $U$, we obtain

$$
\begin{aligned}
2 \gamma \beta & =-2 c \gamma \\
2\left(2 \gamma \alpha+\beta^{2}\right) & =-2 c \beta+2 a, \\
6 \alpha \beta & =-2 c \alpha-2(1+a), \\
4 \alpha^{2} & =2
\end{aligned}
$$

From the last equation $\alpha= \pm 1 / \sqrt{2}$. We shall work with the plus sign to keep the notation simpler. Thus

$$
\begin{align*}
\gamma(\beta+c) & =0  \tag{5.6.23}\\
\sqrt{2} \gamma+\beta^{2} & =-c \beta+a  \tag{5.6.24}\\
3 \beta & =-c-\sqrt{2}(1+a) \tag{5.6.25}
\end{align*}
$$

From the last equation $c=-3 \beta-\sqrt{2}(1+a)$ and the first equation becomes

$$
\gamma(2 \beta+\sqrt{2}(1+a))=0
$$

Now, we have to distinguish two cases.
Case 1. $\gamma=0$.
In this case the second equation gives

$$
4 \beta^{2}+2 \sqrt{2}(1+a)+2 a=0
$$

and we obtain solutions

$$
\beta_{1,2}=-\frac{a}{\sqrt{a}},-\frac{1}{\sqrt{2}}
$$

In the first case

$$
\Psi(U)=\frac{1}{\sqrt{2}} U(U-a)
$$

so that $\Psi(U)=U_{z}^{\prime}$ becomes 0 for $U=a$. It means that there is a possibility that the transformation reducing the second order equation to the first order maybe not invertible at some point and thus we discard this solution. The second case gives

$$
\Psi(U)=\frac{1}{\sqrt{2}} U(U-1)
$$

and $\Psi(U) \neq 0$ in the range where $U$ is allowed to change. So, we shall use this solution with

$$
\begin{equation*}
c=\sqrt{2}\left(\frac{1}{2}-a\right) . \tag{5.6.26}
\end{equation*}
$$

Case 2. $\gamma \neq 0$.
In this case

$$
-c=\beta=-\frac{1}{\sqrt{2}}(1+a)
$$

and, from the second equation in (5.6.25), $\gamma=a / \sqrt{2}$. Thus

$$
\Psi(U)=\frac{a}{\sqrt{2}}-\frac{1}{\sqrt{2}}(1+a) U+\frac{1}{\sqrt{2}} U^{2}=\frac{1}{\sqrt{2}}(a-U)(1-U)
$$

and, again, we see that $U^{\prime}$ can vanish for some $U \in(0,1)$. Hence, again we rule this case out.
Thus, we are left with the equation

$$
U^{\prime}=\frac{1}{\sqrt{2}} U(U-1)
$$

This is a separable equation which we solve by partial fractions. For $U \in(0,1)$

$$
K+\frac{z}{\sqrt{2}}=\int \frac{d U}{U}+\int \frac{d U}{1-U}=\ln \frac{U}{1-U}
$$

for some constant $K$, that is,

$$
U(z)=\frac{\bar{K} e^{\frac{z}{\sqrt{2}}}}{1+\bar{K} e^{\frac{z}{\sqrt{2}}}}
$$

and we obtain a travelling wave in the form

$$
u(x, t)=\frac{\bar{K} e^{\frac{x-c t}{\sqrt{2}}}}{1+\bar{K} e^{\frac{x-c t}{\sqrt{2}}}} .
$$

