

# MATHEMATICAL MODELING AND PARTIAL DIFFERENTIAL EQUATIONS

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# Lecture 1

## Basic facts about ordinary differential equations

### 1 What it means to solve an equation

Before we discuss the relatively complex topic of differential equations, let us reflect for a moment on the equations which we have known for a long time, namely, the algebraic equations. So, what really is an equation and what it means to solve it.

Equations arise when we are looking for a quantity the information about which is given in an indirect way. Thus equations are the final step of mathematical modeling and shouldn't be separated from the original problem. The fact that we are practicing solving given equations is because we have to learn basic techniques. However, in real life the equation is seldom given - it is our task to build an equation starting from physical, biological, financial data and later solve this equation, if possible.

**Example 1.1** *The price of a CD after 15% discount is R51. What was the original price of this CD?*

This is a simple problem. We chose as an unknown  $x$  the original price and using the given information we built the equation

$$x - 0.15x = 51,$$

and the solution is

$$x = 51/0.85 = 60.$$

An important, though here a trivial, question is whether this is the solution to our problem. How can we check it? The answer is that we have to return to the very beginning and check that the answer is correct. In our case, we have to check that if we subtract 15% of 60 from 60, we get 51. Doing this, we obtain  $60 - 60 \cdot 0.15 = 51$  which is correct.

Though presented in a simple context, this is one of the most important points of this first lecture.

**To solve a given problem is to find a quantity which satisfies all the conditions of the problem.**

This simple truth is very often forgotten as students tend to apply mechanically steps they learned under "techniques for solving quadratic equations" or "techniques of integration" labels.

The quantity (quantities) we obtain in the process of "solving" equation are in many cases only the suspects and we have to find out the real culprit in the final check. Since the textbook problems are constructed to be simple, the quantity obtained by applying the prescribed steps is often the solution, but the real life is much more complicated and the understanding of the situation described above should be basis for further learning.

Let us consider the following example.

**Example 1.2** Evaluate the integral

$$\int \frac{dx}{\sin x}.$$

Method 1.

$$\begin{aligned} \int \frac{dx}{\sin x} &= \frac{1}{2} \int \frac{dx}{\sin x/2 \cos x/2} = \frac{1}{2} \int \frac{\cos x/2 dx}{\sin x/2 \cos^2 x/2} \\ &= \frac{1}{2} \int \frac{dx}{\tan x/2 \cos^2 x/2} = \int \frac{du}{u} = \ln |\tan x/2| + C \end{aligned}$$

Method 2.

$$\begin{aligned} \int \frac{dx}{\sin x} &= \int \frac{\sin x dx}{\sin^2 x} = \int \frac{\sin x dx}{1 - \cos^2 x} = - \int \frac{du}{1 - u^2} \\ &= \frac{1}{2} \ln \left| \frac{1-u}{1+u} \right| = \frac{1}{2} \ln \left| \frac{1 - \cos x}{1 + \cos x} \right| + C. \end{aligned}$$

Method 3.

$$\begin{aligned} \int \frac{dx}{\sin x} &= \int \operatorname{cosec} x dx = \int \operatorname{cosec} x \frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x} dx \\ &= - \int \frac{du}{u} = - \ln |\operatorname{cosec} x + \cot x| + C \end{aligned}$$

So three different methods resulted in three different answers! But are they really different? To answer this question we must understand what is the meaning of the indefinite integral. It is not playing with substitutions, integration by parts etc till we get an answer without the integral sign. The equality

$$\int f(x) dx = F(x)$$

is equivalent to

$$F'(x) = f(x) \tag{1.1.1}$$

so that to check which answer is correct we have to differentiate the functions obtained by applying different techniques of integration. For subsequent calculations we shall omit the absolute value signs. First we have for, say,  $-\pi < x < \pi$

$$\begin{aligned} \frac{d}{dx} \ln \tan x/2 &= \frac{1}{\tan x/2} \frac{d}{dx} \tan x/2 = \frac{\cos x/2}{\sin x/2} \frac{1}{2 \cos^2 x/2} \\ &= \frac{1}{\sin x} \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{d}{dx} \left( \frac{1}{2} \ln \frac{1 - \cos x}{1 + \cos x} \right) &= \frac{1}{2} \frac{1 + \cos x}{1 - \cos x} \frac{\sin x(1 + \cos x) + \sin x(1 - \cos x)}{(1 + \cos x)^2} \\ &= \frac{\sin x}{1 - \cos^2 x} = \frac{1}{\sin x}. \end{aligned}$$

Similarly, differentiating the third answer, we get

$$\begin{aligned} \frac{d}{dx} (-\ln(\operatorname{cosec} x + \cot x)) &= \frac{d}{dx} (\ln \sin x - \ln(1 + \cos x)) \\ &= \frac{\cos x}{\sin x} + \frac{\sin x}{1 + \cos x} = \frac{\cos x + \cos^2 x + \sin^2 x}{\sin x(1 + \cos x)} \\ &= \operatorname{cosec} x \end{aligned}$$



So, all the answers are correct. But it doesn't mean that they are the same. A theorem from Calculus tells us that any two indefinite integrals can differ by at most constant, that is

$$\ln |\tan x/2| = \frac{1}{2} \ln \left| \frac{1 - \cos x}{1 + \cos x} \right| + C = -\ln |\operatorname{cosec} x + \cot x| + C_1$$

for some constants  $C, C_1$ . The value of constants is to be determined (here they are incidentally equal).

In this example we touched two problems which are very important in the study of differential equations: the representation of the solution and uniqueness of the solution. We see that the solution to a given problem may have several representations (formulas) and that it is not necessarily unique. Also, from (1.1.1) it follows that in fact we have solved the equation:

$$\frac{dF}{dx} = \frac{1}{\sin x}$$

which represents the simplest differential equation.

## 2 What are differential equations?

In an algebraic equations, like linear or quadratic equations, the unknown quantity is a number (or a collection of numbers) and the equation expresses relations between these numbers and certain numerical coefficients arising in the problem. If the data appearing in the problem are variable, then the unknown will be a function and in the modeling usually we have to balance small increments of this function and the data of the problem. The result typically in an equation involving the derivatives of the unknown function. Such an equation is called a differential equation.

Differential equations are divided into several classes. The main two classes are ordinary differential equations (ODEs) and partial differential equations (PDEs). This course is devoted to PDEs but during the first few lectures we shall recall basic facts concerning ODEs (which ideally should have been covered in the second year calculus).

As suggested by the name, ODEs are equations where the unknown function is a function of one variable and the derivatives involved in the equation are ordinary derivatives of this function. Symbolically, the general form of ODE is

$$F(y^{(n)}, y^{(n-1)}, \dots, y', y, t) = 0,$$

where  $F$  is a given function of  $n + 2$  variables. To solve this ODE means to find an  $n$ -times continuously differentiable function  $y(t)$  such that for any  $t$  (from some interval)

$$F(y^{(n)}(t), y^{(n-1)}(t), \dots, y'(t), y(t), t) = 0.$$

In this section we shall be concerned with *first order* ordinary differential equations which are solved with respect to the derivative of the unknown function, that is, with equations which can be written as

$$y' = f(t, y), \tag{1.2.2}$$

where  $f$  is a given function of two variables.

Several comments are in place here. Firstly, even though in such a simplified form, equation (1.2.2) in general has no closed form solution, that is, it is impossible to write the solution in the form

$$y(t) = \text{combination of elementary functions like } \sin t, \cos t, \ln t, \text{polynomials} \dots$$

**Example 2.1** A trivial example is the equation

$$y' = e^{-t^2}.$$

We know that the solution must be

$$y(t) = \int e^{-t^2} dt$$

but, on the other hand, it is known that this integral cannot be expressed as a combination of elementary functions.

If a solution to a given equation can be written in terms of integrals of elementary functions (as above), then we say that the equation is *solvable in quadratures*. Since we know that every continuous function has an antiderivative (though often we cannot find this antiderivative explicitly), it is almost as good as finding the explicit solution to the equation. However, there are many instances when we cannot solve an equation even in quadratures. How do we know then that the equation has a solution? The answer is that if the right hand side of the equation, that is the function  $f$ , is continuous, then there is at least one solution to (1.2.2). This result is called the Peano Theorem and involves some more advanced calculus. Thus, we can safely talk about solutions to ODEs of the form (1.2.2) even without knowing their explicit formulae.

Another important problem is related to the uniqueness of solutions, that is, whether there is only one solution to a given ODE. A quick reflection shows that clearly not: for the simplest equation

$$y' = 0,$$

the solutions are

$$y(t) = C,$$

where  $C$  is an arbitrary constant; thus there are infinitely many solutions. The uniqueness question, however, hasn't been properly posed. In fact, what we are looking for is usually a solution passing through a specified point.

**Example 2.2** Assume that a point is moving along the horizontal line with speed given by  $v(t) = t$ . Find the position of the point at  $t = 5$ . To solve this problem let us recall that  $v(t) = \frac{ds}{dt}$  where  $s$  is the distance traveled. Thus the problem results in an equation of the type discussed above:

$$v(t) = \frac{ds}{dt} = t$$

and

$$s(t) = 0.5t^2 + C$$

where  $C$  is an arbitrary constant. Hence  $s(5) = 12.5 + C$  and there is no proper answer. In this physical setting the original question is clearly wrongly posed. What we need to give the proper answer is the information about the position of the point at some other time  $t$ , say,  $t = 1$ . If we know that (with respect to a fixed origin)  $s(1) = 2$ , then also  $s(1) = 0.5 + C$  and  $C = 1.5$ . Therefore  $s(5) = 12.5 + 1.5 = 14$ .

From this example (and from physical or other considerations) it follows that if we are interested in getting a unique answer, we not only need the equation (which reflects usually some natural law) but also the state of the system (that is, the value of the solution) at some specified point. Thus, the complete *Cauchy* or *initial value* problem would be to solve

$$\begin{aligned} y' &= f(t, y), & \text{for all } t \in [t_1, t_2] \\ y(t_0) &= y_0, & \text{for some } t_0 \in [t_1, t_2]. \end{aligned} \tag{1.2.3}$$

Once again we emphasize that to solve (1.2.3) is to find a continuously differentiable function  $y(t)$  such that

$$\begin{aligned} y'(t) &= f(t, y(t)) & \text{for all } t \in [t_1, t_2] \\ y(t_0) &= y_0, & \text{for some } t_0 \in [t_1, t_2]. \end{aligned}$$

**Example 2.3** Check that the function  $y(t) = \sin t$  is the solution to the problem

$$\begin{aligned} y' &= \sqrt{1-y^2}, \quad t \in [0, \pi/2] \\ y(\pi/2) &= 1 \end{aligned}$$

**Solution.** LHS:  $y'(t) = \cos t$ , RHS:  $\sqrt{1-y^2} = \sqrt{1-\sin^2 t} = |\cos t| = \cos t$  as  $t \in [0, \pi/2]$ . Thus the equation is satisfied. Also  $\sin \pi/2 = 1$  so the "initial" condition is satisfied.

Note that the function  $y(t) = \sin t$  is not a solution to this equation on a larger interval.

Returning to our uniqueness question we ask whether the problem (1.2.3) has always a unique solution. The answer is negative.

**Example 2.4** The Cauchy problem

$$\begin{aligned} y' &= \sqrt{y}, \quad t \geq 0 \\ y(0) &= 0, \end{aligned}$$

has at least two solutions:  $y \equiv 0$  and  $y = \frac{1}{4}t^2$ .

However, there is a large class of functions  $f$  for which (1.2.3) has exactly one solution. All continuous functions  $f(t, y)$  having bounded derivative  $\frac{\partial f}{\partial y}$  in some neighbourhood of  $(t_0, y_0)$  give rise to (1.2.3) with exactly one solution (at least close to  $t_0$ ). This result is known as the Picard Theorem which we state below.

**Theorem 2.1** Let  $f$  and  $\partial f / \partial y$  be continuous in the rectangle  $R : |t - t_0| \leq a, |y - y_0| \leq b$  for some  $a, b > 0$ . Compute

$$M = \max_{(t,y) \in R} |f(t, y)|$$

and define  $\alpha = \min\{a, b/M\}$ . Then the initial value problem (1.2.3) has exactly one solution at least on the interval  $t_0 - \alpha \leq t \leq t_0 + \alpha$ .

We illustrate the use of this theorem on several examples.

**Example 2.5** Show that the solution  $y(t)$  of the initial value problem

$$\begin{aligned} y' &= t^2 + e^{-y^2}, \\ y(0) &= 0, \end{aligned}$$

exists for  $0 \leq t \leq 0.5$ , and in this interval,  $|y(t)| \leq 1$ .

Let  $R$  be the rectangle  $0 \leq t \leq 0.5, |y| \leq 1$ . The function  $f(t, y) = t^2 + e^{-y^2}$  is continuous and has continuous derivative  $f_y$ . We find

$$M = \max_{(t,y) \in R} |f(t, y)| \leq (1/2)^2 + e^0 = 5/4,$$

thus the solution exists and is unique for

$$0 \leq t \leq \min\{1/2, 5/4\} = 1/2,$$

and of course in this interval  $|y(t)| \leq 1$ .

**Example 2.6** The solution of the initial value problem

$$\begin{aligned} y' &= 1 + y^2, \\ y(0) &= 0, \end{aligned}$$

is given by  $y(t) = \tan t$ . This solution is defined only for  $-\pi/2 < t < \pi/2$ . Let us check this equation against the Picard Theorem. We have  $f(t, y) = 1 + y^2$  and  $f_y(t, y) = 2y$  and both functions are continuous on the whole plane. Let  $R$  be the rectangle  $|t| \leq a$ ,  $|y| \leq b$ , then

$$M = \max_{(t,y) \in R} |f(t, y)| = 1 + b^2,$$

and the solution exists for

$$|t| \leq \alpha = \min\{a, \frac{b}{1+b^2}\}.$$

Since  $a$  can be arbitrary, the maximal interval of existence predicted by the Picard Theorem is the maximum of  $b/(1+b^2)$  which is equal to  $1/2$ .

This shows that it may happen that the Picard theorem sometimes does not give the best possible answer - that is why it is sometimes called "the local existence theorem".

**Example 2.7** Suppose that  $|f(t, y)| \leq K$  in the whole plane  $\mathbb{R}^2$ . Show that the solution of the initial value problem

$$\begin{aligned} y' &= f(t, y), \\ y(t_0) &= y_0, \end{aligned}$$

where  $t_0$  and  $y_0$  are arbitrary, exists for all  $t \in \mathbb{R}$ .

Let  $R$  be the rectangle  $|t - t_0| \leq a$ ,  $|y - y_0| \leq b$  for some  $a, b$ . The quantity  $M$  is given by

$$M = \max_{(t,y) \in R} |f(t, y)| = K,$$

and the quantity

$$|t - t_0| \leq \alpha = \min\{a, \frac{b}{K}\},$$

can be made as large as we wish by choosing  $a$  and  $b$  sufficiently large. Thus the solution exists for all  $t$ .

**Example 2.8** We have seen in Example 2.4 that there are two solutions to the problem

$$\begin{aligned} y' &= \sqrt{y}, \quad t \geq 0 \\ y(0) &= 0. \end{aligned}$$

In this case  $f(t, y) = \sqrt{y}$  and  $f_y = 1/2\sqrt{y}$ ; obviously  $f_y$  is not continuous in any rectangle  $|t| \leq a$ ,  $|y| \leq b$  and we may expect troubles.

Another example of nonuniqueness is offered by

$$\begin{aligned} y' &= (\sin 2t)y^{1/3}, \quad t \geq 0 \\ y(0) &= 0, \end{aligned} \tag{1.2.4}$$

Direct substitution shows that we have 3 different solutions to this problem:  $y_1 \equiv 0$ ,  $y_2 = \sqrt{8/27} \sin^3 t$  and  $y_3 = -\sqrt{8/27} \sin^3 t$ . These are shown at the picture below.

*Fig1.1. Multiple solutions of the problem (1.2.4).*

### 3 ODEs which can be solved in a closed form

#### 3.1 Linear ordinary differential equations of first order

**Definition 3.1** *The general first order linear differential equation is*

$$\frac{dy}{dt} + a(t)y = b(t). \quad (1.3.1)$$

*Functions  $a$  and  $b$  are known continuous functions of time.*

We call this equation "linear", because the dependent variable  $y$  appears by itself in the equation. In other words,  $y'$  and  $y$  appear in the equation only possibly multiplied by a known function and not in the form  $yy'$ ,  $\sin y$  or  $(y')^3$ . For instance, the equation

$$\frac{dy}{dt} + \sin y = b(t)$$

is not linear.

**Remark 3.1** In all applications linear problems are usually much easier than the nonlinear ones. Unfortunately, the Nature is by and large nonlinear. The study of linear equations is, however, justified as very often solutions to linear equations provide very good approximation to solutions of "related" nonlinear equations, especially for some ranges of parameters. ■

It is not immediate how to solve (1.3.1), therefore we shall simplify it even further by putting  $b(t) = 0$ . The resulting equation

$$\frac{dy}{dt} + a(t)y = 0, \quad (1.3.2)$$

is called the *homogeneous* first order linear differential equation (accordingly (1.3.1) is referred to as the *nonhomogeneous* equation). Homogeneous equation can be solved easily. We write (1.3.2) in the form

$$\frac{\frac{dy}{dt}}{y} = -a(t),$$

and observe that if  $y(t) \neq 0$  for any  $t$

$$\frac{\frac{dy}{dt}}{y(t)} = \frac{d}{dt} \ln |y(t)|,$$

thus we reduced the problem to the equation almost the same as (1.1.1)

$$\frac{d}{dt} \ln |y(t)| = -a(t),$$

which can be solved by direct integration

$$\ln |y(t)| = - \int a(t) dt + c_1$$

where  $c_1$  is an arbitrary constant of integration. Taking exponentials of both sides yields

$$|y(t)| = \exp \left( - \int a(t) dt + c_1 \right) = c_2 \exp \left( - \int a(t) dt \right)$$

where  $c_2$  is an arbitrary positive constant:  $c_2 = \exp c_1 > 0$ . We have to get rid of the absolute value bars at  $y(t)$ . To do this observe that in the derivation we required that  $y(t) \neq 0$  for any  $t$ , thus  $y$  being a continuous function, must be of constant sign. Thus we have

$$y(t) = \pm c_2 \exp \left( - \int a(t) dt \right) = c_3 \exp \left( - \int a(t) dt \right) \quad (1.3.3)$$

where this time  $c_3$  can be either positive or negative.

Are these all possible solutions to (1.3.2)? Solution (1.3.3) was derived under provision that  $y \neq 0$ . We clearly see that  $y \equiv 0$  is a solution to (1.3.2) but, fortunately, this solution can be incorporated into (1.3.3) by allowing  $c_3$  to be zero.

However, we still haven't ruled out the possibility that the solution can cross the  $x$ -axis at one or more points. To prove that this is impossible, we must resort to the Picard theorem. First of all we note that the function  $f(t, y)$  is here given by

$$f(t, y) = a(t)y$$

and  $\partial f / \partial y = a(t)$ . Since the rectangle  $R$  can be taken as large as we wish, we can claim that for any  $t_0, y_0$  the initial value problem

$$\begin{aligned} \frac{dy}{dt} + a(t)y &= 0, \\ y(t_0) &= y_0, \end{aligned} \tag{1.3.4}$$

has a unique solution  $y(t)$  defined at least for  $|t - t_0| \leq \min\{a, b / \sup_{|t-t_0| \leq a} |a(t)|\}$ . Therefore, if there is a solution  $y(t)$  to (1.3.2) such that for some  $t_0$  we have  $y(t_0) = 0$ , then this solution is also the solution of the problem

$$\begin{aligned} \frac{dy}{dt} + a(t)y &= 0, \\ y(t_0) &= 0. \end{aligned}$$

However, we know that  $\tilde{y}(t) \equiv 0$  is a solution to this problem thus from the Picard theorem  $y(t) = \tilde{y}(t) \equiv 0$ . In other words, if a solution to (1.3.2) is zero at some point, then it is identically zero.

After this considerations we can claim that all the solutions to (1.3.2) are of the form

$$y(t) = c \exp\left(-\int a(t)dt\right), \tag{1.3.5}$$

where  $c$  is an arbitrary real constant.

How this solution can help us with solving the nonhomogeneous equation

$$\frac{dy}{dt} + a(t)y = b(t)? \tag{1.3.6}$$

If we could repeat the trick used in the solution of (1.3.2) and write the above equation in the form

$$\frac{d}{dt}(\text{"something"}) = b(t),$$

then the solution would be easy. However, the expression  $dy/dt + a(t)y$  does not appear to be a derivative of any simple expression and we have to help it a little bit. We shall multiply both sides of (1.3.6) by some continuous nonzero function  $\mu$  (for a time being unknown) to get the equivalent equation

$$\mu(t) \frac{dy}{dt} + \mu(t)a(t)y = \mu(t)b(t). \tag{1.3.7}$$

and ask the question: for which function  $\mu$  the left-hand side of (1.3.7) is a derivative of some simple expression? We note that the first term on the left-hand side comes from

$$\frac{d\mu(t)y}{dt} = \mu(t) \frac{dy}{dt} + \frac{d\mu(t)}{dt}y,$$

thus if we find  $\mu$  in such a way that

$$\mu(t) \frac{dy}{dt} + \frac{d\mu(t)}{dt}y = \mu(t) \frac{dy}{dt} + \mu(t)a(t)y,$$

that is

$$\frac{d\mu(t)}{dt}y = \mu(t)a(t)y,$$

then we are done. Note that an immediate choice is to solve the equation

$$\frac{d\mu(t)}{dt} = \mu(t)a(t),$$

but this is a homogeneous equation the general solution of which is given by (1.3.5). Since we need only one such function, we may take

$$\mu(t) = \exp\left(\int a(t)dt\right).$$

The function  $\mu$  is called an *integrating factor* of the equation (1.3.6). With such function, (1.3.6) can be written as

$$\frac{d}{dt}\mu(t)y = \mu(t)b(t),$$

thus

$$\mu(t)y = \int \mu(t)b(t)dt + c$$

where  $c$  is an arbitrary constant of integration. Finally

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)} \left( \int \mu(t)b(t)dt + c \right) \\ &= \exp\left(-\int a(t)dt\right) \left( \int b(t) \exp\left(\int a(t)dt\right) dt + c \right) \end{aligned} \quad (1.3.8)$$

It is worthwhile to note that the solution consists of two parts: the general solution to the homogeneous equation associated with (1.3.6)

$$c \exp\left(-\int a(t)dt\right)$$

and, what can be checked by direct differentiation, a solution to the nonhomogeneous equation.

If we want to find a particular solution satisfying  $y(t_0) = y_0$ , then we write (1.3.8) using definite integrals

$$y(t) = \exp\left(-\int_{t_0}^t a(s)ds\right) \left( \int_{t_0}^t b(s) \exp\left(\int_{t_0}^s a(r)dr\right) ds + c \right)$$

and use the fact that  $\int_{t_0}^{t_0} f(s)ds = 0$  for any function  $f$ . This shows that the part of the solution satisfying the nonhomogeneous equation:

$$y_b(t) = \exp\left(-\int_{t_0}^t a(s)ds\right) \int_{t_0}^t b(s) \exp\left(\int_{t_0}^s a(r)dr\right) ds$$

takes on the zero value at  $t = t_0$ . Thus

$$y_0 = y(t_0) = c$$

and the solution to the initial value problem is given by

$$y(t) = y_0 \exp\left(-\int_{t_0}^t a(s)ds\right) + \exp\left(-\int_{t_0}^t a(s)ds\right) \int_{t_0}^t b(s) \exp\left(\int_{t_0}^s a(r)dr\right) ds. \quad (1.3.9)$$

Once again we emphasize that the first term of the formula above solves the homogeneous ( $b(t) = 0$ ) equation with the desired initial value ( $y(0) = y_0$ ) whereas the second solves the nonhomogeneous equation with the

initial value equal to zero. This approach will prove extremely useful also in dealing with partial differential equations.

Again, Picard's theorem shows that there are no more solutions to than those given by (1.3.9). Why? Let  $y_1(t)$  be a solution to (1.3.6). For some  $t = t_0$  this will take on the value  $y_1(t_0)$ . But we know that there is a solution to (1.3.6) given by

$$y(t) = y_1(t_0) \exp \left( - \int_{t_0}^t a(s) ds \right) + \exp \left( - \int_{t_0}^t a(s) ds \right) \int_{t_0}^t b(s) \exp \left( \int_{t_0}^s a(r) dr \right) ds,$$

and by Picard's theorem (this time  $f(t, y) = -a(t)y + b(t)$  but the assumptions are still satisfied),  $y_1(t) = y(t)$ .

**Example 3.1** Find the general solution of the equation

$$y' - 2ty = t.$$

Here  $a(t) = -2t$  so that

$$\mu(t) = \exp \left( -2 \int t dt \right) = e^{-t^2}.$$

Multiplying both sides of the equation by  $\mu$  we obtain

$$e^{-t^2} y' - 2te^{-t^2} = te^{-t^2},$$

which can be written as

$$\frac{d}{dt}(e^{-t^2} y) = te^{-t^2}.$$

Upon integration we get

$$e^{-t^2} y = \int te^{-t^2} dt = -\frac{e^{-t^2}}{2} + c.$$

Thus the general solution is given by

$$y(t) = -\frac{1}{2} + ce^{t^2}.$$

**Example 3.2** Find the solution of the equation

$$y' + 2ty = t,$$

satisfying  $y(1) = 2$ . Here  $a(t) = 2t$  so that

$$\mu(t) = \exp \left( 2 \int t dt \right) = e^{t^2}.$$

As above, we obtain

$$e^{t^2} y' + 2te^{t^2} = \frac{d}{dt}(e^{t^2} y) = te^{t^2}. \quad (1.3.10)$$

General solution is

$$y(t) = \frac{1}{2} + ce^{-t^2}$$

and to find  $c$  we have

$$2 = y(1) = \frac{1}{2} + ce^{-1}$$

which gives

$$c = \frac{3e}{2}$$



and

$$y(t) = \frac{1}{2} + \frac{3}{2}e^{1-t^2}.$$

Alternatively, we can integrate (1.3.10) from 1 to  $t$  to get

$$e^{s^2} y(s) \Big|_1^t = \frac{e^{s^2}}{2} \Big|_1^t$$

and further

$$e^{t^2} y - 2e = e^{t^2} 2 - \frac{e}{2}.$$

Thus again

$$y(t) = \frac{1}{2} + \frac{3}{2}e^{1-t^2}.$$

### 3.2 Separable equations

In the previous section solved the first order linear homogeneous equation

$$y' + a(t)y = 0$$

by re-writing it in the form

$$\frac{1}{y(t)} \frac{dy(t)}{dt} = -a(t)$$

and observing that the left-hand-side can transformed into

$$\frac{d}{dt} \ln |y(t)| = -a(t).$$

Integrating, we found  $\ln |y|$  and taking the inverse function (in this case exponentiating) we found  $y$ .

This approach is not limited to functions linear in  $y$ . Consider the equation

$$\frac{dy}{dt} = \frac{g(t)}{h(y)}, \tag{1.3.11}$$

where  $g$  and  $h$  are known functions. This equation, and any other which could be put into this form, is called the *separable* equation. To solve it, we multiply both sides of (1.3.11) by  $h(y)$  to get

$$h(y) \frac{dy}{dt} = g(t)$$

and observe that if  $H(y) = \int h(y)$ , then the left-hand side can be transformed into

$$\frac{d}{dt} (H(y(t))) = g(t)$$

and upon integration we obtain

$$H(y(t)) = \int g(t) dt + c, \tag{1.3.12}$$

where  $c$  is an arbitrary constant of integration. The next step depends on the properties of  $H$ : for instance, if  $H : \mathbb{R} \rightarrow \mathbb{R}$  is monotonic, then we can find  $y$  explicitly for all  $t$  as

$$y(t) = H^{-1} \left( \int g(t) dt + c \right).$$

Otherwise, we have to do it locally, around the initial values. To explain this, we solve the initial value problem for separable equation.

$$\frac{dy}{dt} = \frac{g(t)}{h(y)}, \quad (1.3.13)$$

$$y(t_0) = y_0, \quad (1.3.14)$$

Using the general solution (1.3.12) (with definite integral) we obtain

$$H(y(t)) = \int_{t_0}^t g(s)ds + c,$$

we obtain

$$H(y(t_0)) = \int_{t_0}^{t_0} a(s)ds + c,$$

which gives

$$c = H(y(t_0)),$$

so that

$$H(y(t)) = \int_{t_0}^t g(s)ds + H(y(t_0)).$$

We are interested in the existence of the solution at least close to  $t_0$ , which means that  $H$  should be invertible close to  $y_0$ . From the Implicit Function Theorem we obtain that this is possible if  $H$  is differentiable in a neighbourhood of  $y_0$  and  $\partial H/\partial y(y_0) \neq 0$ . But  $\partial H/\partial y(y_0) = h(y_0)$ , so we are back at Picard's theorem: if  $h(y)$  is differentiable in the vicinity of  $y_0$  with  $h(y_0) \neq 0$  (and  $g$  is continuous, then  $f(t, y) = g(t)/h(y)$  satisfies the assumptions of the theorem in some rectangle  $R$  about  $t_0, y_0$ ).

**Example 3.3** Find the general solution of the equation

$$y' = t^2/y^2.$$

This equation is equivalent to

$$\frac{d}{dt} \left( \frac{y(t)^3}{3} \right) = t^2,$$

hence  $y^3(t) = t^3 + c$ , where  $c$  is an arbitrary constant and, since the cubic function is monotonic,

$$y(t) = (t^3 + c)^{1/3}.$$

**Remark 3.2** Note that Picard's theorem gives only a sufficient condition for the existence of the unique solution. In the example above the assumptions are obviously violated at  $t = 0$  and  $y = 0$ , that is, the theorem doesn't give an answer as to whether there exists a unique solution to the problem

$$y' = t^2/y^2, \quad y(0) = 0.$$

However, direct computation shows that  $y(t) = t$  is the unique solution to this problem. This is possible due to the cancelation of singularities.

On the other hand, consider similar problem

$$y' = t/y, \quad y(0) = 0.$$

Then the general solution is given by

$$y^2(t) = t^2 + c,$$

and with the initial condition we obtain

$$y^2(t) = t^2.$$

This time the quadratic function is not invertible and it produces two solutions  $y(t) = t$  and  $y(t) = -t$ .

**Example 3.4** *Solve the initial value problem*

$$y' = 1 + y^2, \quad y(0) = 0.$$

We transform the equation as

$$\frac{d}{dt} \tan^{-1} y(t) = 1$$

which gives

$$\tan^{-1} y = t + c,$$

and from the initial condition  $c = 0$ . Therefore, the solution is given by

$$y = \tan t.$$

We considered this problem in Example 2.6. Once again we point out that  $y \rightarrow \pm\infty$  as  $t \rightarrow \pm\pi/2$ . In other words the solution exists only on the interval  $]-\pi/2, \pi/2[$  but there is seemingly nothing at all in the form of the equation which would suggest such a behaviour. The explanation comes again from the Picard theorem. If we consider  $R : |y| \leq b, |t| \leq a$  ( $y_0 = 0, t_0 = 0$ ), then

$$M = \max_{(t,y) \in R} |f(t,y)| = \max_{(t,y) \in R} |1 + y^2| = 1 + b^2$$

and the interval  $[-\alpha, \alpha]$  of the existence of the solution is determined by  $\alpha = \min\{a, b/(1 + b^2)\}$ . Maximum value which the fraction can attain is  $1/2$ , so we see that Picard's theorem ensures the existence of the solution on the interval at most  $[-1/2, 1/2]$  and the fact that the solution exists on  $]-\pi/2, \pi/2[$  is rather a bonus.

The next example shows that there might be another way for a solution to cease to exist.

**Example 3.5** *Find the solution to the following initial value problem*

$$yy' + (1 + y^2) \sin t = 0, \quad y(0) = 1.$$

In a standard way we obtain

$$\int_1^y \frac{r dr}{1 + r^2} = - \int_0^t \sin s ds,$$

which gives

$$\frac{1}{2} \ln(1 + y^2) - \frac{1}{2} \ln 2 = \cos t - 1.$$

Solving this equation for  $y(t)$  gives

$$y(t) = \pm(2e^{-4 \sin^2 t/2} - 1)^{1/2}.$$

To determine which sign we should take we note that  $y(0) = 1 > 0$ , thus the solution is given by

$$y(t) = (2e^{-4 \sin^2 t/2} - 1)^{1/2}.$$

Clearly, this solution is only defined when

$$2e^{-4 \sin^2 t/2} - 1 \geq 0,$$

that is

$$e^{4 \sin^2 t/2} \leq 2.$$

Since the natural logarithm is increasing we may take logarithms of both sides preserving the direction of inequality. We get this way

$$4 \sin^2 t/2 \leq \ln 2$$

and consequently

$$\left| \frac{t}{2} \right| \leq \sin^{-1} \frac{\sqrt{\ln 2}}{2}.$$

Therefore, the solution  $y(t)$  exists only on the open interval  $]-2 \sin^{-1} \frac{\sqrt{\ln 2}}{2}, 2 \sin^{-1} \frac{\sqrt{\ln 2}}{2}[$ . However, contrary to the previous example, the solution does not blow up at the end-points, but simply vanishes.

Fig 1.2. The graph of the solution in Example 3.5.

In the last example we shall see that sometimes it is sensible to adopt a little different approach to the solutions of a differential equation.

**Example 3.6** Find all solutions of the differential equation

$$\frac{dy}{dt} = -\frac{t}{y}. \quad (1.3.15)$$

Standard approach gives

$$\int \frac{d}{dt} \left( \frac{1}{2} y^2(t) \right) dt = - \int t dt$$

which gives

$$y^2 + t^2 = c^2. \quad (1.3.16)$$

The curves described by equation (1.3.16) are closed (they are, in fact, circles), thus we don't have single valued solutions. However, if (1.3.15) describes a motion of a point in  $(t, y)$ -plane, then we can interpret (1.3.16) as traces of this motion and therefore solution in such an implicit form has a physical sense.

Thus, it is not always necessary or desirable to look for the solution in functional form  $y = y(t)$  as, depending on the problem, the solution in the implicit form  $F(y, t) = c$  may be the proper one. In such a case curves  $F(y, t) = c$  are called *solution curves* of the equation.

## 4 Population models

In this section we will study first order differential equations which appear in the population growth theory. At first glance it appears that it is impossible to model the growth of species by differential equations since the population of any species always change by integer amounts. Hence the population of any species can never be a differentiable function of time. However, if the population is large and it increases by one, then the change is very small compared to a given population. Thus we make the approximation that large populations changes continuously (and even differentiable) in time.

Let  $p(t)$  denote the population of a given isolated species at time  $t$  and let  $\Delta t$  be a small time interval. Then the population at time  $t + \Delta t$  can be expressed as

$$p(t + \Delta t) - p(t) = \text{number of births in } \Delta t - \text{number of deaths in } \Delta t.$$

It is reasonable to assume that the number of births and deaths in a short time interval is proportional to the population at the beginning of this interval and proportional to the length of this interval. Taking  $a(t, p)$  to be the difference between the birth and death rate coefficients we obtain

$$p(t + \Delta t) - p(t) = a(t, p(t))\Delta t p(t).$$

Dividing by  $\Delta t$  and passing with  $\Delta t \rightarrow 0$  gives the equation

$$\frac{dp}{dt} = a(t, p)p. \quad (1.4.1)$$

Because of the unknown coefficient  $a(t, p)$ , depending on the unknown function  $p$ , this equation is impossible to solve. The form of  $a$  has to be deduced by other means.

The simplest possible  $a(t, p)$  is constant and in fact such a model is used in short-term population forecasting. So let us assume that  $a(t, p(t)) = a$  and solve the resulting equation which is both homogeneous linear and separable. Thus, from

$$p' = ap$$

we obtain

$$p(t) = p(t_0)e^{a(t-t_0)}. \quad (1.4.2)$$

To be able to give some numerical illustration to this equation we need the coefficient  $a$  and the population at some time  $t_0$ . We use the data of the U.S. Department of Commerce: it was estimated that the Earth population in 1965 was 3.34 billion and that the population was increasing at an average rate of 2% per year during the decade 1960-1970. Thus  $p(t_0) = p(1965) = 3.34 \times 10^9$  and  $a = 0.02$  and (1.4.2) takes the form

$$p(t) = 3.34 \times 10^9 e^{0.02(t-1965)}.$$

To test the accuracy of this formula let us calculate when the population of the Earth is expected to double. To do this we solve the equation

$$p(T + t_0) = 2p(t_0) = p(t_0)e^{0.02T},$$

thus

$$2 = e^{0.02T}$$

and

$$T = 50 \ln 2 \approx 34.6 \text{ years}.$$

This is an excellent agreement with the present observed value of the Earth population and also gives a good agreement with the observed data if we don't go too far into the past (see Fig.1.3).

*Fig 1.3. Comparison of actual population figures (points) with those obtained from equation (3.5).*

On the other hand, if we try to extrapolate this model into a distant future, then we see that, say, in the year 2515, the population will reach  $199980 \approx 200000$  billion. To realize what it means, let us recall that the Earth total surface area 167400 billion square meters, 80% of which is covered by water, thus we have only 3380 billion square meters to our disposal and there will be only  $0.16m^2$  ( $40cm \times 40cm$ ) per person.

Therefore we can only hope that this model is not valid for all times. Indeed, it is observed that the linear model for the population growth is satisfactory as long as the population is not too large. When the population gets very large (with regard to its habitat), these models cannot be very accurate, since they don't reflect the fact that the individual members have to compete with each other for the limited living space, resources and food available. It is reasonable that a given habitat can sustain only a finite number  $N$  of individuals, and the closer the population is to this number, the slower is its growth. The simplest way to take this into account is to take  $a(t, p) = r(N - p)$  and then we obtain the so-called *logistic model*

$$\frac{dp}{dt} = r(N - p)p, \quad (1.4.3)$$

which proved to be one of the most successful models for describing a single species population.

Equation (1.4.3) is no longer linear, but it is separable, thus we can solve it explicitly. Let us consider the initial value problem

$$\begin{aligned} \frac{dp}{dt} &= r(N - p)p, \\ p(t_0) &= p_0 \end{aligned} \quad (1.4.4)$$

Separating variables and integrating we obtain

$$\frac{1}{r} \int_{p_0}^p \frac{ds}{(N - s)s} = t - t_0.$$

To integrate the left-hand side we use partial fractions

$$\frac{1}{(N - s)s} = \frac{1}{N} \left( \frac{1}{s} + \frac{1}{N - s} \right)$$

which gives

$$\begin{aligned} \frac{1}{r} \int_{p_0}^p \frac{ds}{(N - s)s} &= \frac{1}{rN} \int_{t_0}^t \left( \frac{1}{s} + \frac{1}{N - s} \right) ds \\ &= \frac{1}{rN} \ln \frac{p}{p_0} \left| \frac{N - p_0}{N - p} \right|. \end{aligned}$$

From the above equation we see that  $p(t)$  cannot reach  $N$  in any finite time, so if  $p_0 < N$ , then  $p(t) < N$  for any  $t$ , and if  $p_0 > N$ , then  $p(t) > N$  for all  $t > 0$  (note that if  $p_0 = N$ , then  $p(t) = N$  for all  $t$  – this follows from Picard's theorem). Therefore  $(N - p_0)/(N - p(t))$  is always positive and

$$rN(t - t_0) = \ln \frac{p}{p_0} \frac{N - p_0}{N - p}.$$

Exponentiating, we get

$$e^{rN(t-t_0)} = \frac{p(t)}{p_0} \frac{N - p_0}{N - p(t)}$$

or

$$p_0(N - p(t))e^{rN(t-t_0)} = p(t)(N - p_0).$$

Bringing all the terms involving  $p$  to the left-hand side and multiplying by  $-1$  we get

$$p(t) \left( p_0 e^{rN(t-t_0)} + N - p_0 \right) = p_0 N e^{rN(t-t_0)},$$

thus finally

$$p(t) = \frac{p_0 N}{p_0 + (N - p_0)e^{-rN(t-t_0)}}. \quad (1.4.5)$$

Let us examine (1.4.5) to see what kind of population behaviour it predicts. First observe that we have

$$\lim_{t \rightarrow \infty} p(t) = N,$$

hence our model correctly reflects the initial assumption that  $N$  is the maximal capacity of the habitat. Next, we obtain

$$\frac{dp}{dt} = \frac{rp_0N^2(N - p_0)e^{-rN(t-t_0)}}{(p_0 + (N - p_0)e^{-rN(t-t_0)})^2}$$

thus, if  $p_0 < N$ , the population monotonically increases, whereas if we start with the population which is larger than the capacity of the habitat, then such a population will decrease until it reaches  $N$ . Also

$$\frac{d^2p}{dt^2} = r \frac{d}{dt}(p(N - p)) = p'(N - 2p) = p(N - p)(N - 2p)$$

from which it follows that, if we start from  $p_0 < N$ , then the population curve is convex down for  $p < N/2$  and convex up for  $p > N/2$ . Thus, as long as the population is small (less than half of the capacity), then the rate of growth increases, whereas for larger population the rate of growth decreases. This results in the famous *logistic* or *S-shaped* curve which is presented below for particular values of parameters  $r = 0.02$ ,  $N = 10$  and  $t_0 = 0$  resulting in the following function:

$$p(t) = \frac{10p_0}{p_0 + (10 - p_0)e^{-0.2t}}.$$

*Fig 1.4. Logistic curves with  $p_0 < N$  (dashed line) and  $p_0 > N$  (solid line) for  $N = 10$  and  $r = 0.02$ .*

To show how this curve compares with the real data and with the exponential growth we take the experimental coefficients  $N = 10.76$  billion and  $r = 2.695 \times 10^{-12}$ . Then the logistic equation for the growth of the Earth population will read

$$p(t) = \frac{p_0(10.76 \times 10^9)}{p_0 + ((10.76 \times 10^9) - p_0)e^{-0.029(t-t_0)}}.$$

We use this function with the value  $p_0 = 3.34 \times 10^9$  at  $t_0 = 1965$ . The comparison, analogous to that given on Fig. 1.3, is shown below.

*Fig 1.5. Human population on Earth. Comparison of observational data (points), exponential growth (solid line) and logistic growth (dashed line).*

## 4.1 Exponential growth and money matters

In this subsection we shall briefly show that the exponential growth model (1.4.2) offers a quick approximation to the calculations related to compounded interest problem.

If we deposit the amount of  $N$  rands at a bank, the bank will pay us the interest on this deposit so that the amount will increase over time. There are two parameters related to this process: the annual interest  $r$  and how often the interest is added (compounded) to our account. The more frequent the interest is compounded, the better, as the money earned as the interest start working as the capital. To illustrate this, consider two banks having the same annual interest rate of 15%. The first bank compounds the interest yearly, the other daily. Suppose that we deposited the amount of R10 000 at both banks. After one year the amount at the first bank is

$$R10000 + 0.15 \times R10000 = R11500.$$

How to calculate the amount of money in the second bank? Since the interest is added daily, if at the beginning of day  $n$  we had  $N_n$  rands, then at dawn of the day  $n + 1$  we will have  $N_{n+1} = (1 + 0.15/365)N_n$  rands. This is however, the definition of geometric progression, thus

$$N_n = 10000(1 + 0.15/365)^n$$

and

$$N_{365} = 11618.$$

It is now easy to see that in general if the interest is compounded  $k$  times per year, then after  $t$  years the amount of money will be

$$N(t) = (1 + r/k)^{kt} = \left( (1 + r/k)^{k/r} \right)^{rt}.$$

From Calculus we know that

$$\lim_{k \rightarrow \infty} (1 + r/k)^{k/r} = e,$$



and the sequence is monotonically increasing. Thus, if the interest is compounded very often (almost continuously), then we can approximate

$$N(t) = e^{rt}$$

where  $r$  is the interest rate per annum and  $t$  is time in years. Typically, the exponential can be calculated even on a simple calculator. To compare, if the interest were compounded continuously, then after 1 year we would have

$$N(1) = 10000e^{0.15} = 11618.3$$

so that the difference between the exact formula and the exponential approximation is negligible.



# Lecture 2

## Repetition of some necessary facts from Calculus

In this lecture we shall recall basic calculus facts which will be needed in the course of differential equations.

### 1 Basic facts from the theory of integration

In the course we shall frequently need several facts from the integration theory. We list them here as theorems, though we shall not prove them during the lecture. Some easier proofs should be done as exercises.

**Theorem 1.1** *Let  $f$  be a continuous function in  $\bar{\Omega}$ , where  $\Omega \subset \mathbb{R}^d$  is a bounded domain. Assume that  $\forall \mathbf{x} \in \bar{\Omega} \ f(\mathbf{x}) \geq 0$  and  $\int_{\Omega} f(\mathbf{x}) d\mathbf{x} = 0$ . Then  $f \equiv 0$  in  $\bar{\Omega}$ .*

Another "vanishing function" theorem reads as follows.

**Theorem 1.2** *Let  $f$  be a continuous function in a domain  $\Omega$  such that  $\int_{\Omega_0} f(\mathbf{x}) d\mathbf{x} = 0$  for any  $\Omega_0 \subset \Omega$ . Then  $f \equiv 0$  in  $\Omega$ .*

In the proofs of both theorems the crucial role is played by the theorem on local sign preservation by continuous functions: if  $f$  is continuous at  $\mathbf{x}_0$  and  $f(\mathbf{x}_0) > 0$ , then there exists a ball centered at  $\mathbf{x}_0$  such  $f(\mathbf{x}) > 0$  for  $\mathbf{x}$  from this ball.

Next we shall consider perhaps the most important theorem of multidimensional integral calculus which is Green's-Gauss'-Divergence-.... Theorem. Before entering into details, however, we shall devote some time to the geometry of domains in space.

In one-dimensional space  $\mathbb{R}^1$  typical sets with which we will be concerned are open intervals  $]a, b[$ , where  $-\infty \leq a < b \leq +\infty$ . For  $-\infty < a < b < +\infty$ , by  $[a, b]$  we will denote the closed interval with endpoints  $a, b$ . In this case, we say that  $]a, b[$  is the interior of the interval,  $[a, b]$  is its closure and the two-point set consisting of  $\{a\}$  and  $\{b\}$  constitutes the boundary.

In general, for a set  $\Omega$ , we denote by  $\partial\Omega$  its boundary.

The situation in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  is much more complicated. Let us consider first the two-dimensional situation. Then, in most cases, the boundary  $\partial\Omega$  of a two-dimensional region  $\Omega$  is a closed curve. The two most used analytic descriptions of curves in  $\mathbb{R}^2$  are:

- a) as a level curve of a function of two variables

$$F(x_1, x_2) = c,$$

b) using two functions of a single variable

$$\begin{aligned}x_1(t) &= f(t), \\x_2(t) &= g(t),\end{aligned}$$

where  $t \in [t_0, t_1]$  (parametric description). Note that since the curve is to be closed, we must have  $f(t_0) = f(t_1)$  and  $g(t_0) = g(t_1)$ .

In many cases the boundary is composed of a number of arcs so that it is impossible to give a single analytical description applicable to the whole boundary.

**Example 1.1** Let us consider the elliptical region  $x_1^2/a^2 + x_2^2/b^2 \leq 1$ . The boundary is then the ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

This is the description of the curve (ellipse) as a level curve of the function  $F(x_1, x_2) = x_1^2/a^2 + x_2^2/b^2$  (with the constant  $c = 1$ ). Equivalently, the boundary can be written in parametric form as

$$x_1(t) = a \cos t, \quad x_2(t) = b \sin t,$$

with  $t \in [0, 2\pi]$ . ■

In three dimensions the boundary of a solid  $\Omega$  is a two-dimensional surface. This surface can be analytically described as a level surface of a function of three variables

$$F(x_1, x_2, x_3) = c,$$

or parametrically by, this time, three functions of two variables each:

$$\begin{aligned}x_1(t, s) &= f(t, s), \\x_2(t, s) &= g(t, s), \\x_3(t, s) &= h(t, s), \quad t \in [t_0, t_1], \quad s \in [s_0, s_1].\end{aligned}$$

As in two dimensions, it is possible that the boundary is made up of several patches having different analytic descriptions.

**Example 1.2** Consider the domain  $\Omega$  bounded by the ellipsoid

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1.$$

The boundary here is directly given as the level surface of the function  $F(x_1, x_2, x_3) = x_1^2/a^2 + x_2^2/b^2 + x_3^2/c^2$ :

$$F(x_1, x_2, x_3) = 1.$$

The same boundary can be described parametrically as  $\mathbf{r}(t, s) = (f(t, s), g(t, s), h(t, s))$ , where

$$\begin{aligned}f(t, s) &= a \cos t \sin s \\g(t, s) &= b \sin t \sin s \\h(t, s) &= c \cos s\end{aligned}$$

where  $t \in [0, 2\pi], s \in [0, \pi]$ . ■

One of the most important concepts in partial differential equations is that of the *unit outward normal vector* to the boundary of the set. For a given point  $\mathbf{p} \in \partial\Omega$  this is the vector  $\mathbf{n}$ , normal (perpendicular) to the boundary at  $p$ , pointing outside  $\Omega$ , and having unit length.

If the boundary of (two or three dimensional) set  $\Omega$  is given as a level curve of a function  $F$ , then the vector given by

$$\mathbf{N}(\mathbf{p}) = \nabla F|_{\mathbf{p}},$$

is normal to the boundary at  $\mathbf{p}$ . However, it is not necessarily unit, nor outward. To make it a unit vector, we divide  $\mathbf{N}$  by its length; then the unit outward normal is either  $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$ , or  $\mathbf{n} = -\mathbf{N}/\|\mathbf{N}\|$  and the proper sign must be selected by inspection.

*Fig 2.1. Domain  $\Omega$  with boundary  $\partial\Omega$  showing a surface element  $dS$  with outward normal  $\mathbf{n}$  and flux vector  $\phi$ .*

**Example 1.3** Find the unit outward normal to the ellipsoid

$$x_1^2 + \frac{x_2^2}{4} + \frac{x_3^2}{9} = 1,$$

at the point  $\mathbf{p} = (1/\sqrt{2}, 0, 3/\sqrt{2})$ .

Since  $F(x_1, x_2, x_3) = x_1^2 + \frac{x_2^2}{4} + \frac{x_3^2}{9}$ , we obtain  $\nabla F = (2x_1, x_2/2, 2x_3/9)$ . Therefore

$$\mathbf{N}(\mathbf{p}) = \nabla F|_{\mathbf{p}} = (2/\sqrt{2}, 0, 2/3\sqrt{2}).$$

Furthermore

$$\|\mathbf{N}(\mathbf{p})\| = \sqrt{2 + 2/9} = \sqrt{20/9} = 2\sqrt{5}/3,$$

thus

$$\mathbf{n}(\mathbf{p}) = \pm(3/\sqrt{10}, 0, 1/\sqrt{10}).$$

To select the proper sign let us observe that the vector pointing outside the ellipsoid must necessarily point away from the origin. Since the coordinates of the point  $\mathbf{p}$  are nonnegative, a vector rooted at this point and pointing away from the origin must have positive coordinates, thus finally

$$\mathbf{n}(\mathbf{p}) = (3/\sqrt{10}, 0, 1/\sqrt{10}).$$

■

If the boundary is given in a parametric way, then the situation is more complicated and we have to distinguish between dimensions 2 and 3.

Let us first consider the boundary  $\partial\Omega$  of a two-dimensional domain  $\Omega$ , described by  $\mathbf{r}(t) = (x_1(t), x_2(t)) = (f(t), g(t))$ . It is known that the derivative vector

$$\frac{d}{dt}\mathbf{r}(t_p) = (f'(t_p), g'(t_p))$$

is tangent to  $\partial\Omega$  at  $\mathbf{p} = (f(t_p), g(t_p))$ . Since any normal vector at  $\mathbf{p} \in \Omega$  is perpendicular to any tangent vector at this point, we immediately obtain that

$$\mathbf{N}(\mathbf{p}) = (-g'(t_p), f'(t_p)). \quad (2.1.6)$$

Therefore the unit outward normal is given by

$$\mathbf{n}(\mathbf{p}) = \pm \frac{\mathbf{N}(\mathbf{p})}{\|\mathbf{N}(\mathbf{p})\|},$$

where the sign must be decided by inspection so that  $\mathbf{n}$  points outside  $\Omega$ .

If the domain  $\Omega$  is 3-dimensional, then its boundary  $\partial\Omega$  is a surface described by the 3-dimensional vector function of 2 variables:

$$\mathbf{r}(t, s) = (x_1(t, s), x_2(t, s), x_3(t, s)) = (f(t, s), g(t, s), h(t, s)).$$

In this case, at each point  $\partial\Omega \ni \mathbf{p} = \mathbf{r}(t_p)$ , we have two derivative vectors  $\mathbf{r}'_t(t_p)$  and  $\mathbf{r}'_s(t_p)$  which span the two dimensional tangent plane to  $\partial\Omega$  at  $\mathbf{p}$ . Any normal vector must be thus perpendicular to both these vectors and the easiest way to find such a vector is to use the cross-product:

$$\mathbf{N}(\mathbf{p}) = \mathbf{r}'_t(t_p) \times \mathbf{r}'_s(t_p).$$

Again, the unit outward normal is given by

$$\mathbf{n}(\mathbf{p}) = \pm \frac{\mathbf{N}(\mathbf{p})}{\|\mathbf{N}(\mathbf{p})\|} = \pm \frac{\mathbf{r}'_t(t_p) \times \mathbf{r}'_s(t_p)}{\|\mathbf{r}'_t(t_p) \times \mathbf{r}'_s(t_p)\|},$$

where the sign must be decided by inspection.

**Example 1.4** Find the outward unit normal to the ellipse  $\mathbf{r}(t) = (\cos t, 2 \sin t)$  at the point  $\mathbf{p} = \mathbf{r}(\pi/4)$ . Differentiating, we obtain  $\mathbf{r}'_t = (-\sin t, 2 \cos t)$ ; this is a tangent vector to ellipse at  $\mathbf{r}(t)$ . Thus,

$$\mathbf{N} = (-2 \cos t, -\sin t).$$

Next,  $\|\mathbf{N}\| = \sqrt{4 \cos^2 t + \sin^2 t}$  and

$$\mathbf{n} = \pm \frac{1}{\sqrt{4 \cos^2 t + \sin^2 t}} (2 \cos t, \sin t).$$

At the particular point  $\mathbf{p}$  we have  $\cos \pi/4 = \sin \pi/4 = \frac{\sqrt{2}}{2}$ , thus  $\|\mathbf{N}\| = \sqrt{5/2}$  and

$$\mathbf{n}(\mathbf{p}) = \pm 2/\sqrt{5} (1, 1/2).$$

Since the normal must point outside the ellipse, we must chose the "+" sign and finally

$$\mathbf{n}(\mathbf{p}) = 2/\sqrt{5} (1, 1/2).$$

■

Another important concept related to the normal is the *normal derivative* of a function. Let us recall that if  $\mathbf{u}$  is any unit vector and  $f$  is a function, then the derivative of  $f$  at a point  $\mathbf{p}$  in the direction of  $\mathbf{u}$  is defined as

$$f_{\mathbf{u}}(\mathbf{p}) = \lim_{t \rightarrow 0^+} \frac{f(\mathbf{p} + t\mathbf{u}) - f(\mathbf{p})}{t}.$$

Application of the Chain Rule produces the following handy formula for the directional derivative:

$$f_{\mathbf{u}}(\mathbf{p}) = \nabla f|_{\mathbf{p}} \cdot \mathbf{u}.$$

Let now  $f$  be defined in a neighbourhood of a point  $\mathbf{p} \in \partial\Omega$ . The *normal derivative* of  $f$  at  $\mathbf{p}$  is defined as the derivative of  $f$  in the direction of  $\mathbf{n}(\mathbf{p})$ :

$$\frac{\partial f}{\partial n}(\mathbf{p}) = f_{\mathbf{n}}(\mathbf{p}) = \nabla f|_{\mathbf{p}} \cdot \mathbf{n}(\mathbf{p}).$$

**Example 1.5** Let us consider the spherical coordinates

$$\begin{aligned} x_1 &= r \cos \theta \sin \phi, \\ x_2 &= r \sin \theta \sin \phi, \\ x_3 &= r \cos \phi \end{aligned}$$

and let  $f(x_1, x_2, x_3)$  be a function of three variables. This function can be expressed in the spherical coordinates as the function of  $(r, \theta, \phi)$

$$F(r, \theta, \phi) = f(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi) = f(x_1, x_2, x_3).$$

Using the Chain Rule we have

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial r} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial r} + \frac{\partial f}{\partial x_3} \frac{\partial x_3}{\partial r}.$$

Since, for  $i = 1, 2, 3$ ,  $\partial x_i / \partial r = x_i / r$ , we can write

$$\frac{\partial F}{\partial r} = \frac{1}{r} \nabla f \cdot \mathbf{r}. \quad (2.1.7)$$

Assume now that  $f$  (and thus  $F$ ) be given in some neighbourhood of the sphere

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 = R^2.$$

To find the outward unit normal to this sphere we note that  $\nabla F = (2x_1, 2x_2, 2x_3)$  and  $\|\nabla F\| = 2\sqrt{x_1^2 + x_2^2 + x_3^2} = 2R$ . Thus,

$$\mathbf{n} = \frac{1}{R}(x_1, x_2, x_3) = \frac{1}{R}\mathbf{r}. \quad (2.1.8)$$

Geometrically,  $\mathbf{n}$  is parallel to the radius of the sphere but has unit length.

Combining (1.1.15) with (1.1.16), we see that the normal derivative of  $f$  at any point of the sphere is given by

$$\frac{\partial f}{\partial n} = \nabla f \cdot \mathbf{n} = \frac{\partial F}{\partial r}.$$

In other words, the normal derivative of any function at the surface of a sphere is equal to the derivative of this function (expressed in spherical coordinates) with respect to  $r$ . ■

Next we shall discuss yet another important concept: the flux of a vector field. Let us recall that a vector field is a function  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^d$ , where  $\Omega \subset \mathbb{R}^d$ , where  $d = 1, 2, 3, \dots$ . In other words, a vector field assigns a vector to each point of a subset of the space.

**Definition 1.1** The flux of the vector field  $\mathbf{f}$  across the boundary  $\partial\Omega$  of a domain  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$  is

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} d\sigma.$$

Here, if  $d = 2$ , then  $\partial\Omega$  is a closed curve and the integral above is the line integral (of the second kind). The arc length element  $d\sigma$  is to be calculated according to the description of  $\partial\Omega$ . The easiest situation occurs if  $\partial\Omega$  is described in a parametric form by  $\mathbf{r}(t) = (f(t), g(t))$ ,  $t \in [t_0, t_1]$ ; then  $d\sigma = \sqrt{(f')^2 + (g')^2} dt$  and

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} d\sigma = \int_{t_0}^{t_1} \mathbf{f} \cdot \mathbf{n}(t) \sqrt{(f')^2(t) + (g')^2(t)} dt.$$

When  $d = 3$ , then  $\partial\Omega$  is a closed surface and the integral above is the surface integral (of the second kind). The surface element  $d\sigma$  is again the easiest to calculate if  $\partial\Omega$  is given in a parametric form  $\mathbf{r}(t, s) = (f(t, s), g(t, s), h(t, s))$ ,  $t \in [t_0, t_1]$ ,  $s \in [s_0, s_1]$ . Then  $d\sigma = |\mathbf{r}_t \times \mathbf{r}_s| dt ds$  and

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} d\sigma = \int_{t_0}^{t_1} \int_{s_0}^{s_1} \mathbf{f} \cdot \mathbf{n}(t, s) |\mathbf{r}_t \times \mathbf{r}_s| ds dt.$$

**Remark 1.1** With a little imagination the definition of the flux can be used also in one dimensional case. To do this, we have to understand that the integration is something like a summation of the integrand over all the points of the boundary. In one-dimensional case we have  $\Omega = [a, b]$  with  $\partial\Omega = \{a\} \cup \{b\}$ . A vector field in one-dimension is just a scalar function. The unit outward normal at  $\{a\}$  is  $-1$ , and at  $\{b\}$  is  $1$ . Thus  $\mathbf{f}\mathbf{n}(a) = f(a)(-1)$  and  $\mathbf{f}\mathbf{n}(b) = f(b)(1)$  and the flux across the boundary of  $\Omega$  is

$$\mathbf{f} \cdot \mathbf{n}(a) + \mathbf{f} \cdot \mathbf{n}(b) = f(b) - f(a). \quad (2.1.9)$$

■

**Example 1.6** To understand the meaning of the flux let us consider a fluid moving in a certain domain of space. The standard way of describing the motion of the fluid is to associate with any point  $\mathbf{p}$  of the domain filled by the fluid its velocity  $\mathbf{v}(\mathbf{p})$ . In this way we have the *velocity field* of the fluid.

Consider first the one-dimensional case (one can think about a thin pipe). If at a certain point  $x$  we have  $v(x) > 0$ , then the fluid flows to the right, and if  $v(x) < 0$ , then it flows to the left. Let the points  $x = a$  and  $x = b$  be the end-points of a section of the pipe and consider the new field  $f(x) = \rho(x)v(x)$ , where  $\rho$  is the (linear) density of the fluid in point  $x$ . The flux of  $f$ , as defined by (1.1.17), is

$$f(b) - f(a) = \rho(b)v(b) - \rho(a)v(a).$$

For instance, if both  $v(b)$  and  $v(a)$  are positive, then at  $x = b$  the fluid leaves the pipe with velocity  $v(b)$  and at  $x = a$  it enters the pipe with velocity  $v(a)$ . In a small interval of time  $\Delta t$  mass of fluid which left the segment through  $x = b$  is equal  $\rho(b)v(b)\Delta t$  and the mass which entered the segment through  $x = a$  is  $\rho(a)v(a)\Delta t$ , thus, the net rate at which the mass leaves (enters) the segment is equal to  $\rho(b)v(b) - \rho(a)v(a)$ , that is, exactly to the flux of the field  $f$  across the boundary.

This idea can be generalized to more realistic case of three dimensional space. Let us consider a fluid with possibly variable density  $\rho(\mathbf{p})$  filling a portion of space and moving with velocity  $\mathbf{v}(\mathbf{p})$ . We define the mass-velocity field  $\mathbf{f} = \rho\mathbf{v}$ . Let us consider now the domain  $\Omega$  with the boundary  $\Omega$ . Imagine a small portion  $\Delta S$  of  $\partial\Omega$ , which could be considered flat, having the area  $\Delta\sigma$ . Let  $\mathbf{n}$  be the unit normal to this surface and consider the rate at which the fluid crosses  $\Delta S$ . We decompose the velocity  $\mathbf{v}$  into the component parallel to  $\mathbf{n}$ , given by  $(\mathbf{v} \cdot \mathbf{n})\mathbf{n}$  and the tangential component. It is clear that the crossing of the surface can be only due to the normal component (if in time  $\Delta t$  you make two steps perpendicular to the boundary and



two parallel, then you will find yourself only two steps from the boundary despite having done four steps). Therefore the mass of fluid crossing  $\Delta S$  in time  $\Delta t$  is given by

$$\Delta m = \rho \cdot (\mathbf{v} \cdot \mathbf{n}) \Delta t \Delta \sigma.$$

*Fig 2.2. The fluid that flows through the patch  $\Delta S$  in a short time  $\Delta t$  fills a slanted cylinder whose volume is approximately base  $\times$  height  $= \mathbf{v} \cdot \mathbf{n} \Delta \sigma \Delta t$ . The mass of the fluid in the cylinder is then  $\rho \cdot (\mathbf{v} \cdot \mathbf{n}) \Delta t \Delta \sigma$ .*

Thus, the rate at which the fluid is crossing the whole boundary  $\partial\Omega$  (that is, the net rate at which the fluid is filling/leaving the domain  $\Omega$ ) can be approximated by summing up all the contributions  $\Delta m$  over all patches  $\Delta S$  of the boundary which, in the limit as  $\Delta S$  go to zero, is nothing but the flux of  $\mathbf{f}$ :

$$\int_{\partial\Omega} (\rho \mathbf{v} \cdot \mathbf{n}) d\sigma.$$

Thus again we have the identity

Flux of  $\rho \mathbf{v}$  across  $\partial\Omega$  = the net rate at which the mass of fluid is leaving  $\Omega$ .

■

Let us return now to the Green-Gauss.... Theorem. This is a theorem which relates the behaviour of a vector field on the boundary of a domain (flux) with what is happening inside the domain.

Suppose that somewhere inside the domain there is a source of the field (fluid). How can we check this? We can construct a small box around the source and measure whether there is a net outflow, inflow or whether the outflow balances the inflow. If we make this box really small, then we can be quite sure that in the first case there is a source, in the second there is a sink, and in the third case that there is neither sink nor source.

Let us then put this idea into mathematical formulae. Assume that the box  $B$  with one vertex at  $(x_1^0, x_2^0, x_3^0)$  has the edges parallel to the axes of the coordinate system and of lengths  $\Delta x_1$ ,  $\Delta x_2$  and  $\Delta x_3$ .

We calculate the net rate of flow of the vector field  $\mathbf{f} = (f_1, f_2, f_3)$  from the box. Following the calculations given earlier, the flow through the top side is given by

$$\mathbf{f} \cdot \mathbf{n}(x_1^0, x_2^0, x_3^0 + \Delta x_3) \Delta x_1 \Delta x_2 = \mathbf{f} \cdot \mathbf{k}(x_1^0, x_2^0, x_3^0 + \Delta x_3) \Delta x_1 \Delta x_2 = f_3(x_1^0, x_2^0, x_3^0 + \Delta x_3) \Delta x_1 \Delta x_2,$$

and through the bottom

$$\mathbf{f} \cdot \mathbf{n}(x_1^0, x_2^0, x_3^0) \Delta x_1 \Delta x_2 = \mathbf{f} \cdot (-\mathbf{k})(x_1^0, x_2^0, x_3^0) \Delta x_1 \Delta x_2 = -f_3(x_1^0, x_2^0, x_3^0) \Delta x_1 \Delta x_2,$$

thus the net flow through the horizontal sides is

$$(f_3(x_1^0, x_2^0, x_3^0 + \Delta x_3) - f_3(x_1^0, x_2^0, x_3^0)) \Delta x_1 \Delta x_2 \approx \frac{\partial f_3}{\partial x_3}(x_1^0, x_2^0, x_3^0) \Delta x_1 \Delta x_2 \Delta x_3.$$

Similar calculations can be done for the two remaining pairs of the sides and the total flow from the box can be approximated by

$$\left( \sum_{i=1}^3 \frac{\partial f_i}{\partial x_i}(x_1^0, x_2^0, x_3^0) \right) \Delta x_1 \Delta x_2 \Delta x_3.$$

This expression can be considered to be the net rate of the production of the field (fluid) in the box  $B$ . The expression

$$\operatorname{div} \mathbf{f} = \sum_{i=1}^3 \frac{\partial f_i}{\partial x_i}$$

is called the divergence of the vector field  $\mathbf{f}$  and can be considered to be the rate of the production per unit volume (density). To obtain the total net rate of the production in the domain  $\Omega$  we have to add up contributions coming from all the (small) boxes. Thus using the definition of the integral we obtain that the total net rate of the production is given by

$$\iiint_{\Omega} \operatorname{div} \mathbf{f}(x_1, x_2, x_3) dv.$$

Using some common sense reasoning it is easy to arrive at the identity

$$\text{The net rate of production in } \Omega = \text{The net flow across the boundary}$$

The Green–Gauss–... Theorem is the mathematical expression of the above law.

**Theorem 1.3** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$ ,  $d \geq 1$ , with a piecewise  $C^1$  boundary  $\partial\Omega$ . Let  $\mathbf{n}$  be the unit outward normal vector on  $\partial\Omega$ . Let  $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))$  be any  $C^1$  vector field on  $\bar{\Omega} = \Omega \cup \partial\Omega$ . Then*

$$\int_{\Omega} \operatorname{div} \mathbf{f} d\mathbf{x} = \oint_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} d\sigma, \quad (2.1.10)$$

where  $d\sigma$  is the element of surface of  $\partial\Omega$ .

**Remark 1.2** In one dimension this theorem is nothing but the fundamental theorem of calculus:

$$\int_a^b \frac{df}{dx}(x) dx = f(b) - f(a). \quad (2.1.11)$$

In fact, for a function of one variable  $\operatorname{div} \mathbf{f} = \frac{df}{dx}$  and the right-hand side of this equation represents the outward flux across the boundary of  $\Omega = [a, b]$ , as discussed in Remark 1.1.

**Remark 1.3** In two dimensions the most popular form of this theorem is known the Green Theorem which apparently differs from the one given above. To explain this, let the boundary  $\partial\Omega$  of  $\Omega$  be a curve given by parametric equation  $\mathbf{r}(t) = (x_1(t), x_2(t))$ ,  $t \in [t_0, t_1]$  and suppose that if  $t$  runs from  $t_0$  to  $t_1$ , then  $\mathbf{r}(t)$  traces  $\partial\Omega$  in the positive (anticlockwise) direction. Then it is easy to check that the unit outward normal vector  $\mathbf{n}$  is given by

$$\mathbf{n}(t) = \frac{1}{\mathbf{r}'(t)}(x_2'(t), -x_1'(t)),$$

where  $\mathbf{r}'(t) = (x_1'(t), x_2'(t))$ . Thus, if  $\mathbf{f} = (f_1, f_2)$ , then Eq. (1.1.18) takes the form

$$\iint_{\Omega} \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = \int_{t_0}^{t_1} (f_1(t)x_2'(t) - f_2(t)x_1'(t)) dt \quad (2.1.12)$$

On the other hand, the standard version of Green's theorem reads

$$\iint_{\Omega} \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 dx_2 = \oint_{\partial\Omega} f_1 dx_1 + f_2 dx_2 = \int_{t_0}^{t_1} (f_1(t)x'_1(t) + f_2(t)x'_2(t)) dt. \quad (2.1.13)$$

To see that these forms are really equivalent, let us define the new vector field  $\mathbf{g} = (g_1, g_2)$  by :  $g_1 = -f_2, g_2 = f_1$ . Then

$$\operatorname{div} \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2}.$$

The boundary of  $\partial\Omega$  is, as above, parameterized by the function  $\mathbf{r}(t)$ . Thus, if we assume that (1.1.21) holds (for an arbitrary vector field), then

$$\begin{aligned} \iint_{\Omega} \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 &= \iint_{\Omega} \left( \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2} \right) dx_1 dx_2 \\ &= \int_{t_0}^{t_1} (g_1(t)x'_1(t) + g_2(t)x'_2(t)) dt = \int_{t_0}^{t_1} (f_1(t)x'_2(t) - f_2(t)x'_1(t)) dt, \end{aligned}$$

which is (1.1.20). The converse is analogous. ■



# Lecture 1

## Repetition of some necessary facts from Calculus

In this lecture we shall recall basic calculus facts which will be needed in the course of differential equations.

### 1 Basic facts from the theory of integration

In the course we shall frequently need several facts from the integration theory. We list them here as theorems, though we shall not prove them during the lecture. Some easier proofs should be done as exercises.

**Theorem 1.1** *Let  $f$  be a continuous function in  $\bar{\Omega}$ , where  $\Omega \subset \mathbb{R}^d$  is a bounded domain. Assume that  $\forall \mathbf{x} \in \bar{\Omega} \ f(\mathbf{x}) \geq 0$  and  $\int_{\Omega} f(\mathbf{x}) d\mathbf{x} = 0$ . Then  $f \equiv 0$  in  $\bar{\Omega}$ .*

Another "vanishing function" theorem reads as follows.

**Theorem 1.2** *Let  $f$  be a continuous function in a domain  $\Omega$  such that  $\int_{\Omega_0} f(\mathbf{x}) d\mathbf{x} = 0$  for any  $\Omega_0 \subset \Omega$ . Then  $f \equiv 0$  in  $\Omega$ .*

In the proofs of both theorems the crucial role is played by the theorem on local sign preservation by continuous functions: if  $f$  is continuous at  $\mathbf{x}_0$  and  $f(\mathbf{x}_0) > 0$ , then there exists a ball centered at  $\mathbf{x}_0$  such  $f(\mathbf{x}) > 0$  for  $\mathbf{x}$  from this ball.

Next we shall consider perhaps the most important theorem of multidimensional integral calculus which is Green's-Gauss'-Divergence-.... Theorem. Before entering into details, however, we shall devote some time to the geometry of domains in space.

In one-dimensional space  $\mathbb{R}^1$  typical sets with which we will be concerned are open intervals  $]a, b[$ , where  $-\infty \leq a < b \leq +\infty$ . For  $-\infty < a < b < +\infty$ , by  $[a, b]$  we will denote the closed interval with endpoints  $a, b$ . In this case, we say that  $]a, b[$  is the interior of the interval,  $[a, b]$  is its closure and the two-point set consisting of  $\{a\}$  and  $\{b\}$  constitutes the boundary.

In general, for a set  $\Omega$ , we denote by  $\partial\Omega$  its boundary.

The situation in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  is much more complicated. Let us consider first the two-dimensional situation. Then, in most cases, the boundary  $\partial\Omega$  of a two-dimensional region  $\Omega$  is a closed curve. The two most used analytic descriptions of curves in  $\mathbb{R}^2$  are:

- a) as a level curve of a function of two variables

$$F(x_1, x_2) = c,$$

b) using two functions of a single variable

$$\begin{aligned}x_1(t) &= f(t), \\x_2(t) &= g(t),\end{aligned}$$

where  $t \in [t_0, t_1]$  (parametric description). Note that since the curve is to be closed, we must have  $f(t_0) = f(t_1)$  and  $g(t_0) = g(t_1)$ .

In many cases the boundary is composed of a number of arcs so that it is impossible to give a single analytical description applicable to the whole boundary.

**Example 1.1** Let us consider the elliptical region  $x_1^2/a^2 + x_2^2/b^2 \leq 1$ . The boundary is then the ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

This is the description of the curve (ellipse) as a level curve of the function  $F(x_1, x_2) = x_1^2/a^2 + x_2^2/b^2$  (with the constant  $c = 1$ ). Equivalently, the boundary can be written in parametric form as

$$x_1(t) = a \cos t, \quad x_2(t) = b \sin t,$$

with  $t \in [0, 2\pi]$ . ■

In three dimensions the boundary of a solid  $\Omega$  is a two-dimensional surface. This surface can be analytically described as a level surface of a function of three variables

$$F(x_1, x_2, x_3) = c,$$

or parametrically by, this time, three functions of two variables each:

$$\begin{aligned}x_1(t, s) &= f(t, s), \\x_2(t, s) &= g(t, s), \\x_3(t, s) &= h(t, s), \quad t \in [t_0, t_1], \quad s \in [s_0, s_1].\end{aligned}$$

As in two dimensions, it is possible that the boundary is made up of several patches having different analytic descriptions.

**Example 1.2** Consider the domain  $\Omega$  bounded by the ellipsoid

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1.$$

The boundary here is directly given as the level surface of the function  $F(x_1, x_2, x_3) = x_1^2/a^2 + x_2^2/b^2 + x_3^2/c^2$ :

$$F(x_1, x_2, x_3) = 1.$$

The same boundary can be described parametrically as  $\mathbf{r}(t, s) = (f(t, s), g(t, s), h(t, s))$ , where

$$\begin{aligned}f(t, s) &= a \cos t \sin s \\g(t, s) &= b \sin t \sin s \\h(t, s) &= c \cos s\end{aligned}$$

where  $t \in [0, 2\pi], s \in [0, \pi]$ . ■

One of the most important concepts in partial differential equations is that of the *unit outward normal vector* to the boundary of the set. For a given point  $\mathbf{p} \in \partial\Omega$  this is the vector  $\mathbf{n}$ , normal (perpendicular) to the boundary at  $p$ , pointing outside  $\Omega$ , and having unit length.

If the boundary of (two or three dimensional) set  $\Omega$  is given as a level curve of a function  $F$ , then the vector given by

$$\mathbf{N}(\mathbf{p}) = \nabla F|_{\mathbf{p}},$$

is normal to the boundary at  $\mathbf{p}$ . However, it is not necessarily unit, nor outward. To make it a unit vector, we divide  $\mathbf{N}$  by its length; then the unit outward normal is either  $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$ , or  $\mathbf{n} = -\mathbf{N}/\|\mathbf{N}\|$  and the proper sign must be selected by inspection.

**Example 1.3** Find the unit outward normal to the ellipsoid

$$x_1^2 + \frac{x_2^2}{4} + \frac{x_3^2}{9} = 1,$$

at the point  $\mathbf{p} = (1/\sqrt{2}, 0, 3/\sqrt{2})$ .

Since  $F(x_1, x_2, x_3) = x_1^2 + \frac{x_2^2}{4} + \frac{x_3^2}{9}$ , we obtain  $\nabla F = (2x_1, x_2/2, 2x_3/9)$ . Therefore

$$\mathbf{N}(\mathbf{p}) = \nabla F|_{\mathbf{p}} = (2/\sqrt{2}, 0, 2/3\sqrt{2}).$$

Furthermore

$$\|\mathbf{N}(\mathbf{p})\| = \sqrt{2 + 2/9} = \sqrt{20/9} = 2\sqrt{5}/3,$$

thus

$$\mathbf{n}(\mathbf{p}) = \pm(3/\sqrt{10}, 0, 1/\sqrt{10}).$$

To select the proper sign let us observe that the vector pointing outside the ellipsoid must necessarily point away from the origin. Since the coordinates of the point  $\mathbf{p}$  are nonnegative, a vector rooted at this point and pointing away from the origin must have positive coordinates, thus finally

$$\mathbf{n}(\mathbf{p}) = (3/\sqrt{10}, 0, 1/\sqrt{10}).$$

■

If the boundary is given in a parametric way, then the situation is more complicated and we have to distinguish between dimensions 2 and 3.

Let us first consider the boundary  $\partial\Omega$  of a two-dimensional domain  $\Omega$ , described by  $\mathbf{r}(t) = (x_1(t), x_2(t)) = (f(t), g(t))$ . It is known that the derivative vector

$$\frac{d}{dt}\mathbf{r}(t_p) = (f'(t_p), g'(t_p))$$

is tangent to  $\partial\Omega$  at  $\mathbf{p} = (f(t_p), g(t_p))$ . Since any normal vector at  $\mathbf{p} \in \Omega$  is perpendicular to any tangent vector at this point, we immediately obtain that

$$\mathbf{N}(\mathbf{p}) = (-g'(t_p), f'(t_p)). \quad (1.1.14)$$

Therefore the unit outward normal is given by

$$\mathbf{n}(\mathbf{p}) = \pm \frac{\mathbf{N}(\mathbf{p})}{\|\mathbf{N}(\mathbf{p})\|},$$

where the sign must be decided by inspection so that  $\mathbf{n}$  points outside  $\Omega$ .

If the domain  $\Omega$  is 3-dimensional, then its boundary  $\partial\Omega$  is a surface described by the 3-dimensional vector function of 2 variables:

$$\mathbf{r}(t, s) = (x_1(t, s), x_2(t, s), x_3(t, s)) = (f(t, s), g(t, s), h(t, s)).$$

In this case, at each point  $\partial\Omega \ni \mathbf{p} = \mathbf{r}(t_p)$ , we have two derivative vectors  $\mathbf{r}'_s(t_p)$  and  $\mathbf{r}'_t(t_p)$  which span the two dimensional tangent plane to  $\partial\Omega$  at  $\mathbf{p}$ . Any normal vector must be thus perpendicular to both these vectors and the easiest way to find such a vector is to use the cross-product:

$$\mathbf{N}(\mathbf{p}) = \mathbf{r}'_t(t_p) \times \mathbf{r}'_s(t_p).$$

Again, the unit outward normal is given by

$$\mathbf{n}(\mathbf{p}) = \pm \frac{\mathbf{N}(\mathbf{p})}{\|\mathbf{N}(\mathbf{p})\|} = \pm \frac{\mathbf{r}'_t(t_p) \times \mathbf{r}'_s(t_p)}{\|\mathbf{r}'_t(t_p) \times \mathbf{r}'_s(t_p)\|},$$

where the sign must be decided by inspection.

**Example 1.4** Find the outward unit normal to the ellipse  $\mathbf{r}(t) = (\cos t, 2 \sin t)$  at the point  $\mathbf{p} = \mathbf{r}(\pi/4)$ . Differentiating, we obtain  $\mathbf{r}'_t = (-\sin t, 2 \cos t)$ ; this is a tangent vector to ellipse at  $\mathbf{r}(t)$ . Thus,

$$\mathbf{N} = (-2 \cos t, -\sin t).$$

Next,  $\|\mathbf{N}\| = \sqrt{4 \cos^2 t + \sin^2 t}$  and

$$\mathbf{n} = \pm \frac{1}{\sqrt{4 \cos^2 t + \sin^2 t}} (2 \cos t, \sin t).$$

At the particular point  $\mathbf{p}$  we have  $\cos \pi/2 = \sin \pi/2 = \frac{\sqrt{2}}{2}$ , thus  $\|\mathbf{N}\| = \sqrt{5/2}$  and

$$\mathbf{n}(\mathbf{p}) = \pm 2/\sqrt{5} (1, 1/2).$$

Since the normal must point outside the ellipse, we must chose the "+" sign and finally

$$\mathbf{n}(\mathbf{p}) = 2/\sqrt{5} (1, 1/2).$$

■

Another important concept related to the normal is the *normal derivative* of a function. Let us recall that if  $\mathbf{u}$  is any unit vector and  $f$  is a function, then the derivative of  $f$  at a point  $\mathbf{p}$  in the direction of  $\mathbf{u}$  is defined as

$$f_{\mathbf{u}}(\mathbf{p}) = \lim_{t \rightarrow 0^+} \frac{f(\mathbf{p} + t\mathbf{u}) - f(\mathbf{p})}{t}.$$

Application of the Chain Rule produces the following handy formula for the directional derivative:

$$f_{\mathbf{u}}(\mathbf{p}) = \nabla f|_{\mathbf{p}} \cdot \mathbf{u}.$$

Let now  $f$  be defined in a neighbourhood of a point  $\mathbf{p} \in \partial\Omega$ . The *normal derivative* of  $f$  at  $\mathbf{p}$  is defined as the derivative of  $f$  in the direction of  $\mathbf{n}(\mathbf{p})$ :

$$\frac{\partial f}{\partial n}(\mathbf{p}) = f_{\mathbf{n}}(\mathbf{p}) = \nabla f|_{\mathbf{p}} \cdot \mathbf{n}(\mathbf{p}).$$

**Example 1.5** Let us consider the spherical coordinates

$$\begin{aligned} x_1 &= r \cos \theta \sin \phi, \\ x_2 &= r \sin \theta \sin \phi, \\ x_3 &= r \cos \phi \end{aligned}$$

and let  $f(x_1, x_2, x_3)$  be a function of three variables. This function can be expressed in the spherical coordinates as the function of  $(r, \theta, \phi)$

$$F(r, \theta, \phi) = f(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi) = f(x_1, x_2, x_3).$$



Using the Chain Rule we have

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial r} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial r} + \frac{\partial f}{\partial x_3} \frac{\partial x_3}{\partial r}.$$

Since, for  $i = 1, 2, 3$ ,  $\partial x_i / \partial r = x_i / r$ , we can write

$$\frac{\partial F}{\partial r} = \frac{1}{r} \nabla f \cdot \mathbf{r}. \quad (1.1.15)$$

Assume now that  $f$  (and thus  $F$ ) be given in some neighbourhood of the sphere

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 = R^2.$$

To find the outward unit normal to this sphere we note that  $\nabla F = (2x_1, 2x_2, 2x_3)$  and  $\|\nabla F\| = 2\sqrt{x_1^2 + x_2^2 + x_3^2} = 2R$ . Thus,

$$\mathbf{n} = \frac{1}{R}(x_1, x_2, x_3) = \frac{1}{R}\mathbf{r}. \quad (1.1.16)$$

Geometrically,  $\mathbf{n}$  is parallel to the radius of the sphere but has unit length.

Combining (1.1.15) with (1.1.16), we see that the normal derivative of  $f$  at any point of the sphere is given by

$$\frac{\partial f}{\partial n} = \nabla f \cdot \mathbf{n} = \frac{\partial F}{\partial r}.$$

In other words, the normal derivative of any function at the surface of a sphere is equal to the derivative of this function (expressed in spherical coordinates) with respect to  $r$ . ■

Next we shall discuss yet another important concept: the flux of a vector field. Let us recall that a vector field is a function  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^d$ , where  $\Omega \subset \mathbb{R}^d$ , where  $d = 1, 2, 3, \dots$ . In other words, a vector field assigns a vector to each point of a subset of the space.

**Definition 1.1** *The flux of the vector field  $\mathbf{f}$  across the boundary  $\partial\Omega$  of a domain  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$  is*

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} d\sigma.$$

Here, if  $d = 2$ , then  $\partial\Omega$  is a closed curve and the integral above is the line integral (of the second kind). The arc length element  $d\sigma$  is to be calculated according to the description of  $\partial\Omega$ . The easiest situation occurs if  $\partial\Omega$  is described in a parametric form by  $\mathbf{r}(t) = (f(t), g(t))$ ,  $t \in [t_0, t_1]$ ; then  $d\sigma = \sqrt{(f')^2 + (g')^2} dt$  and

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} d\sigma = \int_{t_0}^{t_1} \mathbf{f} \cdot \mathbf{n}(t) \sqrt{(f')^2 + (g')^2} dt.$$

When  $d = 3$ , then  $\partial\Omega$  is a closed surface and the integral above is the surface integral (of the second kind). The surface element  $d\sigma$  is again the easiest to calculate if  $\partial\Omega$  is given in a parametric form  $\mathbf{r}(t) = (f(t, s), g(t, s), h(t, s))$ ,  $t \in [t_0, t_1]$ ,  $s \in [s_0, s_1]$ . Then  $d\sigma = |\mathbf{r}_t \times \mathbf{r}_s| dt ds$  and

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} d\sigma = \int_{t_0}^{t_1} \int_{s_0}^{s_1} \mathbf{f} \cdot \mathbf{n}(t, s) |\mathbf{r}_t \times \mathbf{r}_s| ds dt.$$

**Remark 1.1** With a little imagination the definition of the flux can be used also in one dimensional case. To do this, we have to understand that the integration is something like a summation of the integrand over all the points of the boundary. In one-dimensional case we have  $\Omega = [a, b]$  with  $\partial\Omega = \{a\} \cup \{b\}$ . A vector

field in one-dimension is just a scalar function. The unit outward normal at  $\{a\}$  is  $-1$ , and at  $\{b\}$  is  $1$ . Thus  $\mathbf{f}\mathbf{n}(a) = f(a)(-1)$  and  $\mathbf{f}\mathbf{n}(b) = f(b)(1)$  and the flux across the boundary of  $\Omega$  is

$$\mathbf{f} \cdot \mathbf{n}(a) + \mathbf{f} \cdot \mathbf{n}(b) = f(b) - f(a). \quad (1.1.17)$$

■

**Example 1.6** To understand the meaning of the flux let us consider a fluid moving in a certain domain of space. The standard way of describing the motion of the fluid is to associate with any point  $\mathbf{p}$  of the domain filled by the fluid its velocity  $\mathbf{v}(\mathbf{p})$ . In this way we have the *velocity field* of the fluid.

Consider first the one-dimensional case (one can think about a thin pipe). If at a certain point  $x$  we have  $v(x) > 0$ , then the fluid flows to the right, and if  $v(x) < 0$ , then it flows to the left. Let the points  $x = a$  and  $x = b$  be the end-points of a section of the pipe and consider the new field  $f(x) = \rho(x)v(x)$ , where  $\rho$  is the (linear) density of the fluid in point  $x$ . The flux of  $f$ , as defined by (1.1.17), is

$$f(b) - f(a) = \rho(b)v(b) - \rho(a)v(a).$$

For instance, if both  $v(b)$  and  $v(a)$  are positive, then at  $x = b$  the fluid leaves the pipe with velocity  $v(b)$  and at  $x = a$  it enters the pipe with velocity  $v(a)$ . In a small interval of time  $\Delta t$  mass of fluid which left the segment through  $x = b$  is equal  $\rho(b)v(b)\Delta t$  and the mass which entered the segment through  $x = a$  is  $\rho(a)v(a)\Delta t$ , thus, the net rate at which the mass leaves (enters) the segment is equal to  $\rho(b)v(b) - \rho(a)v(a)$ , that is, exactly to the flux of the field  $f$  across the boundary.

This idea can be generalized to more realistic case of three dimensional space. Let us consider a fluid with possibly variable density  $\rho(\mathbf{p})$  filling a portion of space and moving with velocity  $\mathbf{v}(\mathbf{p})$ . We define the mass-velocity field  $\mathbf{f} = \rho\mathbf{v}$ . Let us consider now the domain  $\Omega$  with the boundary  $\partial\Omega$ . Imagine a small portion  $\Delta S$  of  $\partial\Omega$ , which could be considered flat, having the area  $\Delta\sigma$ . Let  $\mathbf{n}$  be the unit normal to this surface and consider the rate at which the fluid crosses  $\Delta S$ . We decompose the velocity  $\mathbf{v}$  into the component parallel to  $\mathbf{n}$ , given by  $(\mathbf{v} \cdot \mathbf{n})\mathbf{n}$  and the tangential component. It is clear that the crossing of the surface can be only due to the normal component (if in time  $\Delta t$  you make two steps perpendicular to the boundary and two parallel, then you will find yourself only two steps from the boundary despite having done four steps). Therefore the mass of fluid crossing  $\Delta S$  in time  $\Delta t$  is given by

$$\Delta m = \rho(\mathbf{v} \cdot \mathbf{n})\Delta t\Delta\sigma.$$

Thus, the rate at which the fluid is crossing the whole boundary  $\partial\Omega$  (that is, the net rate at which the fluid is filling/leaving the domain  $\Omega$ ) can be approximated by summing up all the contributions  $\Delta m$  over all patches  $\Delta S$  of the boundary which, in the limit as  $\Delta S$  go to zero, is nothing but the flux of  $\mathbf{f}$ :

$$\int_{\partial\Omega} (\rho\mathbf{v} \cdot \mathbf{n})d\sigma.$$

Thus again we have the identity

$$\text{Flux of } \rho\mathbf{v} \text{ across } \partial\Omega = \text{the net rate at which the mass of fluid is leaving } \Omega.$$

■

Let us return now to the Green-Gauss.... Theorem. This is a theorem which relates the behaviour of a vector field on the boundary of a domain (flux) with what is happening inside the domain.

Suppose that somewhere inside the domain there is a source of the field (fluid). How can we check this? We can construct a small box around the source and measure whether there is a net outflow, inflow or whether the outflow balances the inflow. If we make this box really small, then we can be quite sure that in the first case there is a source, in the second there is a sink, and in the third case that there is neither sink nor source.

Let us then put this idea into mathematical formulae. Assume that the box  $B$  with one vertex at  $(x_1^0, x_2^0, x_3^0)$  has the edges parallel to the axes of the coordinate system and of lengths  $\Delta x_1$ ,  $\Delta x_2$  and  $\Delta x_3$ .

We calculate the net rate of flow of the vector field  $\mathbf{f} = (f_1, f_2, f_3)$  from the box. Following the calculations given earlier, the flow through the top side is given by

$$\mathbf{f} \cdot \mathbf{n}(x_1^0, x_2^0, x_3^0 + \Delta x_3) \Delta x_1 \Delta x_2 = \mathbf{f} \cdot \mathbf{j}(x_1^0, x_2^0, x_3^0 + \Delta x_3) \Delta x_1 \Delta x_2 = f_3(x_1^0, x_2^0, x_3^0 + \Delta x_3) \Delta x_1 \Delta x_2,$$

and through the bottom

$$\mathbf{f} \cdot \mathbf{n}(x_1^0, x_2^0, x_3^0) \Delta x_1 \Delta x_2 = \mathbf{f} \cdot (-\mathbf{j})(x_1^0, x_2^0, x_3^0) \Delta x_1 \Delta x_2 = -f_3(x_1^0, x_2^0, x_3^0) \Delta x_1 \Delta x_2,$$

thus the net flow through the horizontal sides is

$$(f_3(x_1^0, x_2^0, x_3^0 + \Delta x_3) - f_3(x_1^0, x_2^0, x_3^0)) \Delta x_1 \Delta x_2 \approx \frac{\partial f_3}{\partial x_3}(x_1^0, x_2^0, x_3^0) \Delta x_1 \Delta x_2 \Delta x_3.$$

Similar calculations can be done for the two remaining pairs of the sides and the total flow from the box can be approximated by

$$\left( \sum_{i=1}^3 \frac{\partial f_i}{\partial x_i}(x_1^0, x_2^0, x_3^0) \right) \Delta x_1 \Delta x_2 \Delta x_3.$$

This expression can be considered to be the net rate of the production of the field (fluid) in the box  $B$ . The expression

$$\operatorname{div} \mathbf{f} = \sum_{i=1}^3 \frac{\partial f_i}{\partial x_i}$$

is called the divergence of the vector field  $\mathbf{f}$  and can be considered to be the rate of the production per unit volume (density). To obtain the total net rate of the production in the domain  $\Omega$  we have to add up contributions coming from all the (small) boxes. Thus using the definition of the integral we obtain that the total net rate of the production is given by

$$\iiint_{\Omega} \operatorname{div} \mathbf{f}(x_1, x_2, x_3) d\mathbf{v}.$$

Using some common sense reasoning it is easy to arrive at the identity

$$\text{The net rate of production in } \Omega = \text{The net flow across the boundary}$$

The Green–Gauss–... Theorem is the mathematical expression of the above law.

**Theorem 1.3** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$ ,  $d \geq 1$ , with a piecewise  $C^1$  boundary  $\partial\Omega$ . Let  $\mathbf{n}$  be the unit outward normal vector on  $\partial\Omega$ . Let  $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))$  be any  $C^1$  vector field on  $\bar{\Omega} = \Omega \cup \partial\Omega$ . Then*

$$\int_{\Omega} \operatorname{div} \mathbf{f} d\mathbf{x} = \oint_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} d\sigma, \quad (1.1.18)$$

where  $d\sigma$  is the element of surface of  $\partial\Omega$ .

**Remark 1.2** In one dimension this theorem is nothing but the fundamental theorem of calculus:

$$\int_a^b \frac{df}{dx}(x) dx = f(b) - f(a). \quad (1.1.19)$$

In fact, for a function of one variable  $\operatorname{div} \mathbf{f} = \frac{df}{dx}$  and the right-hand side of this equation represents the outward flux across the boundary of  $\Omega = [a, b]$ , as discussed in Remark 1.1.

**Remark 1.3** In two dimensions the most popular form of this theorem is known the Green Theorem which apparently differs from the one given above. To explain this, let the boundary  $\partial\Omega$  of  $\Omega$  be a curve given by parametric equation  $\mathbf{r}(t) = (x_1(t), x_2(t))$ ,  $t \in [t_0, t_1]$  and suppose that if  $t$  runs from  $t_0$  to  $t_1$ , then  $\mathbf{r}(t)$  traces  $\partial\Omega$  in the positive (anticlockwise) direction. Then it is easy to check that the unit outward normal vector  $\mathbf{n}$  is given by

$$\mathbf{n}(t) = \frac{1}{\mathbf{r}'(t)}(x_2'(t), -x_1'(t)),$$

where  $\mathbf{r}'(t) = (x_1'(t), x_2'(t))$ . Thus, if  $\mathbf{f} = (f_1, f_2)$ , then Eq. (1.1.18) takes the form

$$\iint_{\Omega} \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = \int_{t_0}^{t_1} (f_1(t)x_2'(t) - f_2(t)x_1'(t)) dt \quad (1.1.20)$$

On the other hand, the standard version of Green's theorem reads

$$\iint_{\Omega} \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 dx_2 = \oint_{\partial\Omega} f_1 dx_1 + f_2 dx_2 = \int_{t_0}^{t_1} (f_1(t)x_1'(t) + f_2(t)x_2'(t)) dt. \quad (1.1.21)$$

To see that these forms are really equivalent, let us define the new vector field  $\mathbf{g} = (g_1, g_2)$  by :  $g_1 = -f_2, g_2 = f_1$ . Then

$$\operatorname{div} \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2}.$$

The boundary of  $\partial\Omega$  is, as above, parameterized by the function  $\mathbf{r}(t)$ . Thus, if we assume that (1.1.21) holds (for an arbitrary vector field), then

$$\begin{aligned} \iint_{\Omega} \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 &= \iint_{\Omega} \left( \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2} \right) dx_1 dx_2 \\ &= \int_{t_0}^{t_1} (g_1(t)x_1'(t) + g_2(t)x_2'(t)) dt = \int_{t_0}^{t_1} (f_1(t)x_2'(t) - f_2(t)x_1'(t)) dt, \end{aligned}$$

which is (1.1.20). The converse is analogous. ■

## 2 Infinite series of functions

Very often the solution of a differential equation (ordinary or partial) is given by an infinite series of simple functions, like monomials or trigonometric functions. It is important to realize that in general a function, given as a sum of an infinite series of functions, even if exists, can have completely different properties than each term. Here we shall discuss conditions which ensure the existence and regularity of the sum of an infinite series.

Let us consider first an infinite series of numbers  $\sum_{n=1}^{\infty} a_n$ , where  $a_n$  can be possibly complex numbers. By partial sums of the series we understand sums  $S_N = \sum_{n=1}^N a_n$ . The series is said to be convergent, if the sequence of numbers  $(S_N)_{N \in \mathbb{N}}$  converges to a finite number, say,  $S$ ; then we say that  $S$  is the sum of the series and write

$$S = \sum_{n=1}^{\infty} a_n.$$

With every series of this form we associate the series of positive numbers  $\sum_{n=1}^{\infty} |a_n|$ . The sequence  $(S_N)_{N \in \mathbb{N}}$  is then increasing and therefore it either converges, or diverges to  $+\infty$ . If  $\sum_{n=1}^{\infty} |a_n|$  converges, then we say

that  $\sum_{n=1}^{\infty} a_n$  converges absolutely. If  $\sum_{n=1}^{\infty} |a_n|$  diverges, but  $\sum_{n=1}^{\infty} a_n$  converges, we say that  $\sum_{n=1}^{\infty} a_n$  is conditionally convergent. If  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then we can change order of summation in the series in an arbitrary way. If the series is only conditionally convergent, then one has to be much more careful—there is a theorem of calculus which states that by rearranging the order of summation one can make a conditionally convergent series converge to any number.

If we consider a series of functions, defined on a common domain  $D \subset \mathbb{R}^d$ :  $\sum_{n=1}^{\infty} f_n(\mathbf{x})$ , then for each  $\mathbf{x} \in D$  we have a series of numbers, and therefore we can apply the consideration of the previous paragraph to define the concept of *pointwise convergence* of a functional series. That is, we say that the series  $\sum_{n=1}^{\infty} f_n(\mathbf{x})$  converges pointwise in  $D$  to a function  $f$ , if for each  $\mathbf{x} \in D$  the sequence of partial sums  $\sum_{n=1}^N f_n(\mathbf{x})$  converges, as a sequence of numbers, to  $f(\mathbf{x})$ . In  $\epsilon - \delta$  language we write

$$\forall \mathbf{x} \in D \quad \forall \epsilon > 0 \quad \exists N_0 \in \mathbb{N} \quad \forall N > N_0 \quad \left| f(\mathbf{x}) - \sum_{n=1}^N f_n(\mathbf{x}) \right| < \epsilon. \quad (1.2.22)$$

Note that  $N_0$  can depend here on both  $\epsilon$  and  $\mathbf{x}$  which makes this convergence very weak in the sense that it doesn't preserve important properties of functions. We may have a series of continuous functions converging pointwise everywhere to a discontinuous function. A classical example is offered by the series  $\sum_{n=1}^{\infty} f_n(x)$  with  $f_n(x) = x^n - x^{n+1}$  for  $x \in [0, 1]$  which has partial sums given by

$$S_N(x) = x - x^{N+1}$$

and they converge to  $f(x) = x$  for  $x \in [0, 1[$  and  $f(1) = 0$ .

The way out of this is to make  $N_0$  independent of  $\mathbf{x}$ . This leads to the concept of *uniform convergence* of the functional series which reads: the series  $\sum_{n=1}^{\infty} f_n(\mathbf{x})$  converges uniformly to  $f(\mathbf{x})$  on  $D$  if

$$\forall \epsilon > 0 \quad \exists N_0 \in \mathbb{N} \quad \forall N > N_0 \quad \sup_{\mathbf{x} \in D} \left| f(\mathbf{x}) - \sum_{n=1}^N f_n(\mathbf{x}) \right| < \epsilon. \quad (1.2.23)$$

The difference between (1.2.22) and (1.2.23) can be explained by saying that to check the uniform convergence we chose the biggest difference over all values of  $\mathbf{x}$  and only then take the limit. It is easy to see that in fact the above statement is equivalent to saying that for each  $\epsilon$  we can select  $N_0$  independent of  $\mathbf{x}$  such that (1.2.22) holds.

In many applications it is not necessary to prove the uniform convergence from the definition (1.2.23). Very often one can get away with the relatively simple Weierstrass criterion which we formulate as the following theorem.

**Theorem 2.1** *If there is a sequence of non-negative numbers  $(c_n)_{n \in \mathbb{N}}$  such that the  $\sum_{n=1}^{\infty} c_n < +\infty$  and*

$$\forall \mathbf{x} \in D, n \in \mathbb{N} \quad |f_n(\mathbf{x})| \leq c_n,$$

*then  $\sum_{n=1}^{\infty} f_n(\mathbf{x})$  converges uniformly and absolutely in  $D$ .*

The importance of the uniform convergence stems from the fact that most important properties of the functions are preserved under uniform limit. For example, if  $f$  is a uniform limit of a sequence or series of continuous functions, then  $f$  itself is continuous. We shall formulate two other theorems on uniformly convergent series of functions which will be needed in the sequel.

**Theorem 2.2** *If  $D \subset \mathbb{R}^d$  is a bounded and closed set, functions  $f_n, n \in \mathbb{N}$  are continuous in  $D$  and  $f(\mathbf{x}) = \sum_{n=1}^{\infty} f_n(\mathbf{x})$  uniformly in  $D$ , then  $f$  is continuous in  $D$  and*

$$\sum_{n=1}^{\infty} \int_D f_n(\mathbf{x}) d\mathbf{x} = \int_D f(\mathbf{x}) d\mathbf{x}. \quad (1.2.24)$$

*Another way of putting this statement is to say that a uniformly convergent series can be term by term (or termwise) integrated.*

The question of termwise differentiation is considerably more delicate.

**Theorem 2.3** *If  $D \subset \mathbb{R}^d$  is a bounded and closed set, functions  $f_n, n \in \mathbb{N}$  are differentiable in  $D$ , and if the series  $\sum_{n=1}^{\infty} f_n(c)$  converges for some  $c \in D$  and the sequence partial derivatives  $\sum_{n=1}^{\infty} \frac{\partial f_n}{\partial x_i}$  converges uniformly in  $D$  for some  $1 \leq i \leq n$ , then  $\sum_{n=1}^{\infty} f_n(\mathbf{x})$  converges uniformly to a function  $f$  which has partial derivative with respect to  $x_i$  and*

$$\sum_{n=1}^{\infty} \frac{\partial f_n(\mathbf{x})}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i}. \quad (1.2.25)$$

### 3 Differentiation under the sign of integral

In many cases the solution of a partial differential equation is given in terms of an improper integral involving parameters which in fact are independent variables of the solution, and we require differentiability of the integral with respect to these parameters.

To simplify presentation we shall focus here on improper integrals over the real line - the generalization of the presented results to higher dimensions is obvious.

Let us recall that the improper integral of a function  $f$  is defined as

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{M, N \rightarrow \infty} \int_{-M}^N f(x) dx \quad (1.3.1)$$

Clearly, if the above limit exists, then we can also write

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$$

but the converse is not true in general.

Let us consider a function  $f$  of two variables  $(x, t) \in \mathbb{R} \times I$  where  $I \subset \mathbb{R}$  is an open interval. If for each  $t \in I$  the improper integral  $\int_{-\infty}^{\infty} f(x, t) dx$  exists, then we can define a new function of  $t$  by

$$F(t) = \int_{-\infty}^{\infty} f(x, t) dx. \quad (1.3.2)$$

Our interest is to determine under which condition  $F$  is a differentiable function. To proceed we need a new concept. We say that the improper integral (1.3.2) converges absolutely and uniformly in  $t \in I$  if for any

$\epsilon > 0$  there is  $R_\epsilon$  such that for any  $R > R_\epsilon$  and any  $t \in I$  we have

$$\int_{R_\epsilon}^R |f(x, t)| < \epsilon \quad \text{and} \quad \int_{-R}^{-R_\epsilon} |f(x, t)| < \epsilon. \quad (1.3.3)$$

Now we can formulate the main result of this section.

**Theorem 3.1** *Let  $f(x, t)$  and  $f_t(x, t)$  be continuous on  $\mathbb{R} \times I$  and assume that for each  $t \in I$  both improper integrals  $\int_{-\infty}^{\infty} f(x, t) dx$  and  $\int_{-\infty}^{\infty} f_t(x, t) dx$  exist, and moreover that the second integral converges uniformly and absolutely. Then for any  $t \in I$*

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} f(x, t) dx = \int_{-\infty}^{\infty} \frac{\partial f}{\partial t}(x, t) dx. \quad (1.3.4)$$

**Proof.** We have to prove that for any  $t_0 \in I$

$$\lim_{h \rightarrow 0} \left( \lim_{R \rightarrow \infty} \int_{-R}^R (h^{-1}(f(x, t_0 + h) - f(x, t_0) - f_t(x, t_0))) dx \right) = 0 \quad (1.3.5)$$

so it is enough if we prove that for any  $\epsilon > 0$  there are  $h_0, R_0$  such that for any  $0 < h < h_0, R > R_0$

$$\int_{-R}^R h^{-1} |(f(x, t_0 + h) - f(x, t_0) - f_t(x, t_0))| dx < \epsilon \quad (1.3.6)$$

To do this we shall split the integral into two parts. Let us fix  $\epsilon > 0$ . Since  $f_t$  is uniformly integrable, we find  $R_\epsilon$  such that for any  $R > R_\epsilon$  and any  $t \in I$

$$\left( \int_{-R}^{-R_\epsilon} + \int_{R_\epsilon}^R \right) |f_t(x, t)| dx < \epsilon,$$

hence in particular

$$\left( \int_{-R}^{-R_\epsilon} + \int_{R_\epsilon}^R \right) |f_t(x, t_0)| dx < \epsilon.$$

On the other hand, we have

$$\begin{aligned} \left( \int_{-R}^{-R_\epsilon} + \int_{R_\epsilon}^R \right) h^{-1} |(f(x, t_0 + h) - f(x, t_0))| dx &= h^{-1} \left( \int_{-R}^{-R_\epsilon} + \int_{R_\epsilon}^R \right) \left| \int_{t_0}^{t_0+h} f_u(x, u) du \right| dx \\ &\leq h^{-1} \int_{t_0}^{t_0+h} \left( \left( \int_{-R}^{-R_\epsilon} + \int_{R_\epsilon}^R \right) |f_u(x, u)| dx \right) du \leq h^{-1} \int_{t_0}^{t_0+h} \epsilon du \\ &= \epsilon \end{aligned}$$

where we used the fact that  $f_t$  is a continuous function to change the order of integration, and uniform integrability of  $f_t$  so that the same  $R_\epsilon$  and the same  $\epsilon$  could be used for all  $f_u(x, u)$ ,  $u \in [t_0, t_0 + h] \subset I$ . Note, that this calculations were completely independent of  $h$ .

Having fixed  $R_\epsilon$  so that the estimates above are valid, we have by Taylor's formula

$$\begin{aligned} \int_{-R_\epsilon}^{R_\epsilon} h^{-1} |f(x, t+h) - f(x, t) - hf_t(x, t)| dx &= \int_{-R_\epsilon}^{R_\epsilon} |f_t(x, t + \theta_x h) - f_t(x, t)| dx \\ &\leq \epsilon, \end{aligned}$$

where we used the fact that  $f_t$  is (uniformly) continuous so that for  $\eta$  small enough we have  $|f_t(x, t + \eta) - f_t(x, t)| \leq \epsilon/2R_\epsilon$  (remember that  $R_\epsilon$  is fixed and we change only  $h$  which does not affect  $R_\epsilon$ ).

Combining these two estimates we obtain that for any  $\epsilon > 0$  we can find  $R_0 = R_\epsilon$  and  $h_0$  such that for  $0 < h < h_0, R > R_0$

$$\begin{aligned} \left| \int_{-R}^R h^{-1} (f(x, t+h) - f(x, t) - hf_t(x, t)) dx \right| &\leq \int_{-R}^R |h^{-1} (f(x, t+h) - f(x, t) - hf_t(x, t))| dx \\ &= \left( \int_{R_\epsilon}^R + \int_{-R}^{-R_\epsilon} \right) |h^{-1} (f(x, t+h) - f(x, t) - hf_t(x, t))| dx + \int_{-R_\epsilon}^{R_\epsilon} h^{-1} |f(x, t+h) - f(x, t) - hf_t(x, t)| dx \\ &\leq 2\epsilon \end{aligned}$$

which proves (9.2.4). ■



# Lecture 3

## Origins of partial differential equations

In this chapter we shall derive and discuss several most important partial differential equations of mathematical physics. Firstly, however, we should make sure that we all know what we are talking about.

### 1 What is a partial differential equation?

Let us consider the evolution of the temperature of a body. In the first year Calculus the so-called Newton Law of Cooling was discussed. In mathematical terms this was expressed by the linear first order ordinary differential equation

$$\frac{du}{dt} = k(u - u_s) \quad (3.1.7)$$

where  $t \rightarrow u(t)$  is the temperature of an object at the time  $t$ ,  $u_s$  is the temperature of the surrounding medium, and  $k$  is the proportionality constant. Thus, the temperature was assumed to be a function of only one variable: time  $t$ . A short reflection shows that this model in general cannot be correct. Typically, in any object there are regions of lower and higher temperature, so the model described by (3.1.7) could be correct under very special physical assumption: that the temperature is uniform throughout the body.

In the real life we can hardly expect such a uniformity – most functions depend on 4 variables: the time and three spatial coordinates. For example, the temperature in the room changes in time, but also changes when we move from one point to another. It means that to describe properly the temperature we have to use function of four variables. As we shall see later, the time and position dependence of the temperature will introduce partial derivatives into the equation governing the temperature and make the equation a partial differential equation.

This is the key identifying property of partial differential equations: there are more than one independent variable  $x_1, x_2, \dots$ . We have also one dependent variable, that is, unknown function  $u(x_1, x_2, \dots)$  (it is also possible to have more than one unknown function, which would lead to a system of PDEs, but we will not discuss such a possibility). A PDE is an identity which relates the independent variables, dependent variable  $u$  and the partial derivatives of  $u$ . For example

$$F(x_1, x_2, u(x_1, x_2), u_{x_1}(x_1, x_2), u_{x_2}(x_1, x_2)) = 0,$$

where  $F : \mathbb{R}^5 \rightarrow \mathbb{R}$  is any function, is the most general PDE in two independent variables of *first order* (the highest order of partial derivatives).

Similarly, we can write the most general form of second order PDE in three variables as

$$F(x_1, x_2, x_3, u, u_{x_1}, u_{x_2}, u_{x_3}, u_{x_1x_1}, u_{x_1x_2}, u_{x_1x_3}, u_{x_2x_3}, u_{x_2x_2}, u_{x_3x_3}) = 0$$

and so on.

Some examples of PDEs (in various variants of notation) are:

1.  $\rho_t + \mathbf{a} \nabla \rho = 0$  (continuity equation)
2.  $u_t - \Delta u = 0$  (heat or diffusion equation)
3.  $u_{tt} - \Delta u = 0$  (waves of small amplitude)
4.  $\Delta u = 0$  (electric potential)
5.  $u_t = -i\Delta u$  (Schrödinger equation of quantum mechanics)
6.  $u_t + u \cdot u_x + u_{xxx} = 0$  (dispersive wave equation)
7.  $u_{tt} + u_{xxxx} = 0$  (vibrating beam equation)

We shall see how to derive some of these equations from physical principles; some of them require, however, a high degree of sophistication and we will leave them aside.

First of all let us note that, contrary to the general notation, we shall distinguish one variable  $t$ . This is dictated by the physical interpretation of the quantities appearing in the equations. Later it will become also clear that the properties of the equations and their solutions with respect to this variable are different from those with respect to the remaining variables reflecting the physical fact that the time is a little different variable from the spatial variables.

Let us look at these equations and try to find some common features. Equations 6. and 7. are equations in two variables, the remaining ones can have any number of independent variables as the general notation for the gradient

$$\nabla u = (u_1, \dots, u_n)$$

and the Laplacian

$$\Delta u = \sum_{n=i}^n u_{x_i x_i}$$

does not specify the number of variables.

Equation 1. is a first order equations, Equations 2. through 5. are second order equations, Equation 6. is a third order and Equation 7. is a fourth order equation.

One of the most important property of a differential equation is *linearity*. A rough definition of a linear differential equation is that the function and its derivatives appear in the equation only at most in first (algebraic) power, that is, linearly.

A more sophisticated definition involves some linear algebra terminology. We recall that the set of (continuous, differentiable) functions is a linear space, that is, a sums and scalar multiples of such functions are again functions having the same properties. In such a space one can define operators, like super-functions, which take functions and transform them into other functions. Such an operator can be defined by e.g.

$$L[u] = u_{x_1} + u_{x_2},$$

that is,  $L$  takes any differentiable function of two variables, calculates both first partial derivatives and adds them together. This operator is a linear operator as

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2]$$

for any scalars  $c_1, c_2$  and functions  $u_1$  and  $u_2$  (for which it is defined). On the other hand, the operator related to Equation 6. is not linear: if we take the function  $u = x$ , then

$$L[u] = x$$

but applied to the function  $2u$  we have

$$L[2u] = 4x \neq 2L[u].$$

Using the concept of an operator we can write any differential equation in the form

$$L[u] = f,$$

by grouping all the terms containing the unknown function and/or its derivatives on the left-hand side. We then say that the differential equation is linear if  $L$  is a linear operator in  $u$ . We also say that the linear equation is homogeneous if  $f \equiv 0$ ; otherwise the equation is said to be non-homogeneous.

The advantage of linearity for the equation  $L[u] = 0$  is that if  $u$  and  $v$  are solution of this equation, then so is  $u + v$ . More generally, any linear combination of solutions is itself a solution; this is sometimes called the *superposition principle*.

**Example 1.1** Knowing that  $u_1(t, x) = 2t + x^2$  satisfies the heat equation  $u_t = u_{xx}$  with the initial condition  $u(0, x) = x^2$  and  $u_2(t, x) = tx^2$  satisfies  $u_t = u_{xx} + x^2 - 2t$  with the initial condition  $u(0, x) = 0$ , find the solution of the problem

$$u_t = u_{xx} + 2x^2 - 4t, \quad u(0, x) = -x^2. \quad (3.1.8)$$

We shall use the superposition principle and look for the solution in the form  $u(t, x) = au_1(t, x) + bu_2(t, x)$  with constants  $a, b$  to be determined. We have

$$u_t = (au_1 + bu_2)_t = au_{1t} + bu_{2t} = au_{1xx} + bu_{2xx} + b(x^2 - 2t) = u_{xx} + b(x^2 - 2t).$$

By comparison with (3.1.8), we obtain  $b = 2$ . Also

$$u(0, x) = au_1(0, x) + bu_2(0, x) = ax^2 + 2 \cdot 0 = ax^2,$$

thus  $a = -1$  and the solution is given by

$$u(t, x) = -2t - x^2 + 2tx^2.$$

Solutions of ordinary differential equations depend on arbitrary constants: one for first order equations, two for second order equations, etc. With partial differential equations the situation becomes much more complicated. Let us consider the following example.

**Example 1.2** Solve the PDE

$$u_{x_1 x_2} = 0. \quad (3.1.9)$$

Writing this equation in the form

$$\frac{\partial}{\partial x_2} \left( \frac{\partial u}{\partial x_1} \right) = 0,$$

we see that  $u_1 = f(x_1)$  for any function  $f$  of variable  $x_1$ . Now, we can integrate this equation with respect to  $x_1$  regarding  $x_2$  as fixed. We get  $u(x_1, x_2) = \int f(x_1) dx_1 + G(x_2)$  where  $G$  is an arbitrary function of  $x_2$ . But the indefinite integral of an arbitrary function is again an arbitrary function, so that, the general solution of (3.1.9) is

$$u(x_1, x_2) = F(x_1) + G(x_2)$$

so that the solution of second order partial differential equations depends in general on two arbitrary functions.

## 2 Conservation laws

Many of the fundamental equations that occurring in the natural and physical sciences are obtained from *conservation laws*. Conservation laws express the fact that some quantity is balanced throughout the process.

In thermodynamics, for example, the First Law of Thermodynamics states that the change in the internal energy in a given system is equal to, or is balanced by, the total heat added to the system plus the work done on the system. Therefore the First Law of Thermodynamics is really an energy balance, or conservation, law.

As an another example, consider a fluid flowing in some region of space that consists of chemical species undergoing chemical reaction. For a given chemical species, the rate of change in time of the total amount of that species in the region must equal the rate at which the species flows into the region, minus the rate at which the species flows out, plus the rate at which the species is created or consumed, by the chemical reactions. This is a verbal statement of a conservation law for the amount of the given chemical species.

Similar balance or conservation laws occur in all branches of science. We can recall the population equation from Chapter the derivation of which was based on the observation that the rate of change of a population in a certain region must equal the birth rate, minus the death rate, plus the migration rate into or out of the region.

Such conservation laws in mathematics are translated usually into differential equations and it is surprising how many different processes of real world end up with the same mathematical formulation. These differential equations are called the *governing equations* of the process and dictate how the process evolves in time. Here we discuss the derivation of some governing equations starting from the first principles. We begin with a basic one-dimensional model.

## 2.1 One-dimensional conservation law

Let us consider a quantity  $u = u(x, t)$  that depends on a single spatial variable  $x$  in an interval  $R \subset \mathbb{R}$ , and time  $t > 0$ . We assume that  $u$  is a density, or concentration, measured in an amount per unit volume, where the amount may refer to the population, mass, energy or any other quantity. The fact that  $u$  depends only on one spatial variable  $x$  can be physically justified by assuming e.g. that we consider a flow in a tube, which is uniform (doesn't change in radial direction), or that the tube is very thin so that any change in the radial direction is negligible (later we shall derive the equation for the same conservation law in an arbitrary dimensional space).

*Fig 3.1. Tube  $\mathcal{I}$ .*

Note that the discussion here is related to Remark 1.1 and Example 1.6. We consider the subinterval  $I = [x, x + h] \subset R$ . The total amount of the quantity  $u$  contained at time  $t$  in the section  $\mathcal{I}$  of the tube between  $x$  and  $x + h$  is given by the integral

$$\text{total amount of quantity in } \mathcal{I} = A \int_x^{x+h} u(s, t) ds,$$

where  $A$  is the area of the cross section of  $\mathcal{I}$ .

Assume now that there is motion of the quantity in the tube in the axial direction. We define the *flux* density of  $u$  at time  $t$  at  $x$  to be the scalar function  $\phi(x, t)$  which is equal to the amount of the quantity  $u$  passing through the cross section at  $x$  at time  $t$ , per unit area, per unit time. By convention, the flux density at  $x$  is positive if the flow at  $x$  is in the positive  $x$  direction. Therefore, at time  $t$  the net rate that the quantity is flowing into the section is the rate that it is flowing in at  $x$  and minus the rate that it is flowing out at  $x + h$ , that is

net rate that the quantity flows into  $\mathcal{I} = A(\phi(x, t) - \phi(x + h, t))$ .

Fig 3.2. One dimensional flow through cross-sections  $x$  and  $x + h$ .

This equation should be compared with Example 1.6, where the flux density was easy to understand: it was the rate at which the mass of fluid flows through the boundary. Here we are considering the flux density of an arbitrary quantity (arbitrary one-dimensional vector field).

Finally, the quantity  $u$  may be destroyed or created inside  $\mathcal{I}$  by some internal or external source (e.g. by a chemical reaction if we consider chemical kinetics equations, or by birth/death processes in mathematical biology). We denote this *source function* which is assumed to be local (acts at each  $x$  and  $t$ ) by  $f(x, t, u)$ . This function gives the rate at which  $u$  is created or destroyed at  $x$  at time  $t$ , per unit volume. Note that  $f$  may depend on  $u$  itself (e.g. the rate of chemical reactions is determined by concentration of the chemicals). Given  $f$ , we may calculate the total rate that  $u$  is created/destroyed in  $\mathcal{I}$  by integration:

$$\text{rate that quantity is produced in } \mathcal{I} \text{ by sources} = A \int_x^{x+h} f(s, t, u(s, t)) ds.$$

The fundamental conservation law can be formulated as follows: for any section  $\mathcal{I}$

$$\begin{aligned} & \text{the rate of change of total amount in } \mathcal{I} \\ &= \text{net rate that the quantity flows into } \mathcal{I} \\ &+ \text{rate that the quantity is produced in } \mathcal{I} \end{aligned}$$

Using the mathematical formulas obtained above we get, having canceled the area  $A$

$$\frac{d}{dt} \int_x^{x+h} u(s, t) ds = \phi(x, t) - \phi(x + h, t) + \int_x^{x+h} f(s, t, u) ds. \quad (3.2.1)$$

The equation above is called a *conservation law in integral form* and holds even if  $u, f, \phi$  are not smooth functions. This form is useful in many cases but rather difficult to handle, therefore it is convenient to reduce it to the differential form. This requires assuming that all the functions (including the unknown  $u$ ) are continuously differentiable. Using basic facts from calculus:

$$(i) \quad \int_x^{x+h} \phi_s(s, t) ds = \phi(x + h, t) - \phi(x, t),$$

$$(ii) \quad \frac{d}{dt} \int_x^{x+h} u(s, t) ds = \int_x^{x+h} u_t(s, t) ds,$$

we can rewrite (3.2.1) in the form

$$\int_I (u_t(s, t) + \phi_s(s, t) - f(s, t, u)) ds = 0. \quad (3.2.2)$$

Since this equation is valid for any interval  $I$  we can use Theorem 1.6.15 to infer that the integral must vanish identically; that is (changing the independent variable back into  $x$ ) we must have

$$u_t(x, t) + \phi_x(x, t) = f(x, t, u) \quad (3.2.3)$$

for any  $x \in R$  and  $t > 0$ . Note that in (3.2.3) we have two unknown functions:  $u$  and  $\phi$ ; function  $f$  is assumed to be given. Function  $\phi$  is usually to be determined from empirical considerations. Equations resulting from such considerations, which specify  $\phi$ , are often called *constitutive relations* or *equations of state*.

Before we proceed in this direction, we shall show how the procedure described above works in higher dimensions.

## 2.2 Conservation laws in higher dimensions

It is relatively straightforward to generalize the discussion above to multidimensional space. Let  $\mathbf{x} = (x_1, x_2, x_3)$  denote a point in  $\mathbb{R}^3$  and assume that  $u = u(\mathbf{x}, t)$  is a scalar density function representing the amount per unit volume of some quantity of interest distributed throughout some domain of  $\mathbb{R}^3$ . In this domain, let  $\Omega \subset \mathbb{R}^3$  be an arbitrary region with a smooth boundary  $\partial\Omega$ . As in one-dimensional case, the total amount of the quantity in  $\Omega$  at time  $t$  is given by the integral

$$\int_{\Omega} u(\mathbf{x}, t) d\mathbf{x},$$

and the rate that the quantity is produced in  $\Omega$  is given by

$$\int_{\Omega} f(\mathbf{x}, t, u) d\mathbf{x},$$

where  $f$  is the rate at which the quantity is being produced in  $\Omega$ .

Some changes are required however when we want to calculate the flow of the quantity in or out  $\Omega$ . Because now the flow can occur in any direction, the flux density is given by a vector  $\mathbf{\Phi}$  (in Example 1.6, the flux density was given by  $\rho(\mathbf{x})\mathbf{v}(\mathbf{x})$ ). However, as we noted in this example, not all the flowing quantity which is close to the boundary will leave or enter the region  $\Omega$  - the component of  $\mathbf{\Phi}$  which is tangential to the boundary  $\partial\Omega$  will not contribute to the total loss (see also Fig. 2.1). Therefore, if we denote by  $\mathbf{n}(\mathbf{x})$  the outward unit normal vector to  $\partial\Omega$ , then the net outward flux of the quantity  $u$  through the boundary  $\partial\Omega$  is given by the surface integral

$$\oint_{\partial\Omega} \mathbf{\Phi}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) d\sigma,$$

where  $d\sigma$  denotes a surface element on  $\partial\Omega$ . Finally, the balance law for  $u$  is given by

$$\frac{d}{dt} \int_{\partial\Omega} u d\mathbf{x} = - \oint_{\partial\Omega} \mathbf{\Phi} \cdot \mathbf{n} d\sigma + \int_{\Omega} f d\mathbf{x}. \quad (3.2.4)$$

The minus sign at the flux term occurs because the outward flux decreases the amount of  $u$  in  $\Omega$ .

As before, the integral form can be reformulated as a local differential equation, provided  $\mathbf{\Phi}$  and  $u$  are sufficiently smooth in  $\mathbf{x}$  and  $t$ . To this end we must use the Gauss theorem (Theorem 1.6.16) which gives

$$\oint_{\partial\Omega} \mathbf{\Phi} \cdot \mathbf{n} d\sigma = \int_{\Omega} \operatorname{div} \mathbf{\Phi} d\mathbf{x}.$$

Using this formula, Equation (3.2.4) can be rewritten as

$$\int_{\Omega} (u_t + \operatorname{div} \mathbf{\Phi} - f) d\mathbf{x} = 0$$

for any subregion  $\Omega$ . Using the vanishing theorem (Theorem 1.6.15) we obtain finally the differential form of the general conservation law in 3-dimensional space

$$u_t + \operatorname{div} \Phi = f(\mathbf{x}, t, u). \quad (3.2.5)$$

In the next section we discuss a number of constitutive relations which provide examples of the flux function.

### 3 Constitutive relations and examples

In this section we describe a few constitutive relations which often appear in the applied sciences. It is important to understand that a constitutive relation appears on a different level than conservation law: the latter is a fundamental law of nature – the mathematical expression of the fact that books should balance (at least in a decent(?) enterprise like the Universe), whereas the former is often an approximate equation having its origins in empirics.

#### 3.1 Transport equation

Probably the simplest non-trivial constitutive relation is when the quantity of interest moves together with the surrounding medium with a given velocity, say,  $\mathbf{v}$ . This can be a simple model for a low density pollutant in the air moving only due to wind with velocity  $\mathbf{v}$  or, as we discussed earlier, the fluid with density  $u$  moving with the velocity  $\mathbf{v}$ . Then the flux is given by

$$\Phi = \mathbf{v}u$$

and, if there are no sources or sinks, the transport equation is given by

$$u_t + \operatorname{div}(\mathbf{v}u) = 0. \quad (3.3.1)$$

We can rewrite this equation in a more explicit form

$$u_t + \mathbf{v} \cdot \nabla u + u \operatorname{div} \mathbf{v} = 0,$$

or, if the velocity is constant (more general, divergence free),

$$\partial_t u + \mathbf{v} \cdot \nabla u = 0. \quad (3.3.2)$$

#### 3.2 Diffusion/heat equations

Assume at first that there are no sources, and write the basic conservation law in one-dimension as

$$u_t + \phi_x = 0 \quad (3.3.3)$$

In many physical (and not only) problems it was observed that the substance (represented here by the density  $u$ ) moves from the regions of higher concentration to regions of lower concentration (heat, population etc. offer a good illustration of this). Moreover, the larger the difference the more rapid the flow is observed. At a point, the large difference can be expressed as a large gradient of the concentration  $u$ , so it is reasonable to assume that

$$\phi(x, t) = -F(u_x(x, t)),$$

where  $F$  is an increasing function passing through  $(0, 0)$ . The explanation of the minus sign comes from the fact that if the  $x$ -derivative of  $u$  at  $x$  is positive, that is,  $u$  is increasing, then the flow occurs in the negative direction of the  $x$  axis. Moreover, the larger the gradient, the larger (in magnitude) the flow. Now, the

simplest increasing function passing through  $(0, 0)$  is a linear function with positive leading coefficient, and this assumption gives the so-called *Fick law*:

$$\phi(x, t) = -Du_x(x, t) \quad (3.3.4)$$

where  $D$  is a constant which is to be determined empirically. We can substitute Fick's law (3.3.4) into the conservation law (3.2.3) (assuming that the solution is twice differentiable and  $D$  is a constant) and get the one dimensional *diffusion equation*

$$u_t - Du_{xx} = 0, \quad (3.3.5)$$

which governs conservative processes when the flux is specified by Fick's law.

To understand why this equation governs also the evolution of the temperature distribution in a body, we apply the conservation law (3.2.3) to heat (thermal energy). Then the amount of heat in a unit volume can be written as  $u = c\rho\theta$ , where  $c$  is the specific heat of the medium,  $\rho$  the density of the medium and  $\theta$  is the temperature. The heat flow is governed by Fourier's law which states that the heat flux density is proportional to the gradient of the temperature, with the proportionality constant equal to  $-k$ , where  $k$  is the thermal conductivity of the medium. Assuming  $c, \rho, k$  to be constants, we arrive at (3.3.5) with  $D = k/c\rho$ .

Generalization of Equation (3.3.5) to multidimensional cases requires some assumptions on the medium. To simplify the discussion we assume that the medium is isotropic, that is, the process of diffusion is independent of the orientation in space. Then the Fick law states that the flux is proportional to the gradient of  $u$ , that is,

$$\Phi = -D\nabla u$$

and the diffusion equation, obtained from the conservation law (3.2.5) in the absence of sources, reads

$$u_t = \operatorname{div}(D\nabla u) = D\Delta u, \quad (3.3.6)$$

where the second equality is valid if  $D$  is a constant.

### 3.3 Variations of diffusion equation

In many cases the evolution is governed by more than one process. One of the most widely occurring cases is when we have simultaneously both transport and diffusion. For example, when we have a pollutant in the air or water, then it is transported with the moving medium (due to the wind or water flow) and, at the same time, it is dispersed by diffusion. Without the latter, a cloud of pollution would travel without any change - the same amount that was released from the factory would eventually reach other places with the same concentration. Diffusion, as we shall see later, has a dispersing effect, that is, the pollution will be eventually deposited but in a uniform, less concentrated way, making life (perhaps) more bearable.

Mathematical expression of the above consideration is obtained by combining equations (3.3.1) and (3.3.6):

$$u_t + \operatorname{div}(vu) = D\Delta u, \quad (3.3.7)$$

or, equivalently, by determining the expression for the flux taking both processes simultaneously into account. The resulting equation (3.3.7) is called the *drift-diffusion equation*.

When the sources are present and the constitutive relation is given by Fick's law, then the resulting equation

$$u_t - D\Delta u = f(\mathbf{x}, t, u) \quad (3.3.8)$$

is called the *reaction-diffusion equation*. If  $f$  is independent of  $u$ , then physically we have the situation when the sources are given a priori (like the injection performed by an observer); the equation becomes then a linear non-homogeneous equation. Sources, however, can depend on  $u$ . The simplest case is then when  $f = cu$  with  $c$  constant. Then the source term describes spontaneous decay (or creation) of the substance at the rate that is proportional to the concentration. In such a case Equation (3.3.8) takes the form

$$u_t - D\Delta u = cu \quad (3.3.9)$$



and can be thought of as the combination of the diffusion and the law of exponential growth.

If  $f$  is nonlinear in  $u$ , then Equation (3.3.8) has many important applications, particularly in the combustion theory and mathematical biology.

Here we describe the so called *Fisher equation* which appears as a combination of the one-dimensional logistic process and the diffusion. As we discussed in Section 4, it is reasonable to assume that the population is governed by the logistic law, which is mathematically expressed as the ordinary differential equation

$$\frac{du}{dt} = ru(N - u), \quad (3.3.10)$$

where  $u$  is the population in some region,  $rN > 0$  is the *growth rate* and  $N > 0$  is the carrying capacity of this region. If we are however interested in the spatial distribution of individuals, then we must introduce the density of the population  $u(\mathbf{x}, t)$ , that is, the amount of individuals per unit of volume (or surface) and the conservation law will take the form

$$u_t + \operatorname{div} \Phi = ru(N - u). \quad (3.3.11)$$

As a first approximation, it is reasonable to assume that within the region the population obeys the Fick law, that is, individuals migrate from the regions of higher density to regions of lower density. This is not always true (think about people migrating to cities) and therefore the range of validity of Fick's law is limited; in such cases  $\Phi$  has to be determined otherwise. However, if we assume that the resources are evenly distributed, then Fick's law can be used with good accuracy. In such a case, Equation (3.3.11) takes the form

$$u_t - D\Delta u = ru(N - u), \quad (3.3.12)$$

which is called the Fisher equation. Note, that if the capacity of the region  $N$  is very large, then writing the Fisher equation in the form

$$u_t - D\Delta u = cu \left(1 - \frac{u}{N}\right),$$

where  $c = rN$ , we see that it is reasonable to neglect the small term  $u/N$  and approximate the Fisher equation by the linear equation (3.3.9) which describes a combination of diffusion-type migratory processes and exponential growth.

### 3.4 Pricing of options – the Black–Scholes model

In recent years the diffusion equation found a completely new area of applications – in the economical sciences. In this subsection we shall describe, without any claim to be rigorous, the derivation of the simplest model of this kind - the Black–Scholes model for pricing European options. Before we start with mathematics, we shall describe what this model is about.

#### Economical background

There are many kinds of financial markets in the world, but the most important are

- Stock markets;
- Bond markets;
- Currency markets;
- Commodity markets;
- Futures and option markets.

The idea behind the currency and commodities markets is clear - there either foreign currency or commodities, that is, physical assets such as gold, oil, electricity etc. are traded. The idea behind stocks (shares or equities) is, roughly speaking, that a company that needs to raise money (to build a new factory or develop a new product, for instance) can do it by selling shares in itself to investors. The company is then "owned" by its shareholders; if the company makes a profit, it may be paid out to shareholders as a dividend. If the company is taken over or otherwise wound up, the proceeds, if any, are distributed to shareholders. Shares thus have a value that reflects the views of investors about the likely future dividend payments and capital growth of the company; this value is quantified by the price at which they are bought and sold on stock exchanges.

As markets have become more sophisticated, more complex contracts than simple buy/sell trades have been introduced. Known as financial derivatives, that is, contracts whose payoff structure is contingent on that of another underlying commodity or security, they can give investors of all kinds a great range of opportunities to tailor their dealings to their investment needs.

We shall discuss here the example of option, which is one of the most common examples of a financial derivative.

**Definition 3.1** *A European call option is a contract with the following conditions: at a prescribed time in the future, known as the expiry date, the holder of the option may purchase a prescribed asset, known as the underlying asset, for a prescribed amount, known as the exercise price which is determined at the time of the issue of the call option contract.*

The word "may" in this description implies that for the holder of the option this contract is a *right* and not an *obligation*.

Since the holder of an option has a right with no obligation, the option must have some value. Moreover, it must be paid for at the time of opening the contract. The question is how much should be paid or, in general, what is the value of an option at any point in time.

To gain some intuition we consider the following example.

**Example 3.1** Assume that today's date is 10 March 2000. I have a call option that allows me to purchase on 10 June 2000 one X share for R10.

Let us consider two situations that may occur. If the price of X share on the 10 of June is R15, then I may exercise my option purchasing it for R10 and sell it immediately for R15. Thus I make instantaneous profit of R5 (assuming no transaction costs). However, it may happen that the share price is R5 on 10th June. Then it doesn't make sense to exercise the option and buy the share for R10 since I can buy it elsewhere for R5. In such a case I will not exercise my option and lose what I have paid for the option.

The price of the option should thus reflect the above discussion. If we assume for simplicity that the share can take only these two values with equal probability 0.5, then the expected profit to be made is

$$0.5 \times 0 + 0.5 \times 5 = R2.5$$

Thus it seems reasonable that the value of the option should be around R2.5.

Let us suppose now that I indeed paid R2.5 for the option. If I am lucky and the value of the share is R15, then my profit is  $R5 - R2.5 = R2.5$  which is 100% of the my investment. On the other hand, if the price is less than R10, then I lose R2.5, which is again 100% of the investment.

If, instead of the option, I purchased on 10th of March the share for R10, then my possible profit would be  $R15 - R10 = R5$ , and the possible loss  $R10 - R5 = R5$ , so in both cases only 50% of the investment. This shows that option prices respond in an exaggerated way (the so-called gearing principle) to changes in the underlying asset price which makes it very attractive for speculations. They can bring a fortune very easily, and ruin with an equal ease (e.g. the collapse of Barings Bank). ■

Let us identify now the mathematical problem related to options. Typical situation is that someone (possibly the owner of the asset) fixes the expiry date  $T$  and the exercise price  $E$  at this date (it is possible to give a

number of expiry dates and exercise prices). Suppose that we want to buy a number of options at a certain time  $t$  before the expiry date. What is the price  $V$  per option we should pay? In other words, we are trying to determine the value of the option at time  $t$ . Before we enter into mathematics, let us perform a qualitative analysis of the problem without any claim to be precise and exhaustive.

The preceding example shows that the greater the share price  $S(T)$  on the expiry date, the greater the profit. Unfortunately we don't know this share price in advance. However, it seems reasonable to assume that the higher the share price  $S(t)$  is now (and this is something we do know), then the higher the price of this asset is likely to be in the future. Thus, the value  $V(t)$  of the call option today depends on today's share price in a monotonically increasing way – the higher price  $S(t)$  today, the bigger value  $V(t, S)$ .

Implicit in this is that there is a significant time left for the option to expire. Just before the expiry date we now almost certainly the value of the asset and the value of the option is thus determined almost certainly as  $\max\{S(T) - E, 0\}$ . In fact, if the share price  $S(T)$  is smaller than the exercise price  $E$ , that is,  $S(T) - E < 0$ , then the value of the option is 0 – we can throw it away. On the other hand, if  $S(T) - E > 0$ , then this is the profit we can make buying and immediately selling the share, thus  $S(T) - E$  should be the price of the option.

Thus we see that the value of the option must depend on two independent variables - the asset price and the time left to expiry.

### Derivation of the option pricing model

The changes in shares' and options' values are unpredictable, random. This makes investing risky. To introduce an extra degree of safety into our financial strategy, we set up a portfolio consisting of a cash balance  $M$  which we invest in a riskless enterprise like government bonds, or simply put in the bank, a number  $\Delta$  of assets of value  $S$  each, and a number  $\delta$  of options on these assets. The value of the portfolio at time  $t$  is

$$P = M + \Delta S + \delta V. \quad (3.3.13)$$

Let us consider the change in the value of the portfolio after a small time interval  $dt$ . The cash balance changes due to:

- interest accrued on  $M$ ,
- buying (or selling)  $d\Delta$  of assets at the price  $S$ ,
- buying (or selling)  $d\delta$  of options at the price  $V$ ,

so that

$$dM = rMdt - (d\Delta)S - (d\delta)V \quad (3.3.14)$$

The value of share portion of the portfolio can change due to

- the acquisition of  $d\Delta$  shares at the price  $S$ ,
- the change in the share value  $dS$ ,

that is

$$d(\Delta S) = (d\Delta)S + \Delta dS. \quad (3.3.15)$$

and similarly the value of the option part of the portfolio changes due to

- the acquisition of  $d\delta$  options at the price  $V$ ,
- the change in the option value  $dV$ ,

that is

$$d(\delta V) = d\delta V + \delta dV. \quad (3.3.16)$$

Summing up Eqs. (3.3.14)–(3.3.16) and simplifying we obtain

$$dP = rMdt + \Delta dS + \delta dV. \quad (3.3.17)$$

To proceed we must introduce some external assumptions. The most important and difficult in this context is the one concerning the change of the asset price  $dS$ . The most common is to decompose the relative change  $dS/S$  into two parts. One is a predictable return similar to the return of money invested in a risk-free bank which we will assume to give the contribution

$$\mu dt,$$

to  $dS/S$ , where  $\mu$  is a measure of the average rate of growth of the asset price. We shall assume here that  $\mu$  is a known constant.

Had  $dS/S = \mu dt$ , then the value of  $S$  would have grown exponentially in time and the exact price  $S$  would have been known at any point of time. However, as we know this is not so – asset prices undergo random fluctuation which makes any precise forecasting impossible. This indicates the need to introduce another contribution to  $dS/S$  which would account for unexpected external effects. This contribution is random, that is, it cannot be expressed as a normal function. We write this contribution as

$$\sigma dX,$$

where this time  $dX$  is not an increment in a variable  $X$  but a random variable described to some extent in the remark below. The parameter  $\sigma$  is called the volatility. With this we obtain

$$dS = S\mu dt + S\sigma dX \quad (3.3.18)$$

**Remark 3.1** By saying that  $dX$  is a random variable we mean that even if the state  $X$  at some  $t$  is exactly known, the state at  $t + dt$ ,  $X + dX$ , is not exactly known – we only know that  $X + dX$  can take certain values with some probability, prescribed by the model.

In the modeling of stock prices we usually assume that  $dX = X(t + dt) - X(t)$  has (for any fixed  $dt$ ) the normal distribution with zero mean (expected value) and with the variance  $dt$ . For those who don't know anything about the probability theory, this means that possible values for  $dX$  will spread around zero and, if  $dt$  is small, then most of them will be quite close to zero.

In what follows we shall use the following relation

$$(dX)^2 = O(dt) \quad (3.3.19)$$

for small  $dt$ . This relation can be roughly explained by noticing that the expected value of  $(dX)^2$  is  $dt$ , and the variance (as  $dX$  has normal distribution) is  $2(dt)^2$ . Thus the possible deviations of  $(dX)^2$  from the expected value  $dt$  are of higher order in  $dt$  for  $dt \rightarrow 0$  and thus negligible on  $dt$ -level of considerations. ■

**Remark 3.2** The following figures present the evolution of the probability density of  $S(t)$  with  $\mu = 0.5$ ,  $\sigma = 0.2$  and  $S(0) = 5$ . Note that for small times the density is concentrated close to the value  $S(t) = S(0)e^{0.5t}$  that is the variance is rather small. For larger  $t$  the graph becomes more flat and the probability of  $S(t)$  being close to the deterministic value is much smaller. This shows that the longer time-span, the more difficult is to predict the share prices.

*Fig 5. The evolution of the probability density of the asset price  $S(t)$  which solves the equation (3.3.18).*

The two figures below show the contour lines of the probability density of Fig 5. and the "deterministic" evolution  $S(t) = 5e^{0.5t}$ . It is visible that the general trend follows the exponential curves but the actual values become more and more smeared around it. The darker regions correspond to the higher probability.

*Fig 6. The contour plot of the probability density of the asset price  $S(t)$  which solves the equation (3.3.18).*

*Fig 7. The "deterministic" evolution of the asset price  $S(t) = 5e^{0.5t}$ .*

Having done that we write formally the Taylor expansion of  $V = V(S, t)$ , which gives

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}dt^2 + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{\partial^2 V}{\partial t \partial S}dtdS + \text{higher order terms.} \quad (3.3.20)$$

We are interested in the contribution to  $dV$  of order  $dt$ , as our aim is to derive the formula for the derivative  $\partial V / \partial t$ .

Using Eq. (3.3.18) we obtain

$$\left(\frac{dS}{S}\right)^2 = \mu^2 dt^2 + 2\mu\sigma dXdt + \sigma^2 dX^2.$$

Using (3.3.19) we see that

$$\left(\frac{dS}{S}\right)^2 = \sigma^2 dt + o(dt). \quad (3.3.21)$$

Therefore by (3.3.18) and (3.3.21) we get from (3.3.20)

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt + o(dt). \quad (3.3.22)$$

Combining (3.3.22) with (3.3.17) we obtain

$$dP = rMdt + \Delta dS + \delta \left( \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt \right) + o(dt). \quad (3.3.23)$$

Next we observe that by balancing the number of assets and options according to the formula

$$\Delta = -\delta \frac{\partial V}{\partial S} \quad (3.3.24)$$

we can eliminate any randomness (represented here by  $dS$ ) from our portfolio, creating thus a risk-free portfolio. The procedure described above is known as the hedging. We are left then with the formula

$$dP = rMdt + \delta \left( \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt \right) + o(dt). \quad (3.3.25)$$

To complete the derivation of the Black-Scholes equation we shall introduce the third main assumption. In financial terms it amounts to saying that there is no arbitrage, which is the technical term for non-existence

of "free lunch". In our context this means that no risk-free portfolio can earn more than the (risk-free) bank deposit, that is, we can at most expect

$$dP = rPdt. \quad (3.3.26)$$

Combining (3.3.26) with (3.3.25), dividing by  $dt$ , and passing with  $dt$  to zero, we obtain

$$rP = rM + \delta \frac{\partial V}{\partial t} + \delta \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}.$$

However, by (3.3.13) and (3.3.24) we have

$$P - M = \Delta S + \delta V = \delta \left( -S \frac{\partial V}{\partial S} + V \right),$$

therefore we finally derived the famous Black-Scholes equation for the valuation of option prices

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r \left( V - S \frac{\partial V}{\partial S} \right). \quad (3.3.27)$$

This equation is similar to the diffusion equation with the "diffusion" coefficient  $D = -\sigma^2 S^2/2$  and drift term  $r(V - S\partial V/\partial S)$ . Later, when we will know how to solve diffusion equations, we shall see how to transform the Black-Scholes equation to the diffusion equation and, subsequently, how to solve it.

## 4 Waves

One of the cornerstones in the study of both linear and nonlinear PDEs is the wave propagation. A *wave* is a recognizable signal which is transferred from one part of the medium to another part with a recognizable speed of propagation. Energy is often transferred as the wave propagates, but matter may not be. We mention here a few areas where wave propagation is of fundamental importance.

Fluid mechanics (water waves, aerodynamics)

Acoustics (sound waves in air and liquids)

Elasticity (stress waves, earthquakes)

Electromagnetic theory (optics, electromagnetic waves)

Biology (epizootic waves)

Chemistry (combustion and detonation waves)

In this section we shall describe how the simplest equation describing the wave propagation can be derived. We also discuss other ways of deriving wave-type equations and give a survey of various processes modeled by them in various field of applied sciences. These equations involve usually second derivative of the unknown function with respect to time and can be thought of as multidimensional versions of the Newton second law of dynamics

$$mu'' = F,$$

where  $m$  is the (constant) mass of a body,  $u$  is its position (thus  $u''$  is the acceleration), and  $F$  is the force acting of the body.

### 4.1 Small vibrations of elastic string

Consider a perfectly flexible (not resisting to bending), elastic homogeneous string of length  $l$ , which undergoes relatively small transverse vibrations. Assume that at equilibrium the string occupies the interval  $[0, L]$ . Let  $u(x, t)$  denotes the displacement from the equilibrium position at time  $t$  and position  $x$ .

*Fig 3.3. The displacement  $u(x, t)$  of the string.*

Because the string is flexible, the tension  $\mathbf{T}(x, t)$  is directed tangentially along the string. Let  $T(x, t)$  be the magnitude of this tension vector. Let  $\rho$  be the density (mass per unit length) of the string - it is a constant because the string is homogeneous.

*Fig 3.4. Balance of the forces acting on the string.*

We shall write down Newton's law for the part of string between any two points  $x$  and  $x + \Delta x$  in  $[0, L]$ . The slope of the string at  $x$  is given by  $\tan \alpha = u_x(x, t)$  and therefore at any given point we have the following set of balancing equations:

$$\begin{aligned} \frac{T(x + \Delta x, t)}{\sqrt{1 + (u_x(x + \Delta x, t))^2}} - \frac{T(x, t)}{\sqrt{1 + (u_x(x, t))^2}} &= 0 \\ \frac{u_x(x + \Delta x, t)T(x + \Delta x, t)}{\sqrt{1 + (u_x(x + \Delta x, t))^2}} - \frac{u_x(x, t)T(x, t)}{\sqrt{1 + (u_x(x, t))^2}} &= \int_x^{x+\Delta x} \rho u_{tt}(t, s) ds, \end{aligned} \quad (3.4.1)$$

where the right hand side of the second equation gives the mass times acceleration integrated over this piece of string. Since we have assumed that there is no motion in the longitudinal direction, the right-hand side component of the first equation is equal to zero.

The set of equations (3.4.1) is something like the conservation law - it expresses a fundamental law of nature. Now come the simplifying assumptions, playing the role of constitutive relations. We say that we are interested in small, mild vibrations of the string; mathematically this can be expressed by requiring  $|u_x|^2 \approx 0$ . This assertion allows us to replace the nonlinear term  $\sqrt{1 + (u_x(x, t))^2}$  by 1. Thus from the first equation of (3.4.1) we obtain

$$T(x + \Delta x, t) = T(x, t)$$



for any  $x$  and  $t$ , which shows that  $T(x, t)$  is independent of  $x$ . We also assume that it is independent of  $t$  so that  $T(x, t) \equiv T = \text{const.}$  The second equation gives then

$$T(u_x(x+h, t) - u_x(x, t)) = \int_x^{x+h} \rho u_{tt}(s, t) ds$$

whence, dividing by  $h$ , passing with  $h$  to 0, and assuming all the time that all the operations are admissible, we obtain

$$T u_{xx} = \rho u_{tt}$$

which is usually written in the following form

$$u_{tt} = c^2 u_{xx} \quad (3.4.2)$$

where  $c = \sqrt{T/\rho}$  is an important parameter - as we shall see later, it is equal to the *wave speed*.

Similar considerations lead to the multidimensional wave equation which have the following form

$$u_{tt} = c^2 \Delta u \quad (3.4.3)$$

## 4.2 Variations of the wave equation

As was mentioned above, the wave equation describes a number of phenomena and can occur in various forms. For simplicity, we shall focus here on one dimensional versions.

- (a) If a significant air resistance is present, then we have the following damped wave equation:

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \quad (3.4.4)$$

where  $r$  is a parameter related to the strength of the air resistance.

- (b) If there is a transverse elastic force, then we obtain:

$$u_{tt} - c^2 u_{xx} + k u = 0, \quad (3.4.5)$$

where  $k$  is the elastic coefficient.

- (c) If there is an externally applied force, then the equation becomes inhomogeneous:

$$u_{tt} - c^2 u_{xx} = f(x, t), \quad (3.4.6)$$

where  $\rho f$  is the density of the external forces (force per unit length).

- (d) The relation between the current  $i$  and the voltage  $u$  in the transmission line is described by the following system of equations:

$$\begin{aligned} u_t &= -\frac{1}{C} i_x - \frac{G}{C} u \\ i_t &= -\frac{1}{L} u_x - \frac{R}{L} i, \end{aligned} \quad (3.4.7)$$

where  $C$  is the capacitance,  $L$  is the self-inductance,  $R$  is the resistance and  $G$  is the leakage coefficient (per unit length). We can write the system (3.4.7) in the form of a single equation:

$$u_{tt} = \frac{1}{CL} u_{xx} - \frac{RC + LG}{CL} u_t - \frac{GR}{CL} u. \quad (3.4.8)$$

In the ideal situation with zero resistance  $R$  and perfect insulation ( $G = 0$ ) we see that Eq. (3.4.8) becomes the wave equation.

## 5 Schrödinger equation

Quantum mechanics is a highly abstract branch of physics and is based on a certain system of axioms (statements accepted without proof). These axioms are to great extent phenomenological, that is, they cannot be derived from any first principles and their justification stems from the fact that calculations based on them are in very good agreement with the experimental data.

The full set of axioms of quantum mechanics involves an abstract mathematics, therefore we shall present here some basic ideas which are necessary to understand physics hidden behind the formal calculations of the next two sections.

One of the basic concepts in quantum mechanics is that of an operator. We have introduced this idea earlier by referring to the linear algebra course. We shall need some other notions of this origin.

Let  $A$  denote any linear operator. In the theory of operators the important role is played by the numbers  $\lambda$  for which the equation

$$Au - \lambda u = f \quad (3.5.1)$$

has no solutions at all or, if the solution exists, it is not unique in some admissible class of functions. The set of all such numbers  $\lambda$  is called the *spectrum* of the operator  $A$ . We divide the spectrum into two subsets. We call  $\lambda$  the eigenvalue and say that it belongs to the *discrete spectrum* if the equation

$$Au - \lambda u = 0. \quad (3.5.2)$$

has a non-zero solution. Such solutions are called eigenfunctions of  $A$ .

Consider for example the equation

$$\frac{du}{dt} - \lambda u = 0.$$

For any  $\lambda$  this equation has a non-zero solution  $u(t) = e^{\lambda t}$ , therefore any real number is an eigenvalue of the operator  $A = d/dt$ .

The remaining part of the spectrum is called the continuous spectrum.

The basic assumptions of quantum mechanics can be summarized as follows.

- (a) Any physical system is completely described by a function  $\psi$ , called the *state vector* or the *wave function* which is normalized by

$$\int_D |\phi(\mathbf{x})|^2 d\mathbf{x} = 1 \quad (3.5.3)$$

where  $D$  is the spatial domain in which the system is considered.

- (b) To every observable physical quantity (position, momentum, energy etc.) there corresponds a certain operator and the only allowed physical results of measurement of this observable are the elements of the spectrum of the corresponding operator.

In quantum mechanics we cannot expect any measurement with absolute accuracy, only with certain probability. We can improve our information of the system by making a series of observations and taking their average.

- (c) If the system is described by the state vector  $\psi$  then the expectation value of the observable corresponding to the operator  $A$  can be calculated by

$$\int_D A\psi(\mathbf{x})\bar{\psi}(\mathbf{x})d\mathbf{x}, \quad (3.5.4)$$

where  $\bar{\psi}$  denotes the complex conjugate of  $\psi$ .

For example, the position of the observable is given by the multiplication operator:  $A\psi = \mathbf{x}\psi$ , and the momentum operator is defined by the gradient operator:  $A\psi = -i\hbar\nabla\psi$ , where  $h$  is the Plank constant.

The time evolution of the system is subject to the following axiom.

- (d) For any system, there exists an operator  $H$ , called the Hamiltonian or the energy operator such that the time evolution of the state vector is described by the Schrödinger equation

$$-i\hbar\psi_t = H\psi. \quad (3.5.5)$$

Here  $i$  is the imaginary unit and  $\hbar = h/2\pi$ .

In what follows we concentrate our discussion on a particular case of the Hamiltonian given by

$$H\psi(\mathbf{x}) = \Delta\psi + V(\mathbf{x})\psi(\mathbf{x}) \quad (3.5.6)$$

which gives the energy of a single charged particle (e.g., electron) in the external field with potential  $V$ . We also assume that  $D = \mathbb{R}^3$ , that is, we consider only the whole-space case.

Note that due to the presence of the imaginary unit in (3.5.5), any solution to this equation has no direct physical meaning. However, it can be proved that  $|\psi(\mathbf{x}, t)|^2$  is the density of the probability of finding the electron at the position  $\mathbf{r}$  at time  $t$ . The normalizing condition (3.5.3) tells us simply that the probability of finding the electron somewhere in the space is equal to 1.

Of particular importance are eigenfunctions and eigenvalues of the operator  $H$ . In other words, if it happens that  $\psi(\mathbf{x}, t)$  is an eigenvector of  $H$  corresponding to some  $\lambda$  then

$$H\psi(\mathbf{x}, t) = \lambda\psi(\mathbf{x}, t)$$

and the Schrödinger equation (3.5.5) takes the form

$$-i\hbar\psi_t(\mathbf{x}, t) = \lambda\psi(\mathbf{x}, t),$$

which is an elementary ordinary differential equation with the solution

$$\psi(\mathbf{x}, t) = e^{-\frac{i\lambda t}{\hbar}} \psi(\mathbf{x}, 0).$$

Hence

$$|\psi(\mathbf{x}, t)|^2 = |\psi(\mathbf{x}, 0)|^2,$$

and we see that the system is not changing (the probability of finding the electron at  $\mathbf{x}$  is constant). That is why the eigenfunctions of  $H$  are often called stationary states. Physically this means that the only stable states of the physical system are described by the eigenfunctions of the Hamiltonian of the system. Later in the course we shall see the consequences of this observation for the hydrogen atom.

## 6 Stationary problems

Let us assume that in any of the examples above the process is independent of time. For example, for the heat equation it is physically natural that if the sources are independent of time, then after an initial period the temperature will stabilize and will not change in time (though of course it can change from point to point). Then we say that we have a *stationary distribution* of the temperature. In the case of the wave equation, if the external force is constant, then the string or membrane will not vibrate - it will stay in a deformed shape. Then we have  $u_t \equiv 0$  and/or  $u_{tt} = 0$  and both the diffusion and the wave equations reduce to the equation

$$\Delta u = f, \quad (3.6.7)$$

which is called the *Poisson* equation. If there are no sources ( $f \equiv 0$ ), then the resulting equation is called the *Laplace* equation.

These equations appear also in many other cases. Consider for example a steady fluid flow, that is, such that at each point of the fluid the velocity  $\mathbf{v}$  does not change in time. Assume further that the flow is irrotational (no eddies) and there are no sources nor sinks (think about a slow tranquil river). Then the assumption of irrotationality gives  $\text{curl } \mathbf{v} = 0$  and from the calculus we know that in such a case the velocity field is a potential vector field (provided the region is simply connected). Therefore  $\mathbf{v} = -\nabla\phi$  for some  $\phi$ . On the other hand, the assumption of no sinks/sources is translated mathematically into the condition  $\text{div } \mathbf{v} = 0$ . Combining these two equations we obtain

$$\text{div } \nabla\phi = \Delta\phi = 0,$$

which is the Laplace equation.

Similar argument, based on Maxwell's equations (or first principles) shows that in electrostatics the electric field  $\mathbf{E}$  has a potential  $\phi$ , and the sources are given by the charge density  $\rho$  according to the formula  $\text{div } \mathbf{E} = 4\pi\rho$ . Thus for the electrostatic potential we obtain the Poisson equation

$$\Delta\phi = -4\pi\rho.$$

Another celebrated example comes from complex analysis. If we consider an analytic function of a complex variable:

$$f(z) = \text{Re } u(z) + i\text{Im } u(z) = u(x + iy) + iv(x + iy),$$

where  $i$  is the imaginary unit, and  $z = x + iy$ , then the Cauchy-Riemann equations give  $u_x = v_y$  and  $u_y = -v_x$  which, upon differentiation, gives

$$\Delta u = 0, \quad \Delta v = 0.$$

## 7 Initial and boundary conditions

As we saw earlier, a partial differential equation can have many solutions. The same situation occurs also for ordinary differential equations - then the solution typically depends on as many constants as is the order of the equation. To be able to select a unique solution of an ODE we have to impose an appropriate number of side conditions which typically are the initial conditions specified at a certain point on the real line.

However, we have seen that the solution of a second order PDE depends in general on two arbitrary function so that the specification of the side conditions will be different. Below we shall describe typical choices of such conditions which are motivated mainly by physical considerations.

Most equations which we have described above are supposed to reflect certain physical processes and due to this fact the variables were explicitly divided into time and spatial variable (of course there is rather no mathematical justification for such separation but, as we shall see later, it makes sense also from the mathematical point of view). Then it is clear that the process will take place in some spatial domain, say  $\Omega \subset \mathbb{R}^3$  over some time  $t \in I \subset (t_0, t_1)$  ( $-\infty < t_0 < t_1 \leq +\infty$ ). Thus our equation is to be considered in the region  $\Omega_T = \Omega \times I$ . If we keep to our physical intuition, the process should have started somehow, which means that we must prescribe some initial state. In the case of the wave equation we have also the second derivative in time so, as in ODEs, we expect that we shall need also the initial speed. These conditions are called the *initial conditions*:

$$u(\mathbf{x}, t_0) = \psi(\mathbf{x}) \quad \text{in } \Omega \tag{3.7.8}$$

for equations of first order in time, and

$$u(\mathbf{x}, t_0) = \psi(\mathbf{x}), \quad u_t(\mathbf{x}, t_0) = \phi(\mathbf{x}) \quad \text{in } \Omega \tag{3.7.9}$$

for equations of second order in time, where  $\phi$  and  $\psi$  are prescribed functions.

Initial conditions are usually not sufficient to determine a unique solution. If we think about the heat distribution in the container, mathematically represented by the region  $\Omega$ , it is clear that the distribution of heat inside will be dictated by the conditions on the boundary. For equations of second order in spatial variables, such as heat and wave equations, we have three typical *boundary conditions*

- (D)  $u$  is specified at the boundary  $\partial\Omega$  ("*Dirichlet condition*"),
- (N) the normal derivative  $\partial u/\partial n = \nabla u \cdot \mathbf{n}$  is specified at the boundary  $\partial\Omega$  ("*Neumann condition*"),
- (R) the combination  $\partial u/\partial n + au$  is specified at the boundary  $\partial\Omega$  ("*Robin conditions*"),

where  $a$  is a given function of  $\mathbf{x}, t$ . Each condition is to hold for each  $t \in I$  and  $\mathbf{x} \in \partial\Omega$ .

The boundary conditions are usually written in the form of equation to hold at the boundary. For instance (N) can be written as

$$\frac{\partial u}{\partial n} = g(\mathbf{x}, t),$$

where  $g$  is a given function (at least known on  $\partial\Omega \times I$ ) and is called the *boundary datum*. Any of these boundary conditions is called *homogeneous* if the specified function  $g$  vanishes on the boundary.

**Remark 7.1** Note that if  $\Omega = (a, b) \subset \mathbb{R}$ , then  $\partial\Omega = \{a\} \cup \{b\}$  and at the right endpoint we have  $\partial u/\partial n(b, t) = u_x(b, t)$  and at the left end point we have  $\partial u/\partial n(a, t) = -u_x(a, t)$ .

To make the boundary conditions more understandable, we shall put some physical meaning into them.

## 7.1 The vibrating string

If the string is held fixed at both ends  $x = 0, l$ , then  $u(0, t) = u(l, t) = 0$ , then we have the homogeneous Dirichlet boundary conditions. They can also move at a prescribed fashion, according to, say,  $u(0, t) = \sin t$  etc. Then we have non homogeneous Dirichlet conditions.

On the other hand, if a given force  $\nu(t)$  is acting at, say, the right endpoint of the string in the transversal direction then, using the same argument as when we derived the wave equation, we obtain

$$T \frac{\partial u}{\partial n}(l, t) = T u_x(l, t) = \nu(t).$$

In particular, if the endpoint can move freely, then we have homogeneous Neumann condition

$$u_x(l, t) = 0.$$

Finally, if the endpoint is attached to a elastic element (spring or rubber) which obeys Hook's law (force proportional to the displacement), then instead of  $\nu$  above we would have  $-au$  and  $u$  would satisfy the Robin condition

$$u_x + (a/T)u = 0.$$

## 7.2 The heat equation

The interpretation of all three boundary conditions for the heat equation is slightly different. The Dirichlet condition is easy to explain - we keep the boundary of the body in a fixed constant temperature (e.g. by submerging it in the large container with melting ice - then we will have homogeneous boundary conditions).

To explain Neumann and Robin conditions we should remember that due to Fourier's law (or Fick's law for diffusion) the flux is proportional to the normal derivative. Therefore the homogeneous Neumann condition tells us that nothing escapes through the boundary, that is, the domain is completely insulated.

Assume now that our body is immersed in a reservoir with known temperature  $g$ , then according to Fourier's law, the flux from the domain into the boundary must be proportional to the difference of temperatures. Therefore in this case we arrive at the Robin condition

$$\frac{\partial u}{\partial n}(\mathbf{x}, t) = a(u(\mathbf{x}, t) - g(t))$$

for all  $\mathbf{x} \in \partial\Omega$ .

### 7.3 Conditions at infinity

Though it is rather unphysical, now and then people are solving problems in the whole space  $\mathbb{R}^3$ . The reason for this is that such problems are easier to solve in a closed form, providing on one hand benchmark solutions, and on the other hand, in many cases, approximate solutions. In such cases physics usually provide some indication as to how the solution should behave at infinity. This could be tricky, but a good example is rendered by the Schrödinger equation in the whole space. The solution  $u$  is such that  $|u|^2$  is the probability density of finding the particle. Therefore the solution must satisfy the normalizing condition  $\int_{\mathbb{R}^3} |u|^2 d\mathbf{x} = 1$  (expressing the property that the probability of finding the particle somewhere in the space is equal to 1). The finiteness of this integral means that the solution must sufficiently quickly decay to zero at infinity.

### 7.4 Side condition for the Black-Scholes equation

In Subsection 3.4 we have derived the Black-Scholes equation for pricing of the European call option. The equation has the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r \left( V - S \frac{\partial V}{\partial S} \right),$$

where  $V(t, S)$  is the price of the option at time  $t$  for a given asset price  $S$ . As we noted before, this equation is similar to the diffusion equation, that is, it is of the first order in time and of the second order in the other variable. Thus, we need one initial condition and some boundary conditions.

To derive this conditions let us return to the discussion at the end of Subsubsection 3.4 where we observed that at the expiry time  $T$  the value of the option must be

$$V(S, T) = \max\{S - E, 0\}. \quad (3.7.10)$$

This gives the desired condition. Certainly, it is not the proper initial condition, but with the Black-Scholes equation we move backward in time: from the expiry date to an unspecified time earlier on so that there is no prescribed initial time but the prescribed final time  $T$  and this explains why we impose the final condition (3.7.10) rather than the initial one.

As we shall see later this condition agrees perfectly with the structure of the equation.

Next we have to specify the boundary conditions, that is, we have to prescribe the behaviour of  $V$  for two specified values of  $S$ . To do this let us observe that if  $S$  becomes zero at some time, then it will stay at this level for any time. Clearly, the option on this share is then worthless, that is, we arrived at the reasonable boundary condition

$$V(0, t) = 0, \quad t \leq T. \quad (3.7.11)$$

At the other end, if the asset price becomes very large then almost certainly we shall exercise the option. Moreover, the exercise price (which is fixed beforehand) becomes negligible and the value of the option becomes comparable with the value of the asset. Thus, we can write the condition

$$\lim_{S \rightarrow +\infty} V(S, t)/S = 1, \quad t \leq T. \quad (3.7.12)$$

Combining all these we obtain the following initial-boundary value problem for the Black-Scholes equation

$$\begin{aligned}\frac{\partial V}{\partial t} &= -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r \left( V - S \frac{\partial V}{\partial S} \right), \\ V(S, T) &= \max\{S - E, 0\}, \quad \text{for all } S > 0, \\ V(0, t) &= 0, \quad \text{for all } t \leq T, \\ \lim_{S \rightarrow +\infty} V(S, t)/S &= 1, \quad \text{for all } t \leq T.\end{aligned}\tag{3.7.13}$$

## 8 Well-posed problems

Given an initial-boundary value problem for a PDE, we are faced with finding its solution. Unfortunately, especially in real life problems, we have to revise our understanding of what it means to solve a problem, as for a majority of PDE problems it is impossible to write down a solution in a closed form. So, in the classical sense, PDE problems cannot be solved.

Having realized this, there are in principle two options. First, which can be called a pure mathematician option, is to say: "tough luck" and look only for problems for which the solution can be found. However, practically the whole technology and science depends upon our ability to find solutions to more and more complicated PDEs. This is done by solving them approximately, using numerical methods. It is however important to understand the fundamentals on which the approximate methods stand. Before we embark on solving a problem numerically we have to answer the three basic questions:

- (1) *Existence*: Is there at least one solution of the given problem?
- (2) *Uniqueness*: Is this solution unique?
- (3) *Stability*: Does the solution depend in a stable way on the data? In other words, whether the solution will change only a little, if we change the data a little?

If it can be proved that a given problem has all these three properties, then such a problem is called *well-posed*.

For a physical problem modeled by PDE a scientist usually tries to formulate realistic auxiliary conditions which all together make a well-posed problem. The mathematician tries to prove whether the problem is or is not well-posed. Such a proof in some sense validates the modeling process. If too few conditions are imposed, then there may be more than one solution (*nonuniqueness*) and the problem is called underdetermined. On the other hand, if there are too many side conditions, then there may be no solution at all (*nonexistence*) and the problem is called overdetermined.

The stability property is normally required in models of physical problems and reflects the fact that we can never measure data with absolute precision and we expect (sometimes wrongly) that small measurement errors will introduce only minor distortions of the solution.

Similar reasoning applies to approximating the solution due to the fact that we feed the computer with approximate data.

As an example, let us consider the heat equation. If we know the initial temperature of the body and the temperature of its boundary, then it is natural to expect that we are able to determine the temperature of the body at any future time. This can be proved mathematically and such a problem is well-posed. On the other hand, consider the reverse problem: given the temperature of the body at a given instant, can we tell what was the temperature of it a few minutes ago? Physically, the temperature tends to spread uniformly, so after some time we expect the temperature to be constant throughout the body, independently of the initial distribution, so it is in general impossible to reconstruct the initial temperature. This would contradict the Second Law of Thermodynamics, and we shall prove it later mathematically.

Consider two other examples of ill-posed problems.

**Example 8.1** Consider the Neumann problem for the stationary heat equation:

$$\begin{aligned}\Delta u &= 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 & \text{on } \partial\Omega\end{aligned}$$

It is clear that any  $u = \text{const}$  is a solution to this problem, hence we don't have uniqueness and the problem is ill-posed. Fortunately, this is also physically natural and can be easily remedied.

**Example 8.2** Consider the Laplace equation

$$u_{xx} + u_{yy} = 0 \tag{3.8.14}$$

in the strip  $D = \{-\infty < x < +\infty, 0 < y < \infty\}$  subject to the initial conditions:

$$u(x, 0) = g(x), \quad u_y(x, 0) = h(x).$$

By direct calculations, the functions

$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin x \sinh ny$$

satisfy Eq. (3.8.14) and  $u_n(x, 0) = 0$ ,  $\frac{\partial u_n}{\partial y}(x, 0) = e^{-\sqrt{n}} \sin x$ . Thus the initial data tend to 0 with  $n \rightarrow \infty$ . However, if  $y \neq 0$ , then  $u_n(x, y)$  does not converge to zero. Thus, for arbitrarily small initial values we may have arbitrarily large (in magnitude) solution. Thus the stability condition (3) is violated. This problem is more complicated and shows that imposing "dynamic" (initial) conditions for "stationary" (Laplace) equation leads to ill-posed problems.

## 9 Canonical forms of second order equations in two independent variables

In the discussion above we have concentrated mainly on the Laplace, wave and heat equations. Here we will see that these equations are somewhat typical, though very distinct from each other.

### 9.1 Classification of second order equations

Let us consider the general linear second order PDE in two variables with constant coefficients:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0 \tag{3.9.1}$$

The part  $a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy}$  containing the highest (second) derivatives of the equation is called the *principal part* of the equation; the remaining part is often referred to as the *lower order terms*. The main properties of the equation are determined by its principal part. It follows that, depending on the coefficients, there are only three possible classes of the equations. We have the following theorem.

**Theorem 9.1** *By a linear transformation of the independent variables, the principal part of Equation (3.9.1) can be reduced to one of the three forms, as follows.*

(i) Hyperbolic case: If  $a_{12}^2 > a_{11}a_{22}$ , then the principal part can be transformed into

$$u_{xx} - u_{yy},$$

or

$$u_{xy}.$$



(ii) Parabolic case: If  $a_{12}^2 = a_{11}a_{22}$ , then the principal part can be transformed into

$$u_{xx}.$$

(iii) Elliptic case: If  $a_{12}^2 < a_{11}a_{22}$ , then the principal part can be transformed into

$$u_{xx} + u_{yy}.$$

**Proof.** Consider equation (3.9.1). If  $a_{11} = a_{22} = 0$ , then the equation is already in one of the hyperbolic canonical forms. Therefore we can assume that, say,  $a_{11} = 1$ . We perform the arbitrary linear change of variables:

$$\begin{aligned} r &= \alpha x + \beta y, \\ s &= \gamma x + \delta y. \end{aligned} \quad (3.9.2)$$

Denote

$$u(x, y) = u(x(r, s), y(r, s)) = v(r, s)$$

and differentiate using the chain rule

$$u_x = v_r r_x + v_s s_x = v_r \alpha + v_s \gamma, \quad u_y = v_r r_y + v_s s_y = v_r \beta + v_s \delta.$$

Second differentiation gives

$$\begin{aligned} u_{xx} &= v_{rr} \alpha^2 + 2v_{rs} \alpha \gamma + v_{ss} \gamma^2, \\ u_{xy} &= v_{rr} \alpha \beta + v_{rs} (\alpha \delta + \beta \gamma) + v_{ss} \gamma \delta, \\ u_{yy} &= v_{rr} \beta^2 + 2v_{rs} \beta \delta + v_{ss} \delta^2. \end{aligned} \quad (3.9.3)$$

Substituting to (3.9.1) we obtain

$$\begin{aligned} &v_{rr}(\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2) \\ &+ 2v_{rs}(\alpha\gamma + a_{12}(\alpha\delta + \beta\gamma) + a_{22}\beta\delta) \\ &+ v_{ss}(\gamma^2 + 2a_{12}\gamma\delta + a_{22}\delta^2) \\ &+ v_r(a_1\alpha + a_2\beta) + v_s(a_1\gamma + a_2\delta) + a_0v = 0. \end{aligned} \quad (3.9.4)$$

Next we shall try to adjust parameters  $\alpha, \beta, \gamma$  and  $\delta$  to simplify Equation (3.9.4) by making some coefficients equal to zero. Let us observe that denoting  $t = \alpha/\beta$  the coefficient at  $v_{rr}$  (and for that matter at  $v_{ss}$ ) will be zero if  $t$  solves the following quadratic equation

$$t^2 + 2a_{12}t\beta + a_{22} = 0. \quad (3.9.5)$$

The discriminant of this equation is equal to  $4(a_{12}^2 - a_{22})$  and denoting  $D = a_{12}^2 - a_{22}$  we see that we have three usual cases: a)  $D > 0$ ,  $D = 0$  and  $D < 0$ . Let us start with the hyperbolic case a).

Let  $D > 0$ ; then Equation (3.9.5) has two distinct real solutions

$$t_{1,2} = -a_{12} \pm \sqrt{D}$$

We define the parameters of (3.9.2) to be  $\alpha = t_1, \beta = 1, \gamma = t_2, \delta = 1$ . Then (3.9.2) defines a non-singular change of variables (that is,  $(x, y)$  and  $(r, s)$  are related to each other in a one-to-one way) as the determinant is equal to  $t_1 - t_2 = 2\sqrt{D} \neq 0$ . Hence, if we define such a change of variables, we see that the coefficients multiplying pure derivatives  $v_{rr}$  and  $v_{ss}$  vanish. On the other hand, calculating the coefficient multiplying the mixed derivative we obtain

$$2(t_1 t_2 + a_{12}(t_1 + t_2) + a_{22}) = -2(a_{12}^2 - a_{22} + D) = -4D \neq 0$$

and the equation takes the form

$$-4Dv_{rs} + (a_1t_1 + a_2)v_r + (a_1t_2 + a_2)v_s + v = 0,$$

and, upon dividing by  $-D$  we obtain the principal part in the second form of the hyperbolic case. To get the first form one has to further change the variables to  $r' = r + s, s' = r' - s'$  and use the second equation of (3.9.3). In this case  $\alpha = 1, \beta = 1, \gamma = 1$  and  $\delta = -1$  and we obtain that

$$v_{rs} = v_{r'r'} + v_{r's'}(-1 + 1) - 1v'_{s's'} = v'_{r'r'} - v'_{s's'},$$

and we have the wave equation form.

Let now  $D = 0$ . Then we have only one root of (3.9.5) and therefore we cannot perform the full change of variables. However, putting  $r = t_1x + y, s = x$  we reduce the coefficient at  $v_{rr}$  to zero. The coefficient at  $v_{rs}$  is now given by

$$2(t_1 + a_{12}) = -2(-a_{12} + a_{12}) = 0$$

and the coefficient at  $v_{ss}$  is equal to 1. Thus the equation takes the form

$$v_{ss} + (a_1t_1 + a_2)v_r + a_1v_s + v = 0$$

which has the principal part in the canonical form of the parabolic case.

Let us consider the last case with  $D < 0$ . Then the roots are complex

$$t_{1,2} = -a_{12} \pm i\sqrt{-D} = \eta \pm i\omega,$$

where  $i$  is the imaginary unit, and since we don't want to introduce complex variables (the notion of differentiability is then much more complicated), we have to redefine the change of variables. If we were to follow the hyperbolic case, we would have

$$r = (\eta + i\omega)x + y, \quad s = (\eta - i\omega)x + y.$$

To get rid of the complex part, we introduce

$$r' = \frac{r + s}{2} = \eta x + y, \quad s' = \frac{r - s}{2i} = \omega x$$

The determinant of this transformation is given by  $-\omega \neq 0$ , so the transformation is invertible. Using again (3.9.4) we get

$$\begin{aligned} v_{r'r'}(\eta^2 + 2a_{12}\eta + a_{22}) + 2v_{r's'}(\eta\omega + a_{12}\omega) + v_{s's'}\omega^2 + v_{r'}(a_1\eta + a_2) + v_s'a_1\omega + a_0v \\ = -D(v_{r'r'} + v_{s's'}) + v_{r'}(a_1\eta + a_2) + v_s'a_1\omega + a_0v = 0 \end{aligned} \quad (3.9.6)$$

where we used  $\eta^2 + 2a_{12}\eta + a_{22} = -a_{12}^2 + a_{22} = \omega^2 = -D$  and  $\eta\omega + a_{12}\omega = \omega(-a_{12} + a_{12}) = 0$ . The theorem is now proved. ■

**Remark 9.1** Analogous consideration can be carried out even when coefficients  $a_{11}, a_{22}, a_{12}$  are functions depending on  $x, y$ . However, the technicalities then are much more difficult as we will not have a linear change of variables like in (3.9.2), but a functional one of the form  $r = \phi(x, y), s = \psi(x, y)$  where  $\phi$  and  $\psi$  have to be determined from a system of differential equations derived in a way similar to that leading to (3.9.5). We shall present an example of such calculations in the subsection below in which we shall show that the Black-Scholes equation can be transformed into the diffusion equation.

## 9.2 Reduction of the Black-Scholes equation to the diffusion equation

In this subsection we shall demonstrate an application of a generalization of Theorem 9.1, which results in the reduction of the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r \left( V - S \frac{\partial V}{\partial S} \right). \quad (3.9.7)$$

to the diffusion equation. As the first step we shall change the direction of time so that the final condition (3.7.10) will become the proper initial condition. Thus we put

$$\tau = T - t,$$

and define

$$V(S, t) = \bar{V}(S, \tau).$$

Differentiating, we obtain immediately  $V_S = \bar{V}_S$ ,  $V_{SS} = \bar{V}_{SS}$  but  $V_t = -\bar{V}_\tau$ . Therefore (3.9.7) will transform into

$$\frac{\partial \bar{V}}{\partial \tau} = -r\bar{V} + rS \frac{\partial \bar{V}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \bar{V}}{\partial S^2}. \quad (3.9.8)$$

In the next step we shall use a standard trick to discard the term  $rV$ . To do this we change the unknown function  $\bar{V}$  according to

$$\bar{V}(S, \tau) = e^{-r\tau} \tilde{V}(S, \tau). \quad (3.9.9)$$

We see that with this change we get the equation

$$e^{-r\tau} \frac{\partial \tilde{V}}{\partial \tau} - re^{-r\tau} \tilde{V} = -re^{-r\tau} \tilde{V} + re^{-r\tau} S \frac{\partial \tilde{V}}{\partial S} + e^{-r\tau} \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \tilde{V}}{\partial S^2}. \quad (3.9.10)$$

which, after dividing by  $e^{r\tau}$  and canceling equal terms, takes the form

$$\frac{\partial \tilde{V}}{\partial \tau} = rS \frac{\partial \tilde{V}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \tilde{V}}{\partial S^2}. \quad (3.9.11)$$

In the final step we set

$$\tilde{V}(S, \tau) = G(x, y) = G(x(S, \tau), y(S, \tau)),$$

where  $x = x(S, \tau)$  and  $y = y(S, \tau)$  are the new variables (like (3.9.2), only non-linear) which we are to determine. The procedure is similar to that of Theorem 9.1 but technically more involved.

The application of the Chain Rule gives

$$\begin{aligned} \tilde{V}_S &= G_x x_S + G_y y_S, \\ \tilde{V}_\tau &= G_x x_\tau + G_y y_\tau, \\ \tilde{V}_{SS} &= G_{xx}(x_S)^2 + 2G_{xy}x_S y_S + G_{yy}(y_S)^2 + G_x x_{SS} + G_y y_{SS}. \end{aligned}$$

Substituting the above equations to (3.9.11) and rearranging terms, we obtain

$$\begin{aligned} &\frac{1}{2}\sigma^2 S^2 (x_S)^2 G_{xx} + \sigma^2 S^2 x_S y_S G_{xy} + \frac{1}{2}\sigma^2 S^2 (y_S)^2 G_{yy} \\ &+ G_x \left( \frac{1}{2}\sigma^2 S^2 x_{SS} - x_\tau + rS x_S \right) + G_y \left( \frac{1}{2}\sigma^2 S^2 y_{SS} - y_\tau + rS y_S \right) = 0. \end{aligned}$$

If we want the above equation to be in the form

$$G_y - G_{xx} = 0, \quad (3.9.12)$$

we have to find functions  $x(S, t), y(S, t)$  satisfying

$$\begin{aligned} \sigma^2 S^2 x_S y_S &= 0, \\ \frac{1}{2}\sigma^2 S^2 (y_S)^2 &= 0, \\ \frac{1}{2}\sigma^2 S^2 x_{SS} - x_\tau + rS x_S &= 0, \\ \frac{1}{2}\sigma^2 S^2 (x_S)^2 + \frac{1}{2}\sigma^2 S^2 y_{SS} - y_\tau + rS y_S &= 0. \end{aligned} \quad (3.9.13)$$

From (3.9.13) we see that  $y$  cannot depend on  $S$ , thus  $y = y(\tau)$ . With this observation the above system of equations reduces to

$$\frac{1}{2}\sigma^2 S^2 x_{SS} - x_\tau + r S x_S = 0, \quad (3.9.14)$$

$$\frac{1}{2}\sigma^2 S^2 (x_S)^2 - y_\tau = 0. \quad (3.9.15)$$

From (3.9.15) and  $y = y(\tau)$  we see that  $Sx_S$  does not depend on  $S$ . Thus

$$Sx_S = \beta(\tau)$$

with for some function  $\beta$ , and we have

$$x(S, \tau) = \beta(\tau) \ln S + \gamma(\tau).$$

for some function  $\gamma$  (of  $\tau$  only). Substituting this equation to (3.9.14) we obtain

$$-\frac{1}{2}\sigma^2 \beta(\tau) + r\beta(\tau) - \beta'(\tau) \ln S - \gamma'(\tau) = 0.$$

From the above it follows that  $\beta = \text{const}$  (otherwise dividing by  $\beta'(\tau)$  we would have that the function  $\ln S$  of  $S$  alone is equal to a function of  $\tau$  only). Clearly we can put  $\beta = 1$  and then  $\gamma$  turns out to be a linear function of  $\tau$  which can be taken as

$$\gamma(\tau) = \left(r - \frac{1}{2}\sigma^2\right) \tau.$$

Then from (3.9.15) we obtain that

$$y(\tau) = \frac{1}{2}\sigma^2 \tau,$$

where again we discarded the constant of integration. Therefore we obtained the transformation of variables

$$\begin{aligned} x(S, \tau) &= \ln S + \left(r - \frac{1}{2}\sigma^2\right) \tau, \\ y(\tau) &= \frac{1}{2}\sigma^2 \tau. \end{aligned} \quad (3.9.16)$$

Note that the above transformation of variables is invertible with the inverse given explicitly by

$$\begin{aligned} \tau &= \frac{2}{\sigma^2} y, \\ S &= e^{x - \left(\frac{2r}{\sigma^2} - 1\right) y}. \end{aligned} \quad (3.9.17)$$

To summarize, we have found a way to transform the Black-Scholes equation (3.9.7) into the diffusion equation (3.9.12). As we shall see later, diffusion equation of this form is relatively easy to solve. Suppose that we know the solution  $G(x, y)$  to (3.9.12). To obtain the solution to the Black-Scholes equation we retrace the above consideration and obtain the function  $V(S, t)$  solving (3.9.7) in the following form

$$\begin{aligned} V(S, t) &= \bar{V}(S, T-t) = e^{-r(T-t)} \tilde{V}(S, T-t) = e^{-r(T-t)} G(x(S, T-t), y(S, T-t)) \\ &= e^{-r(T-t)} G\left(\ln S + \left(r - \frac{1}{2}\sigma^2\right)(T-t), \frac{1}{2}\sigma^2(T-t)\right) \end{aligned} \quad (3.9.18)$$

To complete the reduction we must write the final and boundary condition of (3.7.13) as conditions imposed on the function  $G$  in variables  $x$  and  $y$ . let us start with the final condition (3.7.10). From (3.9.16) we see that for  $t = T$  we have  $y = 0$  and  $x = \ln S$  so that the final condition in  $T$  becomes the initial condition in  $y$ . Moreover,

$$\max\{S - E, 0\} = V(S, T) = G(\ln S, 0),$$

and changing the  $S$ -variable into  $x$  we obtain

$$G(x, 0) = \max\{e^x - E, 0\},$$

which is the proper initial condition for  $G$ .

Let us turn now to the boundary conditions. If  $S \rightarrow 0$ , then from (3.9.16) we see that  $x \rightarrow -\infty$ , therefore one of the boundary conditions for  $G$  must be related to its behaviour as  $x \rightarrow -\infty$ . From (3.9.18) we have immediately

$$\lim_{x \rightarrow -\infty} G(x, y) = 0.$$

Let us turn our attention to the behaviour of  $G$  as  $x \rightarrow +\infty$ . By (3.9.18) and (3.9.17) we have

$$\frac{V(S, t)}{S} = \frac{e^{-\frac{2ry}{\sigma^2}} G(x, y)}{e^{x - \frac{2ry}{\sigma^2} + y}} = \frac{G(x, y)}{e^{x+y}}.$$

Therefore from (3.7.12) we obtain the condition

$$\lim_{x \rightarrow +\infty} e^{-(x+y)} G(x, y) = 1,$$

for any  $y \geq 0$ .

Summarizing our discussion, the original problem (3.7.13) for the Black-Scholes equation is equivalent to the following problem for the diffusion equation

$$\begin{aligned} G_y &= G_{xx}, & \text{for } -\infty < x < \infty, y > 0 \\ G(x, 0) &= \max\{e^x - E, 0\}, & \text{for } -\infty < x < \infty, \\ \lim_{x \rightarrow -\infty} G(x, y) &= 0, & \text{for any } y \geq 0, \\ \lim_{x \rightarrow +\infty} e^{-(x+y)} G(x, y) &= 1, & \text{for any } y \geq 0. \end{aligned} \tag{3.9.19}$$

It is worthwhile to note that the two last conditions make the problem a little overdetermined. However, as we shall see later, the unique solution to the diffusion equation on  $\mathbb{R}$  satisfies them and thus it gives, after back transformation, the unique solution to the original problem (3.7.13).



# Lecture 4

## Method of characteristics and one dimensional wave equation

In this section we will see how some differential equations can be solved by using the change of variables discussed in Section 3.9.1.

### 1 First order equations

We start with the simplest transport equation. Assume that  $u$  is a concentration of some substance (pollutant) in a fluid (amount per unit length). This substance is moving to the right with a speed  $c$ . Then the differential equation for  $u$  has the form:

$$u_t + cu_x = 0. \quad (4.1.1)$$

Let us consider more general linear first order partial differential equation (PDE) of the form:

$$au_t + bu_x = 0, \quad t, x \in \mathbb{R} \quad (4.1.2)$$

where  $a$  and  $b$  are constants. This equation can be written as

$$D_{\mathbf{v}}u = 0, \quad (4.1.3)$$

where  $\mathbf{v} = aj + bi$  ( $j$  and  $i$  are the unit vectors in, respectively,  $t$  and  $x$  directions), and  $D_{\mathbf{v}} = \nabla u \cdot \mathbf{v}$  denotes the directional derivative in the direction of  $\mathbf{v}$ . This means that the solution  $u$  is a constant function along each line having direction  $\mathbf{v}$ , that is, along each line of equation  $bt - ax = \xi$ . Along each such a line the value of the parameter  $\xi$  remains constant. However, the solution can change from one line to another, therefore the solution is a function of  $\xi$ , that is the solution to Eq. (1.5.5) is given by

$$u(x, t) = f(bt - ax), \quad (4.1.4)$$

where  $f$  is an arbitrary differentiable function. Such lines are called the *characteristic lines* of the equation.

**Example 1.1** To obtain a unique solution we must specify the initial value for  $u$ . Hence, let us consider the initial value problem for Eq. (1.5.5): find  $u$  satisfying both

$$\begin{aligned} au_t + bu_x &= 0 & x \in \mathbb{R}, t > 0, \\ u(x, 0) &= g(x) & x \in \mathbb{R}, \end{aligned} \quad (4.1.5)$$

where  $g$  is an arbitrary given function. From Eq. (1.5.7) we find that

$$u(x, t) = g\left(-\frac{bt - ax}{a}\right). \quad (4.1.6)$$

We note that the initial shape propagates without any change along the characteristic lines, as seen below for the initial function  $g = 1 - x^2$  for  $|x| < 1$  and zero elsewhere. The speed  $c = b/a$  is taken to be equal to 1.

Fig 8. The graph of the solution in Example 1.1.

**Example 1.2** Let us consider a variation of this problem and try to solve the initial- boundary value problem

$$\begin{aligned} au_t + bu_x &= 0 & x \in \mathbb{R}, t > 0, \\ u(x, 0) &= g(x) & x > 0, \end{aligned} \tag{4.1.7}$$

$$u(0, t) = h(t) \quad t > 0, \tag{4.1.8}$$

for  $a, b > 0$  From Example 1.1 we have the general solution of the equation in the form

$$u(x, t) = f(bt - ax).$$

Putting  $t = 0$  we get  $f(-ax) = g(x)$  for  $x > 0$ , hence  $f(x) = g(-x/a)$  for  $x < 0$ . Next, for  $x = 0$  we obtain  $f(bt) = h(t)$  for  $t > 0$ , hence  $f(x) = h(x/b)$  for  $x > 0$ . Combining these two equations we obtain

$$u(x, t) = \begin{cases} g(-\frac{bt-ax}{a}) & \text{for } x > bt/a \\ h(\frac{bt-ax}{b}) & \text{for } x < bt/a \end{cases}$$

Let us consider now what happens if  $a = 1 > 0, b = -1 < 0$ . Then the initial condition defines  $f(x) = g(-x)$  for  $x < 0$  and the boundary condition gives  $f(x) = h(-x)$  also for  $x < 0$ ! Hence, we cannot specify both initial and boundary conditions in an arbitrary way as this could make the problem ill-posed.

The physical explanation of this comes from the observation that since the characteristics are given by  $\xi = x + t$ , the flow occurs in the negative direction and therefore the values at  $x = 0$  for any  $t$  are uniquely determined by the initial condition. Therefore we see that to have a well-posed problem we must specify the boundary conditions at the point where the medium flows into the region.

The method we have just used is often called the *geometric method*. It is easy to understand, but sometimes difficult to apply, especially for non-homogeneous problems. Fortunately, this method can be easily reformulated in a more analytic language. Let us introduce the change of variables according to  $\xi = \xi(t, x), \eta = \eta(t, x)$ ; then

$$u_t = u_\xi \xi_t + u_\eta \eta_t, \quad u_x = u_\xi \xi_x + u_\eta \eta_x,$$



and the equation can be written as

$$a(u_\xi \xi_t + u_\eta \eta_t) + b(u_\xi \xi_x + u_\eta \eta_x) = u_\xi(a\xi_t + b\xi_x) + u_\eta(a\eta_t + b\eta_x) = 0$$

If we require the coefficient at  $u_\eta$  to be zero, the easiest way is to introduce  $\eta_t = b$ ,  $\eta_x = a$ , that is  $\eta = bt - ax$ . Note, that this is exactly the characteristic direction! However, this is an incomplete change of variables as originally we have had two independent variables  $t, x$  but we have only one new variable  $\eta$ . That means that knowing  $\eta$  alone we are not able to tell values of  $x$  and  $t$ . The trick is to introduce another variable, say,  $\xi = \xi(x, t)$  in such a way that the system

$$\eta = bt - ax, \quad \xi = \xi(x, t)$$

is uniquely solvable. To keep things as simple as possible, we may here take  $\xi(x, t) = ct + dx$  with  $c, d$  picked so as to have the determinant  $bd + ac \neq 0$ . If  $a$  and  $b$  are not equal to zero, then the easiest choice would be either  $\xi = t$  or  $\xi = x$ , respectively. However, sometimes it is more convenient to use the orthogonal lines given by  $\xi = at + bx$ .

We illustrate this approach in the following example.

**Example 1.3** Find the general solution to the following equation

$$u_t + 2u_x - (x + t)u = -(x + t).$$

Introducing new variables according to  $\xi = t$ ,  $\eta = 2t - x$  we transform the equation into

$$v_\xi - (3\xi - \eta)v = -(3\xi - \eta).$$

This equation can be regarded as a linear first order ordinary differential equation in  $\xi$  with a parameter  $\eta$ .

To find the integrating factor we solve the homogeneous equation

$$\mu_\xi = -(3\xi - \eta)\mu;$$

the integration gives

$$\mu(\xi, \eta) = e^{-(\frac{3}{2}\xi^2 - \eta\xi)}.$$

Multiplying both sides of the equation by  $\mu$  and rearranging the terms we obtain

$$\left( e^{-(\frac{3}{2}\xi^2 - \eta\xi)} v(\xi, \eta) \right)_\xi = -(3\xi - \eta)e^{-(\frac{3}{2}\xi^2 - \eta\xi)},$$

hence the solution is given by

$$v(\xi, \eta) = 1 + C(\eta)e^{(\frac{3}{2}\xi^2 - \eta\xi)},$$

where  $C$  is an arbitrary differentiable function of one variable.

In the original variables we obtain

$$u(x, t) = 1 + C(2t - x)e^{(-t^2/2 + tx)}$$

where  $C$  is an arbitrary differentiable function of one variable.

**Example 1.4** Find the solution of the equation

$$u_t + 2u_x - (x + t)u = -(x + t),$$

which satisfies the initial condition:

$$u(x, 0) = f(x), \quad x > 0,$$

and

$$u(0, t) = g(t), \quad t > 0.$$

We use the general solution obtained in the previous example:

$$u(x, t) = 1 + C(2t - x) \exp(-t^2/2 + tx).$$

Thus

$$f(x) = u(x, 0) = 1 + C(-x)$$

for  $x > 0$ . To avoid misunderstanding we introduce the variable  $s$  as the argument of  $C$ . Thus

$$C(s) = f(-s) - 1, \quad s < 0.$$

On the other hand

$$g(t) = u(0, t) = 1 + C(2t) \exp(-t^2/2),$$

thus

$$C(2t) = \exp(t^2/2) (g(t) - 1).$$

Since we need the function  $s \rightarrow C(s)$ , we introduce  $s = 2t$ ; then  $s > 0$  and

$$C(s) = \exp(s^2/8) (g(s/2) - 1).$$

Thus, we have defined  $C$  for all values of the argument by

$$C(s) = \begin{cases} f(-s) - 1 & \text{for } s < 0, \\ \exp(s^2/8) (g(s/2) - 1) & \text{for } s > 0, \end{cases}$$

In the solution to the given problem the function  $C$  is evaluated at  $s = 2t - x$ , so  $C$  has different definitions for  $2t - x > 0$  and  $2t - x < 0$ . Therefore the solution to the given initial-boundary value problem is given by

$$u(x, t) = \begin{cases} 1 + (f(-2t + x) - 1) \exp(-t^2/2 + tx) & \text{for } x > 2t, \\ \exp((x^2 + tx)/8) (g(t - x/2) - 1) & \text{for } x < 2t. \end{cases}$$

For example, if  $f(x) = x$  and  $g(t) = \sin t$ , then the solution is given by

$$u(x, t) = \begin{cases} 1 + (-2t + x - 1) \exp(-t^2/2 + tx) & \text{for } x > 2t, \\ \exp((x^2 + 4tx)/8) (\sin(t - x/2) - 1) & \text{for } x < 2t, \end{cases}$$

In the figures below note how the point where the analytic description of the solution changes across the characteristic  $x - 2t = 0$ .

*Fig 9. The graph of the solution in Example 1.4.*

*Fig 10. The three-dimensional visualization of the solution in Example 1.4.*

The same principle could be used to solve equations with variable coefficients. In fact, consider the equation

$$a(x, t)u_t + b(x, t)u_x = 0. \quad (4.1.9)$$

This equation asserts that the derivative of  $u$  in the direction of the vector  $(b(x, t), a(t, x))$  is equal to zero at each point  $(x, t)$  where this vector is not vanishing. We can consider a family of curves  $t = t(x)$  which are tangent to these vectors at each point. In other words, these curves will have at each point the slope equal to  $a/b$ ; this is equivalent to say that they satisfy the differential equation

$$\frac{dt}{dx} = \frac{a(t, x)}{b(t, x)}$$

Assume that this equation has solutions given by  $\phi(x, t) = \eta$  where  $\eta$  is an arbitrary constant. Note that in principle for each  $\eta$ , the equation  $\phi(x, t) = \eta$  determines a curve and if a point  $(x, t)$  belongs to this curve, the tangent vector at this point is given by  $(b(x, t), a(x, t))$ . From the general theory it follows then that the normal vector to this curve is given by the gradient of  $\phi$ :  $(\phi_x(x, t), \phi_t(x, t))$ . Let us see what this means for our equation. Consider, for an arbitrary differentiable  $f$ , the function

$$u(x, t) = f(\phi(x, t)) \quad (4.1.10)$$

Inserting  $u$  into the equation (4.1.9) we obtain

$$au_t + bu_x = f' \cdot (a\phi_t + b\phi_x) = 0,$$

due to the orthogonality property of  $[\phi_x, \phi_t]$  which was mentioned above. Thus,  $u$  given by Eq. (4.1.10) is the general solution to (4.1.9).

Note that the solution doesn't change along the curves  $\phi(x, t) = C$  which are therefore the characteristic curves of the equation.

The extension of the above considerations to the nonhomogeneous cases can be done along the lines similarly as in the constant coefficient case. However, some difficulties can occur as  $\eta = \phi(x, t)$  not always gives rise to a well-defined change of variables.

**Example 1.5** Find the solution to the following initial value problem

$$\begin{aligned} u_t + xu_x + u &= 0, \quad t > 0, x \in \mathbb{R}, \\ u(x, 0) &= f(x), \end{aligned}$$

where  $f(x) = 1 - x^2$  for  $|x| < 1$ , and  $f(x) = 0$  for  $|x| \geq 1$ .

The differential equation for characteristic curves is

$$\frac{dt}{dx} = \frac{1}{x}$$

which gives  $x = \eta e^t$ , thus  $\eta = xe^{-t}$ . If the equation hadn't contained the zero order term  $u$ , that is, if it had been in the form

$$u_t + xu_x = 0,$$

then the general solution would have had the form

$$u(x, t) = f(xe^{-t}),$$

for arbitrary function  $f$ .

However, since we have the additional term, we have to perform the full change of variables. Fortunately, putting  $\xi = t$  and  $\eta = xe^{-t}$  produces an invertible change of variables, as  $t = \xi$  and  $x = e^\xi \eta$ .

Defining  $v(\xi, \eta) = u(x, t)$  we obtain that

$$\begin{aligned} 0 = u_t + xu_x + u &= v_\xi \xi_t + v_\eta \eta_t + x(v_\xi \xi_x + v_\eta \eta_x) + v \\ &= v_\xi + v \end{aligned}$$

Therefore we obtain the solution in the form

$$v(\xi, \eta) = C(\eta)e^{-\xi}$$

or

$$u(x, t) = C(xe^{-t})e^{-t}$$

where  $C$  is an arbitrary function.

To find the solution of the initial value problem we have

$$f(x) = u(x, 0) = C(x),$$

and according to the definition of  $C(x) = 1 - x^2$  for  $|x| < 1$  and  $C(x) = 0$  for  $|x| \geq 1$ . In the solution the function  $C$  appears composed with  $xe^{-t}$ . Accordingly,  $C(xe^{-t}) = 1 - x^2e^{-2t}$  for  $|xe^{-t}| < 1$  and  $C(xe^{-t}) = 0$  for  $|xe^{-t}| \geq 1$ . Thus we obtain

$$u(x, t) = \begin{cases} (1 - x^2e^{-2t})e^{-t} & \text{for } |t| > \ln|x|, \\ 0 & \text{for } |t| \leq \ln|x|, \end{cases}$$

*Fig 11. Characteristics of the equation in Example 1.5.*

*Fig 12. 3-dimensional visualization of the solution in Example 1.5.*

*Fig 13. The graph of the solution in Example 1.5 for times  $t = 0, 0.5, 1, 1.5, 2, 2.5$ .*

The described procedure is also not restricted to equations in two independent variables. In fact, let us consider the equation

$$au_t + bu_x + cu_y = 0 \quad (4.1.11)$$

This equation expresses the fact that the directional derivative in the direction of the vector  $[a, b, c]$  is equal to zero, that is, that the solution does not change along any line with parametric equation  $t = t_0 + as, x = x_0 + bs, y = y_0 + cs$ . Such a line can be written also as the pair of equations  $ax - bt = \xi, cy - bt = \eta$ . For each pair  $\xi, \eta$  this pair describes a single line parallel to  $[a, b, c]$ , that is, the solution  $u$  can be a function of  $\xi$  and  $\eta$  only. Thus we obtain the following general solution to (4.1.11)

$$u(x, y, t) = f(ax - bt, ay - ct), \quad (4.1.12)$$

where  $f$  is an arbitrary differentiable function of two variables.

**Example 1.6** Find the solution to the following initial value problem:

$$u_t + 2u_x + 3u_y + u = 0, \quad u(x, y, 0) = e^{-x^2 - y^2}.$$

We use the change of coordinates suggested by the characteristics:

$$\xi = x - 2t, \quad \eta = y - 3t$$

supplemented by

$$\alpha = t,$$

so that the change is invertible:

$$t = \alpha, \quad x = \xi + 2\alpha, \quad y = \eta + 3\alpha.$$

Then, putting  $u(x, y, t) = v(\xi, \eta, \alpha)$ , we find that

$$\begin{aligned} u_t &= (-2)v_\xi + (-3)v_\eta + v_\alpha, \\ u_x &= v_\xi, \\ u_y &= v_\eta \end{aligned}$$

so that

$$0 = u_t + 2u_x + 3u_y + u = v_\alpha + v.$$

The last equation has the general solution

$$v(\xi, \eta, \alpha) = C(\xi, \eta)e^{-\alpha}$$

so that

$$u(x, y, t) = C(x - 2t, y - 3t)e^{-t}.$$

To solve the initial value problem we put  $t = 0$  to get

$$e^{-x^2 - y^2} = C(x, y)$$

so that the solution is of the form

$$u(x, y, t) = e^{-(x-2t)^2 - (y-3t)^2} e^{-t}.$$

*Fig14. The graph of the solution in Example 1.6 for times  $t = 0, 0.4, 0.8, 1.2, 1.6, 2$ .*

## 2 One dimensional wave equation

Let us now consider the one-dimensional wave equation (3.4.2)

$$u_{tt} = c^2 u_{xx} \tag{4.2.1}$$

This is clearly hyperbolic equation with  $D = a_{12}^2 - a_{11}a_{22} = 0 + c^2 > 0$ . The characteristic equation is  $t^2 = c^2$ , that is,  $t = \pm c$  and the characteristics are given by

$$\eta = x + ct, \quad \xi = x - ct.$$

We denote

$$u(t, x) = w(\xi(x, t), \eta(x, t)),$$

and using the chain rule we obtain:

$$u_{tt} - c^2 u_{xx} = -4c^2 v_{\eta\xi} = 0.$$

Solving the last equation we obtain:

$$v(r, s) = f(r) + g(s),$$

where  $f$  and  $g$  are arbitrary functions. Returning to the old variables we get:

$$u(x, t) = f(x + ct) + g(x - ct). \quad (4.2.2)$$

As can be seen, the solution  $u$  is the superposition of two waves:  $g(x - ct)$  traveling to the right at speed  $c$  and  $f(x + ct)$  which travels to the left at the same speed  $c$ .

The initial value problem for the wave equation consists in finding the particular solution which satisfies

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \quad (4.2.3)$$

where  $\phi$  and  $\psi$  are arbitrary given functions.

Physically, it corresponds to looking for the evolution of the shape of a string when the initial shape and the initial velocity are given.

Using Eqs. (1.6.17) and (1.6.18) we find, setting  $t = 0$ , that

$$\phi(x) = f(x) + g(x) \quad (4.2.4)$$

and

$$\psi(x) = cf'(x) - cg'(x). \quad (4.2.5)$$

This is a system of two equations with two unknown functions and since the variable  $x$  has been used to denote the original spatial variable, let us for a time being introduce a neutral variable  $s$  and differentiate the first equation to get the following system

$$\begin{aligned} \phi' &= f' + g', \\ \psi &= cf' - cg'. \end{aligned}$$

From it we obtain

$$f' = \frac{1}{2}(\phi' + \frac{\psi}{c}), \quad g' = \frac{1}{2}(\phi' - \frac{\psi}{c}).$$

Integrating, we get

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c} \int_0^s \psi(w)dw + C_1$$

and

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c} \int_0^s \psi(w)dw + C_2$$

Because of (4.2.4) we obtain  $C_1 + C_2 = 0$ . Now, from Eq. (1.6.17) we have the solution in the form

$$\begin{aligned} u(x, t) &= f(x + ct) + g(x - ct) \\ &= \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \left( \int_0^{x+ct} \psi(w)dw - \int_0^{x-ct} \psi(w)dw \right) \end{aligned}$$



which simplifies to

$$u(x, t) = \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds \quad (4.2.6)$$

This equation is known as the d'Alembert formula. If  $\phi$  is twice differentiable and  $\psi$  once, Eq. (1.6.19) give a genuine solution to the initial value problem (4.2.1), (1.6.18) and, as we shall see later, a unique one.

**Example 2.1** (The plucked string) Consider the infinitely long string with the initial position:

$$\phi(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

and the initial velocity  $\psi = 0$ . To obtain the analytic formula for the solution from d'Alembert's formula we must evaluate  $\phi(x + ct) + \phi(x - ct)$  where  $\phi$  is given above. Assume that  $t > 0$ . Note that the solution will be given by four different formulas in four regions:

1.  $|x - ct| < 1, |x + ct| < 1$ ,
2.  $|x - ct| < 1, |x + ct| \geq 1$ ,
3.  $|x - ct| \geq 1, |x + ct| < 1$ ,
4.  $|x - ct| \geq 1, |x + ct| \geq 1$ .

First, let both  $|x - ct| < 1$  and  $|x + ct| < 1$ . Expanding the absolute value function, we see that we must have  $-1 + ct \leq x \leq 1 + ct$  and  $-1 - ct \leq x \leq 1 - ct$  and, since for  $c, t > 0$

$$-1 - ct < -1 + ct < 1 - ct < 1 + ct$$

we obtain that  $-1 + ct \leq x \leq 1 - ct$ . From d'Alembert's formula for such  $x$  and  $t$ , the solution is given by

$$\frac{1}{2}(\phi(x + ct) + \phi(x - ct)) = 1 - \frac{1}{2}(|x - ct| + |x + ct|).$$

Another domain of definition is when  $|x - ct| < 1$  and  $|x + ct| \geq 1$ , that is,  $-1 + ct < x < 1 + ct$  and  $1 - ct \leq x$  or  $x \leq -1 - ct$ . Since  $-1 - ct$  is the smallest number, it is irrelevant here. Thus,  $x < 1 + ct$  and  $x$  must be greater than the bigger number of  $1 - ct$  and  $-1 + ct$ . For  $1 - ct > -1 + ct$ , that is for  $1 - ct > 0$ , this bigger number is  $1 - ct$ , and for  $1 - ct \leq -1 + ct$  (that is  $1 - ct \leq 0$ ), the bigger number is  $-1 + ct = -(1 - ct)$ . Thus in any case the bigger number is  $|1 - ct|$ . Combining, we obtain  $|1 - ct| < x < 1 + ct$  and then

$$\phi(x + ct) + \phi(x - ct) = 1 - |x - ct|,$$

as  $\phi(x + ct) = 0$ . Similarly,  $|x - ct| \geq 1, |x + ct| < 1$  is equivalent to  $x \leq -1 + ct$  or  $x \geq 1 + ct$ , and  $-1 - ct < x < 1 - ct$ . Thus,  $-1 - ct < x$  and  $x$  is smaller than the smallest number of  $1 - ct$  and  $-1 + ct$ . Arguing as above we get  $-1 - ct < x < -|1 - ct|$  and for such  $x$

$$\phi(x + ct) + \phi(x - ct) = 1 - |x + ct|,$$

as  $\phi(x - ct) = 0$ . Finally, in the region 4, where both  $|x - ct|$  and  $|x + ct|$  are greater than 1, the solution is equal to zero. Summarizing, we obtain the solution in the form

$$u(x, t) = \frac{1}{2} \begin{cases} 2 - |x - ct| - |x + ct| & \text{for } -1 + ct \leq x \leq 1 - ct \\ 1 - |x - ct| & \text{for } |1 - ct| < x \leq 1 + ct \\ 1 - |x + ct| & \text{for } -1 - ct \leq x \leq -|1 - ct| \\ 0 & \text{elsewhere.} \end{cases}$$

By substituting several values for  $t$  we can check that the solution consists of two triangular waves traveling to the left and to the right at speed  $c$ . However, for  $t < 1/2c$  these two waves have a common support and

their superposition gives a trapezoidal wave. For  $t > 1/2c$  the waves separates from each other and preserve their triangular shape forever. This is illustrated on the figures below.

*Fig 14. 3 dimensional visualization of the motion of the plucked string.*

*Fig 15. The motion of the plucked string.*

**Example 2.2** (The hammer blow) Consider the infinitely long string with zero initial displacement hit by a hammer. This situation can be modeled by the following initial conditions:

$$\phi(x) = 0,$$

and

$$\psi(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

To obtain the analytic formula for the solution from d'Alembert's formula we must evaluate the integral  $\int_{x-ct}^{x+ct} \psi(s)ds$ . As before, the solution will have different analytic descriptions in different regions.

- If  $-1 < x - ct, x + ct < 1$ , then both  $\psi(s) = 1$  over the whole region of integration and we have

$$\int_{x-ct}^{x+ct} \psi(s)ds = \int_{x-ct}^{x+ct} 1ds = x + ct - (x - ct) = 2ct.$$

- Let now  $-1 < x - ct < 1, x + ct > 1$ . Then between 1 and  $x + ct$  the function  $\psi(x)$  is equal to zero, thus

$$\int_{x-ct}^{x+ct} \psi(s)ds = \int_{x-ct}^1 1ds = 1 - (x - ct) = 1 - x + ct.$$

- Let now  $x - ct < -1, -1 < x + ct < 1$ . Then between  $x - ct$  and  $-1$  the function  $\psi(x)$  is equal to zero, thus

$$\int_{x-ct}^{x+ct} \psi(s)ds = \int_{-1}^{x+ct} 1ds = x + ct + 1.$$

- If  $x - ct < -1$  and  $x + ct > 1$ , then the function  $\psi$  is equal to zero between  $-1$  and  $x - ct$ , and between  $x + ct$  and 1, thus

$$\int_{x-ct}^{x+ct} \psi(s)ds = \int_{-1}^1 1ds = 2.$$

- Finally, if either  $x - ct, x + ct < -1$  or  $x - ct, x + ct > 1$ , then the function  $\psi = 0$  over the whole region of integration and consequently

$$\int_{x-ct}^{x+ct} \psi(s)ds = \int_{x-ct}^{x+ct} 0ds = 0.$$

Using d'Alembert's formula we obtain

$$u(x, t) = \frac{1}{2c} \begin{cases} 2ct & \text{for } -1 < x - ct, x + ct < 1, \\ 1 - x + ct & \text{for } -1 < x - ct < 1, x + ct > 1, \\ 1 + x + ct & \text{for } x - ct < -1, -1 < x + ct < 1, \\ 2 & \text{for } x - ct < -1, x + ct > 1, \\ 0 & \text{elsewhere.} \end{cases}$$

This solution is illustrated at the figures below.

*Fig 16. 3 dimensional visualization of the motion of the string after the hammer blow.*

*Fig 17. The motion of the string after the hammer blow.*

The technique described above can be use to solve also some initial-boundary value problems.

**Example 2.3** (Waves on the half-line) Let us consider the following initial-boundary value problem on a half-line:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && \text{for } x > 0, \quad t > 0, \\ u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x) && \text{for } x > 0, \\ u(0, t) &= 0 && \text{for } t > 0. \end{aligned}$$

This problem can be reduced to the pure initial value problem by extending the initial data in such a way that the resulting solution is equal to zero for  $x = 0$ . The required extension turns out to be the odd one: let  $\phi_{odd}$  and  $\psi_{odd}$  be the odd extensions of  $\phi$  and  $\psi$ , respectively, that is e.g.  $\phi_{odd}(-x) = -\phi_{odd}(x)$ . Define

$v$  to be the solution of the corresponding initial value problem given by the d'Alembert formula:

$$v(x, t) = \frac{1}{2}(\phi_{odd}(x + ct) + \phi_{odd}(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{odd}(s) ds, \quad (4.2.7)$$

and let  $u(x)$  be  $v$  restricted to  $x > 0$ . We have  $u(0, t) = 0$  due to the fact that all the functions are odd, thus  $u$  is a solution of the original boundary-initial value problem. To write down the solution in the form involving only  $\phi$  and  $\psi$  we note that for  $x > ct$  both  $x + ct$  and  $x - ct$  are positive and for positive arguments  $\phi = \phi_{odd}$  and  $\psi = \psi_{odd}$ . For  $x < ct$  we utilize  $\phi_{odd}(x - ct) = -\phi(ct - x)$  and the analogous property of  $\psi$ . Combining, we get:

$$u(x, t) = \frac{1}{2} \begin{cases} (\phi(x + ct) + \phi(x - ct)) + \frac{1}{c} \int_{x-ct}^{x+ct} \psi(s) ds, & \text{for } x > ct, \\ (\phi(ct + x) - \phi(ct - x)) + \frac{1}{c} \int_{ct-x}^{ct+x} \psi(s) ds & \text{for } x < ct. \end{cases} \quad (4.2.8)$$

If we consider the string plucked at  $x = 4$ , fixed at  $x = 0$ , then the initial condition will be given by  $\phi(x) = 1 - |x - 4|$  for  $|x - 4| < 1$ , and  $\phi(x) = 0$  elsewhere. Thus the odd extension is defined as

$$\phi_{odd}(x) = \begin{cases} 1 - |x - 4| & \text{for } |x - 4| < 1, \\ -1 + |x + 4| & \text{for } |x + 4| < 1, \\ 0 & \text{elsewhere} \end{cases}$$

and the solution of the problem (with initial velocity  $\psi(x) = 0$  and  $c = 1$ ) is given by

$$u(x, t) = \frac{1}{2} \begin{cases} (\phi(x + t) + \phi(x - t)) & \text{for } x > t, \\ (\phi(t + x) - \phi(t - x)) & \text{for } x < t. \end{cases}$$

The analytical formulae for  $u$  are straightforward but messy so that we shall show only the graphical representation of the solution.

*Fig 18. 3-dimensional visualization of the motion of the plucked string with reflection at  $x = 0$ .*

*Fig 19. Level curves of the solution  $u(x, t)$ . Dark region correspond to negative values.*

*Fig 20. The motion of the plucked string with reflection at  $x = 0$*

### 3 Miscellaneous examples

In this section we shall show the application of the method of characteristics to solve a more complicated hyperbolic equation and discuss the spherically symmetric wave equation in space.

**Example 3.1** (Superposition principle.) One of the most important properties of linear equations is that the sum of solutions is again a solution. This superposition principle can be used to construct solution of, say, wave equation with complicated initial/boundary data by combining solutions corresponding to simpler data. As an illustration we shall derive the solution to the initial-boundary value problem on the half-line

discussed in Example 6.2:

$$\begin{aligned} u_{tt} &= u_{xx} & \text{for } x > 0, \ t > 0, \\ u(x, 0) &= \phi(x), \ u_t(x, 0) = 0 & \text{for } x > 0, \\ u(0, t) &= 0 & \text{for } t > 0. \end{aligned} \quad (4.3.1)$$

with  $\phi(x) = 1 - |x - 4|$  for  $|x - 4| < 1$  and  $\phi(x) = 0$  for  $|x - 4| > 1$ .

According to the approach of the Example 6.2, this problem is equivalent to the pure initial value problem

$$\begin{aligned} u_{tt} &= u_{xx} & \text{for } -\infty < x < +\infty, \ t > 0, \\ u(x, 0) &= \phi_{\text{odd}}(x), \ u_t(x, 0) = 0 & \text{for } -\infty < x < +\infty, \end{aligned}$$

where  $\phi_{\text{odd}}$  is the odd extension of  $\phi$ , that is,

$$\phi_{\text{odd}}(x) = \begin{cases} 1 - |x - 4| & \text{for } |x - 4| < 1, \\ -1 + |x + 4| & \text{for } |x + 4| < 1, \\ 0 & \text{elsewhere} \end{cases}$$

One can construct the solution directly from the d'Alembert formula but it becomes very messy. To simplify the derivation we use the superposition principle and the knowledge of the plucked string problem

$$u(x, t) = \frac{1}{2} \begin{cases} 2 - |x - t| - |x + t| & \text{for } -1 + t \leq x \leq 1 - t \\ 1 - |x - t| & \text{for } |1 - t| < x \leq 1 + t \\ 1 - |x + t| & \text{for } -1 - t \leq x \leq -|1 - t| \\ 0 & \text{elsewhere.} \end{cases}$$

This solution satisfies the initial condition  $u(x, 0) = \phi_0(x)$ , where  $\phi_0(x) = 1 - |x|$  for  $|x| < 1$  and zero elsewhere. Now, note that the initial condition  $\phi$  of our problem satisfies  $\phi(x) = \phi_0(x - 4)$  and similarly

$$\phi_{\text{odd}}(x) = \phi_0(x - 4) - \phi_0(x + 4).$$

In the next step we use some invariant properties of the wave equation. It is immediate to check that the function

$$U(x, t) = u(x + a, t)$$

satisfies the wave equation along with  $u$ , for any choice of  $a$ . Similarly,  $-u$  is also a solution of the wave equation if  $u$  is. But if the solution  $u$  satisfies the initial condition  $u(x, 0) = \phi(x)$ , then  $U(x, 0) = u(x + a, 0) = \phi(x + a)$  satisfies the shifted initial condition. This is exactly what we need, as  $\phi(x) = \phi_0(x - 4)$  and we know the solution satisfying the initial condition  $u(x, 0) = \phi(x)$ . Thus

$$u_1(x, t) = u(x - 4, t) = \frac{1}{2} \begin{cases} 2 - |x - 4 - t| - |x - 4 + t| & \text{for } -1 + t \leq x - 4 \leq 1 - t \\ 1 - |x - 4 - t| & \text{for } |1 - t| < x - 4 \leq 1 + t \\ 1 - |x - 4 + t| & \text{for } -1 - t \leq x - 4 \leq -|1 - t| \\ 0 & \text{elsewhere.} \end{cases}$$

is the solution to the wave equation satisfying the initial condition  $u_1(x, 0) = 1 - |x - 4|$  for  $|x - 4| < 1$  and  $u_1(x, 0) = 0$  for  $|x - 4| > 1$ .

Similarly

$$u_2(x, t) = -u(x + 4, t) = \frac{1}{2} \begin{cases} 2 - |x + 4 - t| - |x + 4 + t| & \text{for } -1 + t \leq x + 4 \leq 1 - t \\ 1 - |x + 4 - t| & \text{for } |1 - t| < x + 4 \leq 1 + t \\ 1 - |x + 4 + t| & \text{for } -1 - t \leq x + 4 \leq -|1 - t| \\ 0 & \text{elsewhere.} \end{cases}$$

is the solution to the wave equation satisfying the initial condition  $u_2(x, 0) = -1 + |x + 4|$  for  $|x + 4| < 1$  and  $u_2(x, 0) = 0$  for  $|x + 4| > 1$ .

Using now the superposition principle, we obtain that the solution the original initial-boundary value problem (4.3.1) is given by

$$U(x, t) = u_1(x, t) + u_2(x, t) = u(x - 4, t) - u(x + 4, t)$$

To find the analytic formula for  $U$  first we note that we are interested in solution for  $x > 0$  and  $t > 0$ . Then it is clear that the only part of  $u_2$  which is non-zero in this region is

$$u_2(x, t) = 1 - |x + 4 - t|$$

defined for  $|1 - t| < x + 4 \leq 1 + t$ . Let us interpret this geometrically. The latter inequality shows that  $t \geq x + 3$ . The former splits into  $t > -x - 4$  for  $0 < t \leq 1$  or  $t < x + 5$  for  $t > 1$ . However, since  $x > 0, t > 0$ , the first of the last two inequalities is always satisfied, thus  $u_2(x, t)$  in the first quadrant  $t > 0, x > 0$  is supported in the strip  $3 \leq t - x < 5, x > 0$ .

Let us perform similar geometrical considerations for  $u_1$ . The first expression in the formula for  $u_1$  is valid in the triangle bounded by the lines  $t = 0, t = 5 - x$ , and  $t = 5 - x$  and clearly this triangle doesn't intersect the support of  $u_2$ . The second expression is valid in the region  $-5 \geq t - x < -3$  and  $t > -x + 5$ . Also this region has no common points with the support of  $u_2$ . The third expression is valid in  $3 < t + x < 5, x > 0, t > 0, t - x \geq -3$ . Now, this region has a nonempty intersection with the support of  $u_2$  and this intersection is the triangle bounded by  $x = 0, t - x = 3, t + x = 5$ .

To write the solution to the problem 4.3.1 we must divide the region  $t > 0, x > 0$  into six regions: in

- (i)  $3 + t \leq x \leq 5 - t, t \geq 0$ ,
- (ii)  $-5 \geq t - x < -3, t > -x + 5$ , and
- (iii)  $3 < t + x < 5, -3 < t - x \leq 3$

the solution  $U$  is given by  $u_1$ , in

- (iv)  $3 \leq t - x < 5, t + x \geq 5$ ,

it is given by  $-u_2$ , in

- (v)  $x \geq 0, x + 3 \leq t \leq -x + 5$

it is given by  $u_1 - u_2$ , and finally the solution is zero elsewhere.

Writing these consideration in compact analytic form we obtain

$$U(x, t) = \frac{1}{2} \begin{cases} 2 - |x - 4 - t| - |x - 4 + t| & \text{for } 3 + t \leq x \leq 5 - t, t \geq 0, \\ 1 - |x - 4 - t| & \text{for } -5 \geq t - x < -3, t > -x + 5, \\ 1 - |x - 4 + t| & \text{for } 3 < t + x < 5, -3 < t - x \leq 3, \\ -1 + |x + 4 - t| & \text{for } 3 \leq t - x < 5, t > x + 5, \\ |x + 4 - t| - |x - 4 + t| & \text{for } 3 + x \leq t \leq -x + 5, \\ 0 & \text{elsewhere.} \end{cases}$$

**Example 3.2** Find the solution to

$$u_{xx} - 3u_{xt} - 4u_{tt} = t,$$

with the initial conditions  $u(x, 0) = x^2$  and  $u_t(x, 0) = e^x$ . We have  $a_{11} = 1, a_{12} = -3/2, a_{22} = -4$ , thus  $D = a_{12}^2 - a_{11}a_{22} = 25/4 > 0$ , that is, the equation is hyperbolic.

To find the transformation reducing the equation to a canonical form, we solve the quadratic equation

$$r^2 - 3r - 4 = 0,$$



which gives two real roots  $r_{1,2} = 4, -1$ , so that the required change of variables is given by

$$\xi = 4x + t, \quad \eta = -x + t.$$

Defining  $u(x, t) = v(\xi, \eta)$ , we have

$$u_x = v_\xi \xi_x + v_\eta \eta_x = 4v_\xi - v_\eta$$

and

$$u_t = v_\xi \xi_t + v_\eta \eta_t = v_\xi + v_\eta.$$

For the second derivatives we obtain

$$\begin{aligned} u_{xx} &= 16v_{\xi\xi} - 8v_{\xi\eta} + v_{\eta\eta}, \\ u_{xt} &= 4v_{\xi\xi} + 3v_{\xi\eta} - v_{\eta\eta}, \\ u_{tt} &= v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}, \end{aligned}$$

so that

$$u_{xx} - 3u_{xt} - 4u_{tt} = -25v_{\xi\eta} = \frac{\xi + 4\eta}{5},$$

that is, we have to solve

$$v_{\xi\eta} = -\frac{\xi + 4\eta}{125}.$$

Integrating with respect to  $\eta$  gives

$$v_\xi = -\frac{\xi\eta}{125} - \frac{2\eta^2}{125} + F(\xi),$$

where  $F$  is an arbitrary function of  $\xi$ . Integrating with respect to  $\xi$  and denoting  $f(\xi) = \int F(\xi)d\xi$ , we obtain

$$v(\xi, \eta) = -\frac{\xi^2\eta}{250} - \frac{2\eta^2\xi}{125} + f(\xi) + g(\eta).$$

Returning to the original variables we obtain

$$u(x, t) = f(4x + t) + g(-x + t) - \frac{1}{50}(-4x^2t + 3xt^2 + t^3),$$

where  $f$  and  $g$  are arbitrary.

Using the initial conditions we obtain

$$\begin{aligned} u(x, 0) &= f(4x) + g(-x) = x^2, \\ u_t(x, 0) &= f'(4x) + g'(-x) + \frac{2}{25}x^2 = e^x. \end{aligned}$$

Differentiating the first equation, we obtain the following system

$$\begin{aligned} 4f'(4x) - g'(-x) &= 2x, \\ f'(4x) + g'(-x) &= -\frac{2}{25}x^2 + e^x. \end{aligned}$$

Adding these two equations we obtain

$$5f'(4x) = 2x - \frac{2}{25}x^2 + e^x,$$

or, putting  $4x = s$  and dividing by 5

$$f'(s) = \frac{s}{10} - \frac{1}{1000}s^2 + \frac{1}{5}e^{s/4}.$$

Integrating, we get

$$f(s) = \frac{s^2}{20} - \frac{1}{3000}s^3 + \frac{4}{5}e^{s/4}.$$

On the other hand, multiplying the second equation by 4 and subtracting, we obtain

$$5g'(-x) = -\frac{8}{25}x^2 + 4e^x - 2x,$$

that is,

$$g'(s) = -\frac{8}{125}s^2 + \frac{4}{5}e^{-s} + \frac{2}{5}s.$$

Upon integration we obtain

$$g(s) = -\frac{8}{375}s^3 - \frac{4}{5}e^{-s} + \frac{1}{5}s^2.$$

Thus, the final solution is given by

$$\begin{aligned} u(x, t) = & \frac{(4x+t)^2}{20} - \frac{1}{3000}(4x+t)^3 + \frac{4}{5}e^{(4x+t)/4} - \frac{8}{375}(t-x)^3 - \frac{4}{5}e^{x-t} + \frac{1}{5}(t-x)^2 \\ & - \frac{1}{50}(-4x^2t + 3xt^2 + t^3) \end{aligned}$$

### 3.1 Spherically symmetric solution of the wave equation in space

It can be proved that the solution of the initial value problem for the wave equation in three spatial dimensions

$$\begin{aligned} u_{tt} &= c^2(u_{xx} + u_{yy} + u_{zz}), \quad (x, y, z) \in \mathbb{R}^3, t > 0, \\ u(x, y, z, 0) &= \phi(x, y, z), \\ u_t(x, y, z, 0) &= \psi(x, y, z), \end{aligned} \quad (4.3.2)$$

with spherically symmetric initial data  $\psi(x, y, z) = \psi(r)$  and  $\phi(x, y, z) = \psi(r)$ , where  $r = \|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$ , is also spherically symmetric, that is,  $u(x, y, z, t) = u(r, t)$ .

Let us introduce the notation  $\mathbf{r} = (x, y, z)$ . Since the solution must be of the form  $u(\mathbf{r}, t) = u(r, t)$ , we see that it would be useful to write the Laplacian in (4.1.1) in spherical coordinates (see Tutorial 4 (1)), observing in the process that all the derivatives with respect to the angular variables will drop out.

Let the spherical coordinates be given by  $x = r \cos \alpha \sin \beta, y = r \sin \alpha \sin \beta, z = r \cos \beta, r > 0, 0 < \alpha < 2\pi, 0 < \beta < \pi$ . Using the Chain Rule we obtain  $u_x = u_r r_x, u_y = u_r r_y, u_z = u_r r_z$  and

$$u_{xx} = u_{rr}(r_x)^2 + u_r r_{xx}, \quad u_{yy} = u_{rr}(r_y)^2 + u_r r_{yy}, \quad u_{zz} = u_{rr}(r_z)^2 + u_r r_{zz}$$

On the other hand,  $r^2 = x^2 + y^2 + z^2$ , hence  $r_x = r^{-1}x$  and similarly for the other derivatives. Further,  $r_{xx} = r^{-1} - r^{-3}x^2$ . Thus

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= u_{rr}(r_x^2 + r_y^2 + r_z^2) + u_r(r_{xx} + r_{yy} + r_{zz}) \\ &= u_{rr} + 2r^{-1}u_r. \end{aligned}$$

so that the spherically symmetric wave equation can be written in the following form

$$u_{tt} = c^2(u_{rr} + 2r^{-1}u_r). \quad (4.3.3)$$

Upon multiplication by  $r$  and due to the identity  $(ru)_{rr} = ru_{rr} + 2u_r$  we obtain

$$(ru)_{tt} = c^2(ru)_{rr},$$

hence introducing the new function  $v = ru$  we can reduce the initial value problem (4.1.1) to the following boundary-initial value problem for the one-dimensional wave equation

$$\begin{aligned} v_{tt} &= c^2 v_{rr}, \quad r > 0, t > 0, \\ v(0, t) &= 0, \quad t > 0, \\ v(r, 0) &= r\phi(r), \quad r > 0, \\ v_t(r, 0) &= r\psi(r), \quad r > 0. \end{aligned}$$

We dealt with this problem in Example 6.2. Thus it follows that we have to extend  $r\phi(r)$  and  $r\psi(r)$  to the whole line in an odd manner and solve the resulting initial value problem. Specifying for the above case, we shall obtain from d'Alembert's formula the following solution:

$$\begin{aligned} u(r, t) &= \frac{1}{2r}((r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)) + \frac{1}{2rc} \int_{r-ct}^{r+ct} s\psi(s)ds, \text{ for } r > ct, \\ u(r, t) &= \frac{1}{2r}(\phi(ct+r) - \phi(ct-r)) + \frac{1}{2rc} \int_{ct-r}^{ct+r} r\psi(s)ds \text{ for } r < ct. \end{aligned} \quad (4.3.4)$$

**Example 3.3** Solve the following initial value problem for three dimensional wave equation:

$$\begin{aligned} u_{tt} &= \Delta u, \\ u(x, y, z, 0) &= e^{-\sqrt{x^2+y^2+z^2}}, \\ u_t(x, y, z, 0) &= 0. \end{aligned}$$

We see that the initial values are spherically symmetric and therefore using the substitution  $v(r, t) = ru(r, t)$  we reduce the above problem to the one-dimensional initial-boundary value problem on the half-line

$$\begin{aligned} v_{tt} &= v_{rr}, \quad r > 0, t > 0, \\ v(0, t) &= 0, \quad t > 0, \\ v(r, 0) &= re^{-r}, \quad r > 0, \\ v_t(r, 0) &= 0, \quad r > 0. \end{aligned}$$

Odd extension of  $v(r, 0)$  is given by  $v_{odd}(r, 0) = re^{-|r|}$ ,  $r \in \mathbb{R}$ , thus the solution of the above problem has the form

$$v(r, t) = \frac{1}{2} \left( (r+t)e^{-|r+t|} + (r-t)e^{-|r-t|} \right)$$

or, without absolute value bars,

$$v(r, t) = \frac{1}{2} \begin{cases} (r+t)e^{-(r+t)} + (r-t)e^{t-r} & \text{for } r > t, \\ (r+t)e^{-(r+t)} + (r-t)e^{r-t} & \text{for } r < t. \end{cases}$$

Thus, the solution  $u$  of the original problem is of the form

$$u(r, t) = \frac{1}{2r} \begin{cases} (r+t)e^{-(r+t)} + (r-t)e^{t-r} & \text{for } r > t, \\ (r+t)e^{-(r+t)} + (r-t)e^{r-t} & \text{for } r < t, \end{cases}$$

where  $r, t > 0$ .

**Example 3.4** Solve the following initial value problem for three dimensional wave equation:

$$\begin{aligned} u_{tt} &= \Delta u, \\ u(\mathbf{r}, 0) &= 0, \\ u_t(\mathbf{r}, 0) &= 1 \text{ for } \|\mathbf{r}\| \leq 1, \quad u_t(\mathbf{r}, 0) = 0 \text{ for } \|\mathbf{r}\| > 1. \end{aligned}$$

We see that the initial values are spherically symmetric and therefore we can use formula (4.3.4). Thus

$$\begin{aligned} u(r, t) &= \frac{1}{2r} \int_{r-t}^{r+t} s\psi(s)ds, \text{ for } r > t, \\ u(r, t) &= \frac{1}{2r} \int_{t-r}^{t+r} r\psi(s)ds \text{ for } r < t. \end{aligned}$$

Because  $\phi$  is defined by different formulae in different regions, we see that we will have also different representation of  $u$ . Let us consider first the domain where  $r - t > 0$ . Then, if  $r + t < 1$ , then the integration is carried over the whole interval giving

$$\int_{r-t}^{r+t} s ds = \frac{1}{2}((r+t)^2 - (r-t)^2) = 2rt.$$

The next subregion is given by  $0 < r - t < 1$  and  $r + t > 1$ . Then we have

$$\int_{r-t}^1 s ds = \frac{1}{2}(1 - (r-t)^2).$$

The last subregion is given by  $r - t > 1$ ,  $r + t > 1$ , and therefore the integration gives 0. In the region  $t - r > 0$  we have again the case with  $t + r < 1$ , hence

$$\int_{t-r}^{r+t} s ds = \frac{1}{2}((r+t)^2 - (t-r)^2) = 2rt.$$

The next subregion is determined by  $0 < t - r < 1$  and  $t + r > 1$  with the integral

$$\int_{t-r}^1 s ds = \frac{1}{2}(1 - (t-r)^2).$$

Over the last subregion the integral is again zero.

We see that we can combine these integrals so that the solution is given by

$$u(r, t) = \begin{cases} t & \text{for } r + t < 1 \\ \frac{1-(r-t)^2}{4r} & \text{for } |1-t| < r < 1+t \\ 0 & \text{elsewhere} \end{cases}$$

## 4 Energy conservation and uniqueness

In previous sections we have constructed solutions to the initial value problem for the wave equation but we still do not know whether this solution is unique. This is particularly important in the case of spherically symmetric initial data - to get the solution we simply assumed that the solution is also spherically symmetric, but there is no guarantee that there are no other solutions.

To prove that the wave equation is uniquely solvable we apply the so-called energy method.

Let us consider a vibrating elastic homogeneous body occupying at rest a region  $\Omega$  and let  $u(\mathbf{r}, t)$  be the displacement at time  $t$  of the point situated at  $\mathbf{r}$ . It can be calculated that kinetic energy of the body is given by

$$E_K(t) = \frac{\rho}{2} \int_{\Omega} (u_t)^2(\mathbf{r}, t) d\mathbf{r},$$

where  $\rho$  is the density of the body, and the potential (elastic) energy is given by

$$E_P(t) = \frac{T}{2} \int_{\Omega} |\nabla u|^2 d\mathbf{r},$$

where  $T$  is the magnitude of the tension in the body. The total energy of the system at a given time  $t$  is given by

$$E(t) = E_K(t) + E_P(t). \quad (4.4.1)$$

If the wave equation is modeling other physical phenomena (like, for instance, electromagnetic waves), then the meaning of the above integrals is different but they are always related to various forms of energy and (5.1.1) is the total energy of the system.

In what follows we prove that if the solution satisfies either homogeneous Dirichlet, or homogeneous Neumann boundary conditions on a bounded domain  $\Omega$ , or if  $\Omega = \mathbb{R}^3$  and the solution (together with its first derivatives) decays sufficiently fast to zero as  $\|\mathbf{r}\| \rightarrow \infty$ , then  $E(t)$  is independent of time. We multiply Eq. (4.1.1) by  $u_t$ , and using

$$(u_t^2)_t = 2u_t u_{tt}$$

and the following identity:

$$\begin{aligned} \operatorname{div}(u_t \nabla u) &= (u_t u_x)_x + (u_t u_y)_y + (u_t u_z)_z = u_{tx} u_x + u_{ty} u_y + u_{tz} u_z \\ &\quad + u_t(u_{xx} + u_{yy} + u_{zz}) = \frac{1}{2}(u_x^2 + u_y^2 + u_z^2)_t + u_t \Delta u = \frac{1}{2}|\nabla u|^2_t + u_t \Delta u, \end{aligned}$$

we obtain:

$$0 = u_t(u_{tt} - c^2 \Delta u) = \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right)_t - c^2 \nabla \cdot (u_t \nabla u).$$

Integrating this equation over  $\Omega$  and using Green's theorem we get:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (u_t^2 + c^2 |\nabla u|^2)_t d\mathbf{r} - \int_{\Omega} c^2 \nabla \cdot (u_t \nabla u) d\mathbf{r} &= \\ \frac{1}{2} \int_{\Omega} (u_t^2 + c^2 |\nabla u|^2)_t d\mathbf{r} - c^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial n} dS. \end{aligned}$$

Next, if the solution satisfies either homogeneous Dirichlet or homogeneous Neumann condition on the boundary, then the surface integral vanishes. On the other hand, if  $\Omega = \mathbb{R}^3$ , then we use the fact, that  $\int_{\mathbb{R}^3} = \lim_{\|\mathbf{r}\| \rightarrow \infty} \int_{B_r}$ , where  $B_r$  is the ball centered at zero with radius  $r$  and use the postulated convergence to zero of the solution and its derivatives to obtain that the surface integral tends to zero with  $r$  tending to infinity. Therefore we see that

$$\frac{dE}{dt} = \frac{\rho}{2} \int_{\Omega} (u_t^2 + c^2 |\nabla u|^2)_t d\mathbf{r} = \frac{d}{dt} \int_{\Omega} \left( \frac{\rho}{2} u_t^2 + \frac{T}{2} c^2 |\nabla u|^2 \right) d\mathbf{r} = 0$$

(recall that  $c^2 = T/\rho$ ), which shows that the total energy of the system is independent of time. This is the principle of conservation of the energy.

We use this result to prove that the initial value problem

$$\begin{aligned} u_{tt} &= c^2(u_{xx} + u_{yy} + u_{zz}), & (x, y, z) \in \mathbb{R}^3, t > 0, \\ u(x, y, z, 0) &= \phi(x, y, z), \\ u_t(x, y, z, 0) &= \psi(x, y, z), \end{aligned} \quad (4.4.2)$$

(and of course also the corresponding one and two dimensional problems) can have only one solution. Indeed, if  $u_1$  and  $u_2$  are two solutions to (4.3.1), then  $u = u_1 - u_2$  satisfies Eq. (4.1.1) with zero boundary (if applicable) and initial conditions:

$$u(\mathbf{r}, 0) = u_t(\mathbf{r}, 0) = 0. \quad (4.4.3)$$

At any moment of time we have from the principle of energy conservation

$$E(0) = E(t),$$

therefore by Eq. (5.1.2)

$$\begin{aligned} 0 &= \int_{\Omega} \left( \frac{1}{2} u_t^2(\mathbf{r}, 0) + \frac{1}{2} c^2 |\nabla u(\mathbf{r}, 0)|^2 \right) d\mathbf{r} \\ &= \int_{\Omega} \left( \frac{1}{2} u_t^2(\mathbf{r}, t) + \frac{1}{2} c^2 |\nabla u(\mathbf{r}, t)|^2 \right) d\mathbf{r}. \end{aligned}$$

Since all the integrands in the last integral are nonnegative, we obtain

$$u_t = 0, \quad \nabla u = 0$$

which implies  $u = \text{const}$  and, again by Eq. (5.1.2), we infer  $u = 0$ . Hence  $u_1 = u_2$  which shows that the solution is unique.

# Lecture 1

## One dimensional wave equation

### 5 First order equations

To introduce the topic let us consider the one dimensional transport equation. Assume that  $u$  is a concentration of some substance (pollutant) in a fluid (amount per unit length). This substance is moving to the right with a speed  $c$ . Then the differential equation for  $u$  has the form:

$$u_t + cu_x = 0. \quad (1.5.4)$$

Let us consider more general linear first order partial differential equation (PDE) of the form:

$$au_t + bu_x = 0, \quad t, x \in \mathbb{R} \quad (1.5.5)$$

where  $a$  and  $b$  are constants. This equation can be written as

$$D_{\mathbf{v}}u = 0, \quad (1.5.6)$$

where  $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$  and  $D_{\mathbf{v}} = \nabla u \cdot \mathbf{v}$  denotes the directional derivative in the direction of  $\mathbf{v}$ . This means that  $u$  remains constant along each line  $bt - ax = \text{const}$  (which is parallel to  $\mathbf{v}$ ) and the solution to Eq. (1.5.5) is given by

$$u(x, t) = f(bt - ax), \quad (1.5.7)$$

where  $f$  is an arbitrary differentiable function. We adopt the following definition.

**Definition 5.1** *Lines, along which the solution of the equation does not change, are called characteristic lines of the equation.*

Let us consider the initial value problem for Eq. (1.5.5), that is, the solution to

$$\begin{aligned} au_t + bu_x &= 0 & x \in \mathbb{R}, t > 0, \\ u(x, 0) &= g(x) & x \in \mathbb{R}, \end{aligned} \quad (1.5.8)$$

where  $g$  is an arbitrary function. From Eq. (1.5.7) we find that

$$u(x, t) = g\left(-\frac{bt - ax}{a}\right). \quad (1.5.9)$$

We note that the initial shape propagates without any change along the characteristic lines.

## 6 One dimensional wave equation

Let us consider a flexible elastic homogeneous string which undergoes relatively small transverse vibrations. We assume that the string remains in plane. We denote by  $u(x, t)$  the displacement of the string at point  $x$  in the moment of time  $t$ . The equation for  $u$  takes the following form:

$$u_{tt} = \frac{T}{\rho} u_{xx}, \quad (1.6.10)$$

where  $\rho$  is the density of the string (mass per unit length) and  $T$  is the tension in the string. Both parameters are assumed here to be constant. We denote

$$c := \sqrt{\frac{T}{\rho}}. \quad (1.6.11)$$

As we see later,  $c$  is the speed of wave propagation.

Equation (1.6.10) has many variations. For instance:

- (a) If a significant air resistance is present, then we have the following damped wave equation:

$$u_{tt} - c^2 u_{xx} + ru_t = 0. \quad (1.6.12)$$

- (b) If there is a transverse elastic force, then we obtain:

$$u_{tt} - c^2 u_{xx} + ku = 0. \quad (1.6.13)$$

- (c) If there is externally applied force, then the equation becomes inhomogeneous:

$$u_{tt} - c^2 u_{xx} = f(x, t). \quad (1.6.14)$$

- (d) The relation between the current  $i$  and the voltage  $u$  in the transmission line is described by the following system of equations:

$$\begin{aligned} u_t &= -\frac{1}{C} i_x - \frac{G}{C} u \\ i_t &= -\frac{1}{L} u_x - \frac{R}{L} i, \end{aligned} \quad (1.6.15)$$

where  $C$  is the capacitance,  $L$  is the self-inductance,  $R$  is the resistance and  $G$  is the leakage coefficient (per unit length). We can write the system (1.6.15) in the form of a single equation:

$$u_{tt} = \frac{1}{CL} u_{xx} + \frac{RC - LG}{CL} u_t + \frac{GR}{CL} u. \quad (1.6.16)$$

In the ideal situation with zero resistance  $R$  and perfect insulation ( $G = 0$ ) we see that Eq. (1.6.16) becomes the wave equation.

Let us find a solution to the wave equation (1.6.10). We use the so-called characteristic method. We see that the equation factorizes to

$$\left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = 0,$$

so it can be thought as a superposition of two first order equations. Let us introduce new variables related to characteristics of both equations.

$$\xi = \xi(x, t) = x + ct, \quad \eta = \eta(x, t) = x - ct$$



and denote

$$u(t, x) = w(\xi(x, t), \eta(x, t)).$$

Using the chain rule we obtain:

$$u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta} = 0.$$

Solving the last equation we obtain:

$$w(\xi, \eta) = f(\xi) + g(\eta),$$

where  $f$  and  $g$  are arbitrary functions. Returning to the old variables we get:

$$u(x, t) = f(x + ct) + g(x - ct). \quad (1.6.17)$$

As can be seen, the solution  $u$  is the superposition of two waves:  $g(x - ct)$  traveling to the right at speed  $c$  and  $f(x + ct)$  which travels to the left at the same speed  $c$ .

The initial value problem for the wave equation consists in finding the particular solution which satisfies

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \quad (1.6.18)$$

where  $\phi$  and  $\psi$  are arbitrary given functions.

Physically, it corresponds to looking for the evolution of the shape of a string when the initial shape and the initial velocity are given.

Using Eqs. (1.6.17) and (1.6.18) we find that

$$u(x, t) = \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds \quad (1.6.19)$$

This equation is known as the d'Alembert formula. If  $\phi$  is twice differentiable and  $\psi$  once, Eq. (1.6.19) give a genuine solution to the initial value problem (1.6.10), (1.6.18) and, as we shall see later, a unique one.

**Example 6.1** (The plucked string) Consider the infinitely long string with the initial position:

$$\phi(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

and the initial velocity  $\psi = 0$ . From d'Alembert formula we obtain immediately

$$u(x, t) = \frac{1}{2} \begin{cases} 2 - |x - ct| - |x + ct| & \text{for } -1 + ct \leq x \leq 1 - ct \\ 1 - |x - ct| & \text{for } |1 - ct| < x \leq 1 + ct \\ 1 - |x + ct| & \text{for } -1 - ct \leq x \leq -|1 - ct| \\ 0 & \text{elsewhere.} \end{cases}$$

By substituting several values for  $t$  we can check that the solution consists of two triangular waves traveling to the left and to the right at speed  $c$ . However, for  $t < 1/2c$  these two waves have a common support and their superposition gives a trapezoidal wave. For  $t > 1/2c$  the waves separates from each other and preserve their triangular shape forever.

**Example 6.2** (Waves on the half-line) Let us consider the following initial-boundary value problem on a half-line:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} & \text{for } x > 0, \quad t > 0, \\ u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x) & \text{for } x > 0, \\ u(0, t) &= 0 & \text{for } t > 0. \end{aligned}$$

This problem can be reduced to the pure initial value problem by extending the initial data in such a way that the resulting solution is equal to zero for  $x = 0$ . The required extension turns out to be the odd one:

let  $\phi_{odd}$  and  $\psi_{odd}$  be the odd extensions of  $\phi$  and  $\psi$ , respectively, that is e.g.  $\phi_{odd}(-x) = -\phi_{odd}(x)$ . Define  $v$  to be the solution of the corresponding initial value problem given by the d'Alembert formula:

$$v(x, t) = \frac{1}{2}(\phi_{odd}(x + ct) + \phi_{odd}(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{odd}(s) ds, \quad (1.6.20)$$

and let  $u(x)$  be  $v$  restricted to  $x > 0$ . We have  $u(0, t) = 0$  due to the fact that all functions are odd, thus  $u$  is a solution of the original boundary-initial value problem. To write down the solution in the form involving only  $\phi$  and  $\psi$  we note that for  $x > ct$  both  $x + ct$  and  $x - ct$  are positive and for positive arguments  $\phi = \phi_{odd}$  and  $\psi = \psi_{odd}$ . For  $x < ct$  we utilize  $\phi_{odd}(x - ct) = -\phi(ct - x)$  and the analogous property of  $\psi$ . Combining, we get:

$$\begin{aligned} u(x, t) &= \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds, \quad x > ct, \\ u(x, t) &= \frac{1}{2}(\phi(ct + x) - \phi(ct - x)) + \frac{1}{2c} \int_{ct-x}^{ct+x} \psi(s) ds. \quad x < ct. \end{aligned} \quad (1.6.21)$$

# Lecture 4

## The wave equation in space

In this chapter we shall generalize the d'Alembert formula for 2 and 3 spatial dimensions.

### 1 Three dimensional wave equation

Let  $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$ . We consider the equation

$$u_{tt} - c^2 \Delta u = u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0 \quad (4.1.1)$$

subject to the following initial conditions:

$$u(\mathbf{r}, 0) = \phi(\mathbf{r}), \quad u_t(\mathbf{r}, 0) = \psi(\mathbf{r}). \quad (4.1.2)$$

We prove that the solution to the problem (1.5.4)-(1.5.5) is given by the following generalization of the d'Alembert formula:

$$u(\mathbf{r}, t) = \frac{1}{4\pi c^2 t} \int_S \psi(\mathbf{w}) dS + \frac{\partial}{\partial t} \left( \frac{1}{4\pi c^2 t} \int_S \phi(\mathbf{w}) dS \right) \quad (4.1.3)$$

where the integration is carried over the sphere  $S = \{\mathbf{w}; \|\mathbf{r} - \mathbf{w}\| = ct\}$  (here  $\|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$ ). Formula (4.1.3) is called Kirchhoff's formula.

To prove this formula we introduce the average of  $u$  over the sphere  $S_\rho = \{\mathbf{w}; \|\mathbf{w}\| = \rho\}$  defined as

$$\bar{u}(\rho, t) = \frac{1}{4\pi\rho^2} \int_{S_\rho} u(\mathbf{w}, t) dS = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi u(\mathbf{w}, t) \sin\theta d\theta d\varphi. \quad (4.1.4)$$

We note that for continuous  $u$  we have for any  $t$

$$\lim_{\rho \rightarrow 0} \bar{u}(\rho, t) = u(0, t). \quad (4.1.5)$$

In fact, with the formula

$$u(0, t) = \frac{1}{4\pi\rho^2} \int_S u(0, t) dS$$

we obtain

$$|\bar{u}(\rho, t) - u(0, t)| \leq \frac{1}{4\pi\rho^2} \int_S |u(\mathbf{w}, t) - u(0)| d\mathbf{w} \leq \epsilon$$

provided  $\rho$  is small enough to have  $\max_{\|\mathbf{w}\| \leq \rho} |u(\mathbf{w}, t) - u(0)| \leq \epsilon$  (which follows from the continuity of  $u$ ). To find the equation satisfied by  $\bar{u}$  we integrate Eq. (4.1.1) over the solid ball  $D_\rho = \{\mathbf{w}; \|\mathbf{w}\| \leq \rho\}$  and upon using the divergence (Gauss) theorem we obtain:

$$\int_{D_\rho} u_{tt}(\mathbf{w}) d\mathbf{w} = c^2 \int_{D_\rho} \Delta u d\mathbf{w} = c^2 \int_{S_\rho} \frac{\partial u}{\partial n} dS. \quad (4.1.6)$$

Let us calculate the normal derivative  $\frac{\partial u}{\partial n}$  on the surface of the sphere. It follows that the unit outward normal is given by  $\mathbf{n} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x, y, z)$  and therefore

$$\frac{\partial u}{\partial n} = \frac{xu_x + yu_y + zu_z}{\sqrt{x^2 + y^2 + z^2}}.$$

On the other hand, using spherical coordinates

$$x = \rho \cos \phi \sin \psi, \quad y = \rho \sin \phi \sin \psi, \quad z = \rho \cos \psi,$$

$r > 0, 0 < \phi < 2\pi, 0 < \psi < \pi$ , we obtain

$$u_\rho = u_x x_\rho + u_y y_\rho + u_z z_\rho = \frac{xu_x + yu_y + zu_z}{\rho} = \frac{\partial u}{\partial n}.$$

Because of this and Eq. (4.1.4), Eq. (6.1.11) can be written as

$$\int_0^\rho \bar{u}_{tt}(w, t) w^2 dw = c^2 \int_{S_\rho} u_\rho dS = c^2 \rho^2 \frac{\partial}{\partial \rho} \int_{S_\rho} u dS = c^2 \rho^2 \frac{\partial \bar{u}}{\partial \rho}(\rho, t).$$

Differentiating with respect to  $\rho$  we obtain:

$$\rho^2 \bar{u}_{tt} = c^2 \rho^2 \bar{u}_{\rho\rho} + c^2 2\rho \bar{u}_\rho. \quad (4.1.7)$$

Denoting  $v = \rho \bar{u}$  we see that  $v$  satisfies one dimensional wave equation

$$v_{tt} = c^2 v_{\rho\rho} \quad (4.1.8)$$

with the following boundary and initial conditions:

$$\begin{aligned} v(0, t) &= 0, \\ v(\rho, 0) &= \rho \bar{\phi}(\rho), \\ v_t(\rho, 0) &= \rho \bar{\psi}(\rho). \end{aligned} \quad (4.1.9)$$

We dealt with this problem in Example 6.2. From the second equation (1.6.21) we see that for  $0 < x < ct$  we have

$$\begin{aligned} v(\rho, t) &= \frac{1}{2} ((ct + \rho) \bar{\phi}(ct + \rho) - (ct - \rho) \bar{\phi}(ct - \rho) + \int_{ct-\rho}^{ct+\rho} s \bar{\psi}(s) ds) \\ &= \frac{1}{2c} \left( \frac{\partial}{\partial t} \int_{ct-\rho}^{ct+\rho} s \bar{\phi}(s) ds + \int_{ct-\rho}^{ct+\rho} s \bar{\psi}(s) ds \right). \end{aligned}$$

where we used the following Calculus formula:

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(s) ds = f(b(x)) \frac{db}{dx} - f(a(x)) \frac{da}{dx}.$$

By Eq. (4.1.5) and Eq. (4.1.9) we obtain

$$u(0, t) = \lim_{\rho \rightarrow 0} \rho^{-1} v(\rho, t) = \lim_{\rho \rightarrow 0} \frac{v(\rho, t) - v(0, t)}{\rho} = \frac{\partial v}{\partial \rho}(0, t). \quad (4.1.10)$$

Differentiating Eq. (4.1.10) we get:

$$\frac{\partial v}{\partial \rho}(\rho, t) = (ct + \rho)\bar{\psi}(ct + \rho) + (ct - \rho)\bar{\psi}(ct - \rho) + \frac{\partial}{\partial t}((ct + \rho)\bar{\phi}(ct + \rho) + (ct - \rho)\bar{\phi}(ct - \rho)).$$

and using the definition of the average and Eq. (4.1.10) we obtain

$$\begin{aligned} u(0, t) &= t\bar{\psi}(ct) + \frac{\partial \bar{\phi}}{\partial t}(ct) \\ &= \frac{1}{4\pi c^2 t} \int_{\|\mathbf{w}\|=ct} \psi(\mathbf{w}) dS + \frac{\partial}{\partial t} \left( \frac{1}{4\pi c^2 t} \int_{\|\mathbf{w}\|=ct} \phi(\mathbf{w}) dS \right) \end{aligned} \quad (4.1.11)$$

which gives the formula for evolution at  $\mathbf{r} = 0$ . To obtain the evolution for arbitrary  $\mathbf{r}$  we note that for any fixed  $\mathbf{r}$  the function  $U(\mathbf{w}, t) = u(\mathbf{w} + \mathbf{r}, t)$  is the solution of the problem (4.1.1)-(4.1.2) with the initial data  $\phi(\mathbf{w} + \mathbf{r})$  and  $\psi(\mathbf{w} + \mathbf{r})$ . By Eq. (4.1.11) we obtain

$$\begin{aligned} u(\mathbf{r}, t) &= U(0, t) = \\ &= \frac{1}{4\pi c^2 t} \int_{\|\mathbf{w}\|=ct} \psi(\mathbf{w} + \mathbf{r}) dS + \frac{\partial}{\partial t} \left( \frac{1}{4\pi c^2 t} \int_{\|\mathbf{w}\|=ct} \phi(\mathbf{w} + \mathbf{r}) dS \right) \\ &= \frac{1}{4\pi c^2 t} \int_{\|\mathbf{w}-\mathbf{r}\|=ct} \psi(\mathbf{w}) dS + \frac{\partial}{\partial t} \left( \frac{1}{4\pi c^2 t} \int_{\|\mathbf{w}-\mathbf{r}\|=ct} \phi(\mathbf{w}) dS \right) \end{aligned} \quad (4.1.12)$$

which is exactly Eq. (4.1.3).

## 2 The method of descent for two dimensional problems

We solve now the analogous problem in two dimensions: find  $u$  which satisfies the equation

$$u_{tt} - c^2(u_{xx} + u_{yy}) = 0 \quad (4.2.1)$$

and the following initial conditions:

$$u(x, y, 0) = \phi(x, y), \quad u_t(x, y, 0) = \psi(x, y). \quad (4.2.2)$$

The idea is to treat the solution of the two dimensional problem as the solution of the three dimensional problem which is independent of  $z$ . To shorten notation we assume that  $\phi = 0$ . The surface integral in the formula (4.1.11) can be simplified to the double integral over the disc in the following way:

$$\begin{aligned} u(0, 0, t) &= \frac{1}{4\pi c^2 t} \int_{x^2+y^2+z^2=c^2t^2} \psi(x, y) dS \\ &= \frac{1}{2\pi c^2 t} \int_{x^2+y^2 \leq c^2t^2} \psi(x, y) \sqrt{1+z_x^2+z_y^2} dx dy \\ &= \frac{1}{2\pi c} \int_{x^2+y^2 \leq c^2t^2} \frac{\psi(x, y)}{\sqrt{c^2t^2 - x^2 - y^2}} dx dy, \end{aligned}$$

where we used the fact that  $z = \sqrt{c^2 t^2 - x^2 - y^2}$  and therefore

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{\frac{c^2 t^2 - x^2 - y^2 + x^2 + y^2}{c^2 t^2 - x^2 - y^2}} = \frac{ct}{\sqrt{c^2 t^2 - x^2 - y^2}}.$$

The solution to the full original problem (4.2.1)-(4.2.2) can be obtained by translation, as in three dimensional case:

$$\begin{aligned} u(x, y, t) = & \frac{1}{2\pi c} \left( \int_C \frac{\psi(s, r)}{\sqrt{c^2 t^2 - (x-s)^2 - (y-r)^2}} ds dr \right. \\ & \left. + \frac{\partial}{\partial t} \int_C \frac{\phi(s, r)}{\sqrt{c^2 t^2 - (x-s)^2 - (y-r)^2}} ds dr \right), \end{aligned} \quad (4.2.3)$$

where  $C = \{(r, s); (x-s)^2 + (y-r)^2 \leq c^2 t^2\}$ .

### 3 Spherical waves

The Kirchhoff formula (4.1.3) is not immediately useful for calculating solutions to the initial value problem (4.1.1), (4.1.2). It is, however, of a theoretical importance. One example of application will be demonstrated in this section.

Let us assume that the initial values  $\phi(\mathbf{r})$  and  $\psi(\mathbf{r})$  of (4.1.2) are spherically symmetric, that is,  $\phi(\mathbf{r}) = \phi(\|\mathbf{r}\|) = \phi(\sqrt{x^2 + y^2 + z^2})$  and  $\psi(\mathbf{r}) = \psi(\|\mathbf{r}\|) = \psi(\sqrt{x^2 + y^2 + z^2})$ , and let us consider the first integral in the first term in (4.1.3), denoted for a moment by  $u_1$ ,

$$u_1(\mathbf{r}, t) = \int_S \psi(\mathbf{w}) dS.$$

The integration here is carried out over the sphere centered at  $\mathbf{r}$  and having radius  $ct$ . It follows that the value of this integral depends only on the value  $r = \|\mathbf{r}\|$  and not on the direction (provided  $\psi$  depends only on  $r$ ). An intuitive justification of this fact follows from the observation that the integral "collects" all the values of  $\psi$  from the sphere  $\|\mathbf{r} - \mathbf{w}\| = ct$  and, because  $\psi$  is spherically symmetric, this depends only on the distance of  $\mathbf{r}$  from the origin and not on the direction to  $\mathbf{r}$ .

A mathematically rigorous proof of this statement requires a knowledge of some facts from Linear Algebra, which will be stated here without proofs. Let  $A$  be a matrix representing any rotation of  $\mathbb{R}^3$  about the origin (then  $A$  is an orthogonal matrix). Then the following are true:

1.  $A$  is an isometry, that is, for any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ ,  $\|A\mathbf{x} - A\mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$ ,
2.  $A$  is invertible and the inverse  $A^{-1}$  is also an isometry,
3. for any spherically symmetric function  $\psi$  and any  $\mathbf{x} \in \mathbb{R}^3$ ,  $\psi(A\mathbf{x}) = \psi(\mathbf{x})$ ,
4. the Jacobian of the transformation  $A$ , equal to  $|\det A|$ , is equal to 1.

Let us take  $\mathbf{r}_1$  and  $\mathbf{r}_2$  with  $\|\mathbf{r}_1\| = \|\mathbf{r}_2\|$ , and let  $A$  be any rotation such that  $A\mathbf{r}_1 = \mathbf{r}_2$ . We have

$$\begin{aligned} u_1(\mathbf{r}_2, t) &= u_1(A\mathbf{r}_1, t) = \int_{\|\mathbf{w} - A\mathbf{r}_1\| = ct} \psi(\mathbf{w}) dS_w = \int_{\|AA^{-1}\mathbf{w} - A\mathbf{r}_1\| = ct} \psi(\mathbf{w}) dS_w \\ &= \int_{\|A^{-1}\mathbf{w} - \mathbf{r}_1\| = ct} \psi(\mathbf{w}) dS_w = \int_{\|\mathbf{w}' - A\mathbf{r}_1\| = ct} \psi(A\mathbf{w}') dS_{w'} \end{aligned}$$

$$= \int_{\|\mathbf{w}' - \mathbf{r}_1\| = ct} \psi(\mathbf{w}') dS_{\mathbf{w}'} = u_1(\mathbf{r}_1, t)$$

which shows that  $u_1$  is spherically symmetric. Since the same argument is valid for the second term in (4.1.3) we can state the following theorem.

**Theorem 3.1** *The solution of the initial value problem for the wave equation (4.1.1), (4.1.2) with spherically symmetric initial data  $\psi(\mathbf{r}) = \psi(r)$  and  $\phi(\mathbf{r}) = \phi(r)$  is also spherically symmetric.*

Let us consider therefore the following initial value problem: find  $u$  satisfying

$$u_{tt} - c^2 \Delta u = u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0 \quad (4.3.1)$$

and subject to the following initial conditions:

$$u(\mathbf{r}, 0) = \phi(r), \quad u_t(\mathbf{r}, 0) = \psi(r). \quad (4.3.2)$$

Since we know that the solution must be also of the form  $u(\mathbf{r}, t) = u(r, t)$  we see that it would be useful to write the Laplacian in (4.3.1) in spherical coordinates (see Tutorial 2), observing in the process that all the derivatives with respect to the angular variables will drop out.

Let the spherical coordinates be given by  $x = r \cos \alpha \sin \beta, y = r \sin \alpha \sin \beta, z = r \cos \beta, r > 0, 0 < \alpha < 2\pi, 0 < \beta < \pi$ . Using the Chain Rule we obtain  $u_x = u_r r_x, u_y = u_r r_y, u_z = u_r r_z$  and

$$u_{xx} = u_{rr}(r_x)^2 + u_r r_{xx}, \quad u_{yy} = u_{rr}(r_y)^2 + u_r r_{yy}, \quad u_{zz} = u_{rr}(r_z)^2 + u_r r_{zz}$$

On the other hand,  $r^2 = x^2 + y^2 + z^2$ , hence  $r_x = r^{-1}x$  and similarly for the other derivatives. Further,  $r_{xx} = r^{-1} - r^{-3}x^2$ . Thus

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= u_{rr}(r_x^2 + r_y^2 + r_z^2) + u_r(r_{xx} + r_{yy} + r_{zz}) \\ &= u_{rr} + 2r^{-1}u_r. \end{aligned}$$

and the spherically symmetric wave equation can be written in the following form

$$u_{tt} = c^2(u_{rr} + 2r^{-1}u_r). \quad (4.3.3)$$

Upon multiplication by  $r$  and due to the identity  $(ru)_{rr} = ru_{rr} + 2u_r$  we obtain

$$(ru)_{tt} = c^2(ru)_{rr},$$

hence introducing a new function  $v = ru$  we can reduce the initial value problem (4.3.1), (4.3.2) to the following boundary-initial value problem for the one-dimensional wave equation

$$\begin{aligned} v_{tt} &= c^2 v_{rr}, & r > 0, t > 0, \\ v(0, t) &= 0, & t > 0, \\ v(r, 0) &= r\phi(r), & r > 0, \\ v_t(r, 0) &= r\psi(r), & r > 0. \end{aligned}$$

**Remark 3.1** We could obtain the same result noting that for a spherically symmetric function  $u$ , the average  $\bar{u}$ , introduced in (4.1.4), is equal exactly to  $u$ . Therefore the problem (4.1.8) and (4.1.9) coincides with the one above.

We dealt with this problem in Example 6.2. It follows that we have to extend  $r\phi(r)$  and  $r\psi(r)$  to the whole line in an odd manner and solve the resulting initial value problem. Specifying for the above case, we shall

obtain from d'Alembert's formula the following solution:

$$\begin{aligned} u(r, t) &= \frac{1}{2r}((r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)) + \frac{1}{2rc} \int_{r-ct}^{r+ct} s\psi(s)ds, \text{ for } r > ct, \\ u(r, t) &= \frac{1}{2r}(\phi(ct+r) - \phi(ct-r)) + \frac{1}{2rc} \int_{ct-r}^{ct+r} r\psi(s)ds \text{ for } r < ct. \end{aligned} \quad (4.3.4)$$

**Example 3.1** Solve the following initial value problem for three dimensional wave equation:

$$\begin{aligned} u_{tt} &= \Delta u, \\ u(\mathbf{r}, 0) &= 0, \\ u_t(\mathbf{r}, 0) &= 1 \text{ for } \|\mathbf{r}\| \leq 1, \quad u_t(\mathbf{r}, 0) = 0 \text{ for } \|\mathbf{r}\| > 1. \end{aligned}$$

We see that the initial values are spherically symmetric and therefore we can use formula (4.3.4). Thus

$$\begin{aligned} u(r, t) &= \frac{1}{2r} \int_{r-t}^{r+t} s\psi(s)ds, \text{ for } r > t, \\ u(r, t) &= \frac{1}{2r} \int_{t-r}^{t+r} r\psi(s)ds \text{ for } r < t. \end{aligned}$$

Because  $\phi$  is defined by different formulae in different regions, we see that we will have also different representations of  $u$ . Let us consider first the domain where  $r - t > 0$ . Then, if  $r + t < 1$ , then the integration is carried over the whole interval giving

$$\int_{r-t}^{r+t} sds = \frac{1}{2}((r+t)^2 - (r-t)^2) = 2rt.$$

The next subregion is given by  $0 < r - t < 1$  and  $r + t > 1$ . Then we have

$$\int_{r-t}^1 sds = \frac{1}{2}(1 - (r-t)^2).$$

The last subregion is given by  $r - t > 1$ ,  $r + t > 1$ , and therefore the integration gives 0. In the region  $t - r > 0$  we have again the case with  $t + r < 1$ , hence

$$\int_{t-r}^{t+r} sds = \frac{1}{2}((t+r)^2 - (t-r)^2) = 2rt.$$

The next subregion is determined by  $0 < t - r < 1$  and  $t + r > 1$  with the integral

$$\int_{t-r}^1 sds = \frac{1}{2}(1 - (t-r)^2).$$

Over the last subregion the integral is again zero.

We see that we can combine these integrals so that the solution is given by

$$u(r, t) = \begin{cases} t & \text{for } r + t < 1 \\ \frac{1-(r-t)^2}{4r} & \text{for } |1-t| < r < 1+t \\ 0 & \text{elsewhere} \end{cases}$$



**Remark 3.2** The above procedure is unavailable in 2 spatial dimensions. Precisely, in polar coordinates the equation corresponding to (4.3.3) is given by

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r),$$

which cannot be reduced to the one-dimensional wave equation.



# Lecture 5

## Energy and causality

### 1 Principle of conservation of energy

In Lectures 3 and 4 we have constructed solutions to the initial value problem for the wave equation but we still do not know whether this solution is unique. The answer is positive and to prove it we apply the so-called energy method.

Let us consider a vibrating elastic homogeneous body occupying at rest a region  $\Omega$  and let  $u(\mathbf{r}, t)$  be the displacement at time  $t$  of the point situated at  $\mathbf{r}$ . It can be calculated that kinetic energy of the body is given by

$$E_K(t) = \frac{\rho}{2} \int_{\Omega} u_t^2(\mathbf{r}, t) d\mathbf{r},$$

where  $\rho$  is the density of the body, and the potential (elastic) energy is given by

$$E_P(t) = \frac{T}{2} \int_{\Omega} |\nabla u|^2 d\mathbf{r},$$

where  $T$  is the magnitude of the tension in the body. The total energy of the system at a given time  $t$  is given by

$$E(t) = E_K(t) + E_P(t). \quad (5.1.1)$$

If the wave equation is modeling other physical phenomena (like, for instance, electromagnetic waves), then the meaning of the above integrals is different but they are always related to various forms of energy and (5.1.1) is the total energy of the system.

In what follows we prove that if the solution satisfies either homogeneous Dirichlet, or homogeneous Neumann boundary conditions, or  $\Omega = \mathbb{R}^3$  and the solution decays sufficiently fast to zero as  $\|\mathbf{r}\| \rightarrow \infty$ , then  $E(t)$  is independent of time. We multiply Eq. (4.1.1) by  $u_t$ , and using  $(u_t^2)_t = 2u_t u_{tt}$  and the following identity:

$$\begin{aligned} \operatorname{div}(u_t \nabla u) &= (u_t u_x)_x + (u_t u_y)_y + (u_t u_z)_z = u_{tx} u_x + u_{ty} u_y + u_{tz} u_z \\ &+ u_t(u_{xx} + u_{yy} + u_{zz}) = \frac{1}{2}(u_x^2 + u_y^2 + u_z^2)_t + u_t \Delta u = \frac{1}{2}|\nabla u|^2_t + u_t \Delta u, \end{aligned}$$

we obtain:

$$0 = u_t(u_{tt} - c^2 \Delta u) = \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right)_t - c^2 \nabla \cdot (u_t \nabla u).$$

Integrating this equation over  $\Omega$  and using Green's theorem we get:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (u_t^2 + c^2 |\nabla u|^2)_t d\mathbf{r} - \int_{\Omega} c^2 \nabla \cdot (u_t \nabla u) d\mathbf{r} &= \\ \frac{1}{2} \int_{\Omega} (u_t^2 + c^2 |\nabla u|^2)_t d\mathbf{r} - c^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial n} dS. \end{aligned}$$

Next, if the solution satisfies either homogeneous Dirichlet or homogeneous Neumann condition on the boundary, then the surface integral vanishes. On the other hand, if  $\Omega = \mathbb{R}^3$ , then we use the fact, that  $\int_{\mathbb{R}^3} = \lim_{\|\mathbf{r}\| \rightarrow \infty} \int_{B_r}$ , where  $B_r$  is the ball centered at zero with radius  $r$  and use the postulated convergence to zero of the solution and its derivatives to obtain that the surface integral tends to zero with  $r$  tending to infinity. Therefore we see that

$$\frac{dE}{dt} = \frac{\rho}{2} \int_{\Omega} (u_t^2 + c^2 |\nabla u|^2)_t d\mathbf{r} = \frac{d}{dt} \int_{\Omega} \left( \frac{\rho}{2} u_t^2 + \frac{T}{2} c^2 |\nabla u|^2 \right) d\mathbf{r} = 0$$

(recall that  $c^2 = T/\rho$ ), which shows that the total energy of the system is independent of time. This is the principle of conservation of the energy.

We use this result to prove that the initial value problem (4.1.1)-(4.1.2) (and also the corresponding one and two dimensional problems) can have only one solution. Indeed, if  $u_1$  and  $u_2$  are two solutions to (4.1.1)-(4.1.2), then  $u = u_1 - u_2$  satisfies Eq. (4.1.1) with zero boundary (if applicable) and initial conditions:

$$u(\mathbf{r}, 0) = u_t(\mathbf{r}, 0) = 0. \quad (5.1.2)$$

At any moment of time we have from the principle of energy conservation

$$E(0) = E(t),$$

therefore by Eq. (5.1.2)

$$\begin{aligned} 0 &= \int_{\Omega} \left( \frac{1}{2} u_t^2(\mathbf{r}, 0) + \frac{1}{2} c^2 |\nabla u(\mathbf{r}, 0)|^2 \right) d\mathbf{r} \\ &= \int_{\Omega} \left( \frac{1}{2} u_t^2(\mathbf{r}, t) + \frac{1}{2} c^2 |\nabla u(\mathbf{r}, t)|^2 \right) d\mathbf{r}. \end{aligned}$$

Since all the integrands in the last integral are all nonnegative, we obtain

$$u_t = 0, \quad \nabla u = 0$$

which implies  $u = \text{const}$  and again by Eq. (5.1.2) we infer  $u = 0$ . Hence  $u_1 = u_2$  which shows that the solution is unique.

## 2 Causality and Huygens' principle

A closer look at Kirchhoff's formula (4.1.3) shows that the values of the solution  $u(\mathbf{r}, t)$  to the initial value problem (4.1.1)- (4.2.2) at a point  $P(\mathbf{r}, t)$  of the space-time depends only on the initial values of  $u$  on the spherical surface  $S = \{\mathbf{w}; |\mathbf{w} - \mathbf{r}| = ct\}$ . This statement can be inverted in the following way: the initial values  $\phi(\mathbf{r}_0)$  and  $\psi(\mathbf{r}_0)$  at a spatial point  $\mathbf{r}_0$  influence the solution only on the surface  $\{(\mathbf{r}, t); |\mathbf{r} - \mathbf{r}_0| = ct\}$  of the cone that emanates from the point of the space-time  $(\mathbf{r}_0, 0)$ . This is the statement of the Huygens' principle. It means that the solution of the three-dimensional wave equation propagates exactly at speed  $c$ , not faster, and not slower.

This property allows us to see and hear sharply, since acoustic and electromagnetic (light) waves obey the wave equation. The observer sees (or hears) exactly what happened at time  $t - d/c$ , where  $d$  is the distance between him and the source. We shall illustrate this statement by considering how two signals (waves), sent at  $\mathbf{r} = 0$  at times  $t = -2$  and  $t = 0$ , are they registered by an observer situated at some point with  $\|\mathbf{r}\| = 4$ .

**Example 2.1** Let us consider the solution to the wave equation

$$\begin{aligned} u_{tt} &= u_{xx} + u_{yy} + u_{zz}, \\ u(\mathbf{r}, -2) &= 0, \\ u_t(\mathbf{r}, -2) &= \psi(\mathbf{r}), \end{aligned} \quad (5.2.3)$$

where  $\psi(\mathbf{r}) = 1$  for  $\|\mathbf{r}\| \leq 1$  and  $\psi(\mathbf{r}) = 0$  elsewhere. In the case when the same initial value was specified at  $t_0 = 0$ , this problem was solved in Example 3.1 and the solution is given by

$$u(r, t) = \begin{cases} t & \text{for } r + t < 1 \\ \frac{1-(r-t)^2}{4r} & \text{for } |1-t| < r < 1+t \\ 0 & \text{elsewhere} \end{cases}$$

It is easy to check that then  $v(r, t) = u(r, t + 2)$  will be the solution of the problem (5.2.3). Let us consider the situation that somebody wants to send a signal by specifying the value of  $u_t$ , as given above, at  $t_0 = -2$  and  $t_1 = 0$ . The corresponding solution  $w$  will be given for  $t > 0$  by

$$w(\mathbf{r}, t) = v(\mathbf{r}, t) + u(\mathbf{r}, t).$$

In fact, to find this solution we observe that  $w$  must be a solution of the initial value problem with the initial values  $w(\mathbf{r}, 0) = v(\mathbf{r}, 0)$  and  $w_t(\mathbf{r}, 0) = \psi(\mathbf{r}) + v_t(\mathbf{r}, 0)$ . Since the problem is linear, the solution is given by the sum of solutions corresponding to initial values  $w_1(\mathbf{r}, 0) = 0$  and  $w_{1,t}(\mathbf{r}, 0) = \psi(\mathbf{r})$  and  $w_2(\mathbf{r}, 0) = v(\mathbf{r}, 0)$  and  $w_{2,t}(\mathbf{r}, 0) = v_t(\mathbf{r}, 0)$ . Since the solutions to the initial value problems are unique, that is, we cannot have more than one solution to the wave equation having equal values at  $t = 0$ , and with equal values of their time derivatives at  $t = 0$ , we see that  $w_2(\mathbf{r}, t) = v(\mathbf{r}, t)$  and  $w_1(\mathbf{r}, t) = u(\mathbf{r}, t)$ .

Now, let us consider an observer at a distance  $r = 4$  from the origin. He will record the following signal:

$$w(4, t) = v(4, t) + u(4, t) = \begin{cases} 0 & \text{for } 0 < t < 1 \\ \frac{1-(2-t)^2}{16} & \text{for } 1 < t < 3 \\ \frac{1-(4-t)^2}{16} & \text{for } 3 < t < 5 \\ 0 & \text{for } t > 5 \end{cases}$$

that is, he will record two clearly distinct signals corresponding to the signals emitted.

This is not so in one dimension as shown in the following example.

**Example 2.2** Let us consider a situation analogous to that in the previous example, only in one dimension. Suppose that somebody sent two signals by hitting the string at  $-1 < x < 1$  with a hammer twice: at  $t = -2$  and  $t = 0$ , and with velocity 1. The initial value problem related to the event at  $t = 0$  is as follows: find  $u$  satisfying

$$\begin{aligned} u_{tt} &= u_{xx}, \\ u(x, 0) &= 0, \\ u_t(x, 0) &= \psi(x), \end{aligned} \tag{5.2.4}$$

where  $\psi(x) = 1$  for  $|x| < 1$  and  $\psi(x) = 0$  otherwise. Using d'Alembert's formula we calculate that the solution is given by

$$u(x, t) = \begin{cases} t & \text{for } |x| < 1 - t, \\ 1 & \text{for } |x| < t - 1, \\ \frac{1-x+t}{2} & \text{for } |t-1| < x < t+1, \\ \frac{1+x+t}{2} & \text{for } -(t+1) < x < -|t-1|, \\ 0 & \text{elsewhere.} \end{cases}$$

As before the solution corresponding to two consecutive blows at  $t = -2$  and  $t = 0$  is given by

$$w(x, t) = u(x, t) + u(x, t + 2)$$

and the observer at  $x = 4$  will record the following signal

$$w(4, t) = u(4, t) + u(4, t + 2) = \begin{cases} 0 & \text{for } 0 < t < 1 \\ \frac{t-1}{2} & \text{for } 1 < t < 3 \\ \frac{t-3}{2} + 1 & \text{for } 3 < t < 5 \\ 2 & \text{for } t > 5. \end{cases}$$

Hence we see that the two signals become one and the observer will record a part of the first signal forever. Interestingly enough, if these two signals were sent by plucking the string at  $t = -2$  and  $t = 0$ , as in Example 2.2, then the signal received at  $x = 4$  is clear. In fact, using the formula for the solution  $u$  with initial condition at  $t = 0$ :

$$u(x, t) = \frac{1}{2} \begin{cases} 2 - |x - t| - |x + t| & \text{for } -1 + t \leq x \leq 1 - t \\ 1 - |x - t| & \text{for } |1 - t| < x \leq 1 + t \\ 1 - |x + t| & \text{for } -1 - t \leq x \leq -|1 - t| \\ 0 & \text{elsewhere.} \end{cases}$$

and following the previous example, we obtain that the recorded signal in the form

$$w(4, t) = u(4, t) + u(4, t + 2) = \begin{cases} 0 & \text{for } 0 < t < 1 \\ 1 - |2 - t| & \text{for } 1 < t < 3 \\ 1 - |4 - t| & \text{for } 3 < t < 5 \\ 0 & \text{for } t > 5. \end{cases}$$

and we see clearly the separation of the signals.

Note, that the loss of clarity of the signal in the first case of the above example is caused by the fact that the initial data  $\psi$  is averaged over a full spatial (one-dimensional) domain and therefore any individual signal is lost.

Without going into details we note that in two dimensions neither signal created by specifying the initial value, nor specified by the initial velocity can be recorded clearly as in this case both solutions are obtained by averaging over the spatial domains. On the contrary, in three dimensions we average over spheres which have lower dimension than the surrounding space.

# Lecture 5

## Basic properties of diffusion equation

Let us recall that the diffusion (heat) equation is given by the formula (3.3.6)

$$u_t = D\Delta u. \quad (5.0.1)$$

It follows that even the one dimensional diffusion equation (5.0.1) is much harder to solve than the wave equation. We start therefore with analyzing some basic properties of it.

### 1 Maximum principle and its consequences

Let us consider an open bounded spatial domain  $\Omega \subset \mathbb{R}^n$  and the space-time domain  $\Omega_T = \Omega \times (0, T)$  for some  $T > 0$ . In the following theorem, that has far reaching consequences, we shall prove that the solution of (5.0.1) with nonnegative initial and boundary (Dirichlet) data must be nonnegative everywhere.

**Theorem 1.1** *Let  $T$  be any number greater than zero. If  $u$  is a continuous function in the closed set  $\bar{\Omega}_T = \{(\mathbf{x}, t); \mathbf{x} \in \bar{\Omega}, 0 \leq t \leq T\}$  and twice differentiable in  $\Omega_T$ , such that  $u(\mathbf{x}, 0) \geq 0$  for all  $\mathbf{x} \in \bar{\Omega}$  and  $u(\mathbf{x}, t) \geq 0$  for all  $(\mathbf{x}, t) \in \partial\Omega \times [0, T]$ , and  $u$  satisfies the diffusion equation (5.0.1) in the open set  $\Omega_T = \{(\mathbf{x}, t); \mathbf{x} \in \Omega, 0 < t < T\}$ , then  $u \geq 0$  throughout  $\Omega_T$ .*

*Similarly, if  $u(\mathbf{x}, 0) \leq 0$  on  $\bar{\Omega}$  and  $u(\mathbf{x}, t) \leq 0$  on  $\partial\Omega \times [0, T]$ , then  $u \leq 0$  throughout  $\bar{\Omega}_T$ .*

**Proof.** The idea of the proof is as follows. If  $u$  is not non-negative, then it must be negative somewhere. On the other hand, since  $u$  is a continuous function and  $\bar{\Omega}_T$  is bounded and closed, then it must take a minimum in  $\bar{\Omega}_T$  and this minimum must be negative, as  $u$  takes on negative values. The minimum cannot be attained on

$$\Gamma_T = (\bar{\Omega} \times \{t = 0\}) \cup (\partial\Omega \times [0, T]),$$

thus it must be attained either at  $(\mathbf{x}_0, t_0) \in \Omega \times (0, T)$  or  $(\mathbf{x}_0, T) \in \Omega \times \{t = T\}$ . If this minimum is attained in the interior, then using elementary facts from Calculus we have  $u_t(\mathbf{x}_0, t_0) = 0$  and  $u_{x_i x_i}(\mathbf{x}_0, t_0) \geq 0, i = 1, \dots, n$ . Now, if  $u$  is a "nice" function, we would have at least one  $u_{x_i x_i}(\mathbf{x}_0, t_0) > 0$  and at this point

$$u_t - D\Delta u < 0$$

contrary to the assumption that  $u_t - D\Delta u = 0$ . Similarly, if the minimum is attained at  $(\mathbf{x}_0, T)$ , then the only possibility is that  $u_t(\mathbf{x}_0, T) \leq 0$  and the same argument holds. However, it is possible that  $\Delta u = 0$  at the minimum point (e.g.  $u(x, y) = x^4 + y^4$  has this property) and therefore the proof presented above is not complete.

To overcome this difficulty we shall consider a small perturbation of  $u$  which is free of this property. A simple choice is

$$w_\epsilon(\mathbf{x}, t) = u(\mathbf{x}, t) + \epsilon \left( \frac{\|\mathbf{x}\|^2}{2n} + 2t \right),$$

for some  $\epsilon > 0, 0 \leq t \leq T$ . Note that  $w_\epsilon$  is nonnegative on  $\Gamma_T$ , and if it is negative somewhere, then it must have a negative minimum. As before, at this point  $\frac{\partial w_\epsilon}{\partial t} \leq 0$  and  $-\Delta w_\epsilon \leq 0$  so that  $\frac{\partial w_\epsilon}{\partial t} - \Delta w_\epsilon \leq 0$ . On the other hand, we have  $\frac{\partial w_\epsilon}{\partial t} = u_t + 2\epsilon t$  and  $\frac{\partial^2 w_\epsilon}{\partial x_i \partial x_i} = u_{x_i x_i} + \frac{\epsilon}{n}$  for  $i = 1, \dots, n$ , hence  $\frac{\partial w_\epsilon}{\partial t} - \Delta w_\epsilon = \epsilon > 0$ . This contradiction proves that for any  $\epsilon > 0, w_\epsilon \geq 0$ . However, for each  $\mathbf{x}, t, w_\epsilon(\mathbf{x}, t) \rightarrow u(\mathbf{x}, t)$ , which means that  $u(\mathbf{x}, t) \geq 0$  for all  $(\mathbf{x}, t) \in \bar{\Omega}_T$ .

To prove the statement for  $u$  having non-positive values on the boundary, we define  $\bar{u} = -u$  and apply the argument above to  $\bar{u}$ . ■

A similar reasoning allows to prove the corresponding statement for the particular case when  $\Omega = \mathbb{R}^n$ . In this case we assume also that  $u$  is a bounded function.

**Theorem 1.2** *Let  $T$  be any number greater than zero. If  $u$  is a bounded continuous function in the closed set  $\mathbb{R}_T^n = \{(\mathbf{x}, t); \mathbf{x} \in \mathbb{R}^n, 0 \leq t \leq T\}$  and twice differentiable in  $\mathbb{R}^n \times (0, T)$ , such that  $u(\mathbf{x}, 0) \geq 0$  for all  $\mathbf{x} \in \mathbb{R}^n$ , and  $u$  satisfies the diffusion equation (5.0.1) in the open set  $\mathbb{R}^n \times (0, T)$ , then  $u \geq 0$  throughout  $\mathbb{R}_T^n$ . Similarly, if  $u(\mathbf{x}, 0) \leq 0$  on  $\mathbb{R}^n$ , then  $u \leq 0$  throughout  $\mathbb{R}^n \times [0, T]$ .*

**Proof.** The only difference with the previous theorem is that the set  $\Omega$  is unbounded and in general we cannot claim that  $u$  (or rather  $w_\epsilon$ ) attains a minimum. Consider once again the function  $w$

$$w_\epsilon(\mathbf{x}, t) = u(\mathbf{x}, t) + \epsilon \left( \frac{\|\mathbf{x}\|^2}{2n} + 2t \right).$$

Since  $u$  is a bounded function, then there is a constant  $m$  such that for any  $\mathbf{x} \in \mathbb{R}^n$  and  $0 \leq t \leq T$  we have  $u(\mathbf{x}, t) \geq m$ . Therefore for  $\|\mathbf{x}\| \geq \sqrt{2n|m|/\epsilon}$  and  $0 \leq t \leq T$  we have that  $w_\epsilon(\mathbf{x}, t) \geq 0$ , and this leaves only the cylinder  $\|\mathbf{x}\| \leq \sqrt{2n|m|/\epsilon}, 0 \leq t \leq T$  as a possible set where  $w_\epsilon$  can be negative. But this set is bounded and closed and therefore we can apply the argument of the previous proof to claim that  $w_\epsilon \geq 0$  in this set, and therefore  $w_\epsilon \geq 0$  throughout  $\mathbb{R}^n \times [0, T]$ .

The second part of the proof is carried out as before. ■

These two theorems have a number of important corollaries. First we formulate and prove the result which is called the *Maximum Principle*.

**Corollary 1.1** *Let us assume that  $u$  is a continuous function in  $\bar{\Omega}_T$  and twice differentiable in  $\Omega_T$ , and in  $\Omega_T$  it satisfies  $u_t - D\Delta u = 0$ . As before, define*

$$\Gamma_T = (\bar{\Omega} \times \{t = 0\}) \cup (\partial\Omega \times [0, T])$$

*if  $\Omega$  is a bounded set and, if  $\Omega = \mathbb{R}^n$ , we put*

$$\Gamma_T = \mathbb{R}^n \times \{t = 0\}.$$

*In the latter case we assume additionally that  $u$  is a bounded function in  $\mathbb{R}^n \times [0, T]$ . Then*

$$\begin{aligned} \sup_{(\mathbf{x}, t) \in \Omega_T} u(\mathbf{x}, t) &= \sup_{(\mathbf{x}, t) \in \Gamma_T} u(\mathbf{x}, t) \\ \inf_{(\mathbf{x}, t) \in \bar{\Omega}_T} u(\mathbf{x}, t) &= \inf_{(\mathbf{x}, t) \in \bar{\Omega}_T} u(\mathbf{x}, t) \end{aligned} \quad (5.1.2)$$

**Proof.** Let  $\sup_{(\mathbf{x}, t) \in \Gamma_T} u(\mathbf{x}, t) = M$  and  $\inf_{(\mathbf{x}, t) \in \Gamma_T} u(\mathbf{x}, t) = m$ . Define the new function  $U = u - m$ ; then  $U \geq 0$  on  $\Gamma_T$  and it is bounded provided  $u$  was bounded. Since  $U$  also satisfies the diffusion equation, we obtain from the Theorems 1.1 and 1.2 that in both cases  $U \geq 0$  throughout  $\Omega_T$ . Thus  $u \geq m$  in  $\Omega_T$  but since  $\Gamma_T \subset \bar{\Omega}_T$ , we have

$$m = \inf_{(\mathbf{x}, t) \in \Gamma_T} u(\mathbf{x}, t) \leq \inf_{(\mathbf{x}, t) \in \bar{\Omega}_T} u(\mathbf{x}, t) \leq \inf_{(\mathbf{x}, t) \in \Gamma_T} u(\mathbf{x}, t) = m$$

and the second equality in (5.1.2) is proved.



The first one can be proved exactly in the same way by considering the function  $V = M - u$  which is also non-negative on  $\Gamma_T$  and satisfies the diffusion equation. ■

The Maximum Principle agrees perfectly with the physical intuition. Let us have some initial concentration of a substance, then this concentration will surely decrease in time, since the substance will tend to spread uniformly in the medium.

If we take snapshots of the solution to the diffusion equation, then we will see that the initial maxima will be decreasing and the initial minima will be increasing in time, thus the diffusion equation has a smoothing effect on the data contrary to the wave equation where the initial shape travelled undisturbed in time.

This smoothing effect will result in the initial distribution approaching a uniform one as time increases which agrees with the Second Principle of Thermodynamics (the entropy of the system increases).

Another consequence of this fact is that the diffusion equation is irreversible in time. The change of the arrow of time  $t' = -t$  changes equation (5.0.1) into

$$u_{t'} + u_{xx} = 0$$

which has completely different properties. For instance, it cannot be solved for most initial data. Physically, it corresponds to the statement that knowing the concentration of substance at some instant  $t$ , we are usually unable to determine the initial concentration, or that the substance cannot diffuse from the region of lower concentration to the region of higher concentration.

## 2 Uniqueness and stability of the solution to the diffusion equation

We derive two consequences of the Maximum Principle. Let us consider the following initial-boundary value problem for the diffusion equation:

$$\begin{aligned} u_t - D\Delta u &= f \quad \text{for } \mathbf{x} \in \Omega, \quad t > 0, \\ u(\mathbf{x}, t) &= g(t), \quad \text{for } \mathbf{x} \in \partial\Omega, \quad t > 0 \\ u(\mathbf{x}, 0) &= \phi(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega, \end{aligned} \tag{5.2.3}$$

where  $f, \phi, g$  are given functions. As before, if  $\Omega = \mathbb{R}^n$ , then the problem (5.2.3) is to be modified by dropping the boundary conditions; i.e. we consider

$$\begin{aligned} u_t - D\Delta u &= f \quad \text{for } \mathbf{x} \in \mathbb{R}^n, \quad t > 0, \\ u(\mathbf{x}, 0) &= \phi(\mathbf{x}), \quad \text{for } \mathbf{x} \in \mathbb{R}^n, \end{aligned} \tag{5.2.4}$$

**Definition 2.1** We say that a function  $u$  is a solution to the problem if it is continuous in  $\overline{\Omega}_T$ , twice continuously differentiable with respect to  $\mathbf{x}$  and once with respect to  $t$  in  $\Omega_T$  and satisfies (5.2.3) (or (5.2.4)).

We have the following theorem.

**Theorem 2.1** Let  $\Omega$  be a bounded set. Then there exists at most one solution of the problem (5.2.3).

If  $\Omega = \mathbb{R}^n$ , then there exists at most one bounded solution to the problem (5.2.4).

**Proof.** Let us assume that we have two solutions  $u_1$  and  $u_2$  satisfying (5.2.3); then  $u = u_1 - u_2$  satisfies (5.2.3) with  $f = g = \phi = 0$ , that is,  $u = 0$  on  $\Gamma_T$  for any  $T > 0$  be arbitrary. By Theorem 1.1, it must be both nonnegative and nonpositive, therefore  $u(\mathbf{x}, t) = 0$ , and consequently  $u_1 = u_2$ , hence the solution is unique.

The result for  $\Omega = \mathbb{R}^n$  follows from Theorem 1.2 which is available due to the assumption that  $u_1$  and  $u_2$  are bounded (and consequently  $u = u_1 - u_2$  is bounded). ■

The Maximum Principle allows us to establish the stability of the solution to the initial-boundary value problem (5.2.3) and the initial value problem (5.2.4).

Let us recall that the stability is one of the properties of a well-posed problem and roughly speaking means that any two solutions to the problem with the initial and boundary data which differ a little, will remain close to each other for all times. We have

**Theorem 2.2** *Let  $u_1$  and  $u_2$  denote the solutions to (5.2.3) with the data  $f, g_1, \phi_1$  and  $f, g_2, \phi_2$ , respectively. Then, for any  $T > 0$*

$$\sup_{(\mathbf{x}, t) \in \bar{\Omega}_T} |u_1(x, t) - u_2(x, t)| \leq \sup_{\mathbf{x} \in \Omega} |\phi_1(\mathbf{x}) - \phi_2(\mathbf{x})| + \sup_{(\mathbf{x}, t) \in \partial\Omega \times [0, T]} |h_1(\mathbf{x}, t) - h_2(\mathbf{x}, t)| \quad (5.2.5)$$

**Proof.** Under the assumptions of the theorem, the function  $u_1 - u_2$  solves (5.2.3) with the data  $f = 0, g = g_1 - g_2, \phi = \phi_1 - \phi_2$  and by the Maximum Principle we obtain

$$\begin{aligned} \sup_{(\mathbf{x}, t) \in \bar{\Omega}_T} |u_1(x, t) - u_2(x, t)| &= \sup_{(\mathbf{x}, t) \in \Gamma_T} |u_1(\mathbf{x}, t) - u_2(\mathbf{x}, t)| \\ &\leq \sup_{\mathbf{x} \in \Omega} |u_1(\mathbf{x}, 0) - u_2(\mathbf{x}, 0)| + \sup_{(\mathbf{x}, t) \in \partial\Omega \times [0, T]} |u_1(\mathbf{x}, t) - u_2(\mathbf{x}, t)| \\ &= \sup_{\mathbf{x} \in \Omega} |\phi_1(\mathbf{x}) - \phi_2(\mathbf{x})| + \sup_{(\mathbf{x}, t) \in \partial\Omega \times [0, T]} |h_1(\mathbf{x}, t) - h_2(\mathbf{x}, t)|, \end{aligned}$$

which is exactly Eq. (5.2.5). ■

# Lecture 6

## Solution of the diffusion equation

### 1 One dimensional diffusion equation

The purpose of this lecture is to solve the initial value problem:

$$\begin{aligned} u_t &= Du_{xx} & -\infty < x < +\infty, \quad 0 < t < +\infty, \\ u(x, 0) &= \phi(x). \end{aligned} \tag{6.1.1}$$

The solution of (6.1.22) is not easy to derive. We provide two ways of finding the formula of the solution, one depending on certain symmetries exhibited by the equation, the other using Fourier transforms technique.

#### 1.1 Similarity solution

Let us start with listing some general properties of solutions to the diffusion equation. Next step will be to solve (6.1.22) for a particular initial data and finally to build the general solution from this particular one.

We note that the solution to (6.1.22) has the following invariance properties:

- (a) The translate  $u(x - y, t)$  of the solution  $u(x, y)$  is another solution, for any fixed  $y$ .
- (b) Any derivative of a solution is again a solution.
- (c) If  $S(x, t)$  is a solution and  $f$  is any function, then

$$v(x, t) = \int_{-\infty}^{+\infty} S(x - y, t) f(y) dy \tag{6.1.2}$$

is a solution (provided the integral exists).

- (d) If  $u(x, t)$  is a solution, then the dilated function  $u(\sqrt{a}x, at)$  is again a solution, for any  $a > 0$ .

We start with formula (6.1.2). We have noted that  $v$  as defined there is a solution but we need a particular solution satisfying the prescribed initial value. The simplest way is to require that

$$v(x, 0) = \int_{-\infty}^{\infty} S(x - y, 0) \phi(y) dy = \phi(x),$$

that is, for a given initial value  $\phi$  the formula (6.1.2) would give the solution in an explicit way. So our problem is to find the solution  $S$  to the diffusion equation with quite involved initial condition

$$\int_{-\infty}^{\infty} S(x - y, 0) \phi(y) dy = \phi(x) \tag{6.1.3}$$

for any reasonable function  $\phi$  and  $x \in \mathbb{R}$ . Let us try to determine  $S(x - y, 0)$  in a more direct way. Let us assume for a time being that  $\phi$  vanishes at  $\pm\infty$  and start from the well-known formula

$$\int_{-\infty}^x \phi'(y) dy = \phi(x). \quad (6.1.4)$$

Let us observe that it can be written in the form more similar to (6.1.3) by introducing the Heaviside function  $H$  defined as follows

$$H(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0. \end{cases}$$

With the help of this function, (6.1.4) can be written as

$$\int_{-\infty}^{\infty} H(x - y) \phi'(y) dy = \phi(x) \quad (6.1.5)$$

which would be (6.1.3) but for the fact that we have the derivative of  $\phi$  instead of  $\phi$  itself. However, forgetting for a moment that  $H$  is not a differentiable function (it is not even continuous!) we use formally the integration by parts to arrive at the desired formula

$$- \int_{-\infty}^{\infty} \frac{\partial}{\partial y} (H(x - y)) \phi(y) dy = \int_{-\infty}^{\infty} H'(x - y) \phi(y) dy = \phi(x) \quad (6.1.6)$$

So we see that if we could find a solution of the diffusion equation which at  $t = 0$  behaves as the "derivative" of  $H$ , then we would have a good candidate for the solution to the initial value problem (6.1.22) in the form (6.1.2). However, since this "derivative" of  $H$  is rather elusive, we note that we still can use Eq. (6.1.5) to construct the solution. In fact, if we find the solution  $Q(x, t)$  to the problem (6.1.22) with a particular initial value  $Q(x, 0) = H(x)$ , we obtain, at least formally, the solution to (6.1.22) in the form

$$u(x, t) = \int_{-\infty}^{\infty} Q(x - y, t) \phi'(y) dy, \quad (6.1.7)$$

provided  $\phi$  is differentiable and vanishes at  $\pm\infty$ . If for  $t > 0$  the function  $Q(x, t)$  is differentiable with respect to  $x$ , then integrating by parts (6.1.7) and using  $Q_x(x - y, t) = -Q_y(x - y, t)$  we also have

$$u(x, t) = \int_{-\infty}^{\infty} Q_x(x - y, t) \phi(y) dy, \quad (6.1.8)$$

where we expect that in one sense or another we should have  $Q_x(x - y, 0) = H_x(x - y)$ , which would give us the solution in the form (6.1.2). Note that the form (6.1.8) is much less restrictive (it doesn't require  $\phi$  to be differentiable and to vanish at  $\pm\infty$ ). As we shall see later, (6.1.8) provides a genuine solution for a very large class of initial values.

Hence, following the route outlined above we shall start with finding the solution  $Q$  of (6.1.22) which satisfies the following initial condition

$$Q(x, 0) = H(x),$$

where  $H$  is the Heaviside function.

We note that the initial value doesn't change under dilation, that is,  $Q(\sqrt{a}x, 0) = H(\sqrt{a}x) = H(x) = Q(x)$ , hence taking into account the property (d) we expect the solution  $Q$  to be also invariant with respect to dilation:  $Q(\sqrt{a}x, at) = Q(x, t)$ . The simplest combination of  $x$  and  $t$  which is invariant under such a dilation is  $x/\sqrt{t}$ , thus it is plausible to look for the solution in the following form:

$$Q(x, t) = g(p) = g\left(\frac{x}{\sqrt{4Dt}}\right)$$

where the factor  $1/\sqrt{4D}$  has been introduced to simplify the following formulae. Substituting such defined  $Q$  to the diffusion equation yields the ordinary differential equation for  $g$ :

$$g'' + 2pg' = 0. \quad (6.1.9)$$

To solve this equation we first denote  $h = g'$  which will transform (6.1.9) to the first order equation

$$h' + 2ph = 0,$$

with the general solution  $h(p) = c_1 \exp(-p^2)$ , where  $c_1$  is a constant. The function  $g$  is then obtained by direct integration giving

$$Q(x, t) = g(p) = c_1 \int_0^p e^{-s^2} ds + c_2 = c_1 \int_0^{x/\sqrt{4Dt}} e^{-s^2} ds + c_2, \quad (6.1.10)$$

where  $c_1, c_2$  are constants. To determine them we use the initial conditions to get:

$$\begin{aligned} \text{for } x > 0 \quad 1 &= \lim_{t \rightarrow 0} Q(x, t) = c_1 \int_0^{+\infty} e^{-s^2} ds + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2, \\ \text{for } x < 0 \quad 0 &= \lim_{t \rightarrow 0} Q(x, t) = c_1 \int_0^{-\infty} e^{-s^2} ds + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2, \end{aligned}$$

upon which

$$Q(x, t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4Dt}} e^{-s^2} ds, \quad (6.1.11)$$

where we used  $\int_0^{+\infty} e^{-s^2} ds = \sqrt{\pi}/2$  (see Lemma 1.1).

*Fig 21. The graph of the solution  $Q(x, t)$ .*

The function  $Q$ , through Eq. (6.1.7), gives us the solution for all differentiable initial values which vanish at  $\pm\infty$ . Following the procedure outlined above we shall construct the function  $S$  which hopefully will give solutions for a larger class of initial data. Thus, we introduce the function

$$S(x, t) = \frac{\partial Q}{\partial x}(x, t),$$

or, explicitly,

$$S(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}. \quad (6.1.12)$$

This function is called the *source function*, *fundamental solution* or *propagator of the diffusion equation*.

*Fig 22. The graph of the source function  $S(x, t)$ .*

**Remark 1.1** The source function  $S$  is defined for all  $-\infty < x < +\infty$  and  $t > 0$ . It is positive and even. For large  $t$  it is very spread out, and for small  $t$  it is a very tall, thin spike (close to the "delta function"). Moreover, as we shall see later (Eq. (6.1.26))

$$\int_{-\infty}^{+\infty} S(x, t) dx = 1. \quad (6.1.13)$$

The physical interpretation of  $S$  is as follows. If we consider the diffusion of a unit mass of a substance which at the time  $t = 0$  is concentrated exactly at the point  $x = y$ , then the density  $u(x, t)$  of this substance evolves through the diffusion according to the formula

$$u(x, t) = S(x - y, t).$$

Note that (6.1.13) tells us that in the process of diffusion there is neither loss nor gain of the substance - the amount stays the same.

Using the above considerations we can give a physical interpretation of Eq. (6.1.2). It tells us that the solution for a continuous initial distribution of mass, given by  $\phi$ , can be obtained as the sum (remember that the integral is a limiting case of sums) of solutions for point masses.

## 1.2 Fourier transform approach to solving the diffusion equation

The derivation of the solution to the diffusion equation presented in the previous section is elementary but not very obvious. If one is familiar with the method of Fourier transforms, then one can derive this formula in a much more straightforward way. Note however that all the calculation here will be performed at an "engineering" level of rigour and, having obtained the final formula, we have still resort to Theorem 1.1 to prove that this formula provides indeed the sought solution.

Let us recall basic properties of the Fourier transform relevant to our problem. If  $f$  is a continuous integrable function, then the Fourier transform of  $f$  is defined by the formula

$$\mathcal{F}[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

where  $i$  is the imaginary unit. To shorten notation we shall frequently use the notation

$$\mathcal{F}[f] = \hat{f}.$$

It can be proved that  $\mathcal{F}$  is a one-to-one transformation and thus has an inverse, denoted  $\mathcal{F}^{-1}[f]$ , or shortly  $\check{f}$ .

If  $f$  vanishes at  $\pm\infty$ , then integrating by parts we obtain

$$\begin{aligned}\mathcal{F}[f_x](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_x(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} e^{-i\omega x} f(x) \Big|_{-\infty}^{+\infty} + \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \\ &= i\omega \mathcal{F}[f](\omega).\end{aligned}\tag{6.1.14}$$

If also the derivatives of  $f$  vanish at  $\pm\infty$ , then integrating by parts twice we obtain

$$\mathcal{F}[f_{xx}](\omega) = -\omega^2 \mathcal{F}[f](\omega)\tag{6.1.15}$$

If  $f$  and  $g$  are two integrable functions such that  $f(x)g(y-x)$  is integrable for any  $y$ , then the convolution of these two functions is defined as

$$(f * g)(y) = \int_{-\infty}^{+\infty} f(x)g(y-x)dx = \int_{-\infty}^{+\infty} f(y-x)g(x)dx.\tag{6.1.16}$$

As with the differentiation, the Fourier transform turns the convolution into a much simpler operation, namely into an algebraic multiplication of the Fourier transforms of the two functions. To show this we calculate

$$\begin{aligned}\mathcal{F}[f * g](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x)g(y-x)dx \right) e^{-i\omega y} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left( \int_{-\infty}^{\infty} g(y-x) e^{-i\omega y} dy \right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \left( \int_{-\infty}^{\infty} g(y-x) e^{-i\omega(y-x)} dy \right) dx \\ &= \sqrt{2\pi} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(z) e^{-i\omega z} dz \right) dz \\ &= \sqrt{2\pi} \mathcal{F}[f](\omega) \mathcal{F}[g](\omega).\end{aligned}\tag{6.1.17}$$

Thus, clearly

$$\mathcal{F}^{-1}[\hat{f} \cdot \hat{g}] = \frac{1}{\sqrt{2\pi}} f * g.\tag{6.1.18}$$

The last item from the Fourier transform theory we shall need is the transform of the function  $e^{-ax^2}$ ,  $a > 0$ . The evaluation requires some complex integration. We have

$$\begin{aligned}\mathcal{F}[e^{-ax^2}](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-i\omega x} dx \\ &= \frac{e^{-\omega^2/4a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\sqrt{a}x + i\omega/2\sqrt{a})^2} dx = \frac{e^{-\omega^2/4a}}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-(t + i\omega/2\sqrt{a})^2} dt\end{aligned}$$

where we completed the square in the exponent and, for the last equality, we performed the change of variables  $t = \sqrt{a}x$ . The integral

$$\int_{-\infty}^{\infty} e^{-(t+i\omega/2\sqrt{a})^2} dt$$

looks very much like the integral

$$\int_{-\infty}^{\infty} e^{-p^2} dp,$$

the value of which is  $\sqrt{\pi}$  (see Lemma 1.1), however in the latter integral the variable runs along the real line and in the former it runs along the complex line  $t + i\omega/2\sqrt{a}$ ,  $-\infty < t < \infty$  and it is not immediate that the values of both integrals are equal. To ascertain this we consider the complex integral

$$I_R = \int_{\gamma_R} e^{-z^2} dz$$

where  $z$  is a complex variable and  $\gamma_R$  runs along the rectangle consisting of the segment of the real line  $-R < t < R$ , the segment  $t + i\omega/2\sqrt{a}$ ,  $-\infty < t < +\infty$  and two vertical segments  $z = \pm R + it$ ,  $0 < t < \omega/2\sqrt{a}$ , traversed clockwise. Since  $e^{-z^2}$  is an entire function, we have

$$\begin{aligned} 0 &= I_R = - \int_{-R}^R e^{-t^2} dt + \int_0^{\omega/2\sqrt{a}} e^{-(R+it)^2} dt \\ &\quad + \int_{-R}^R e^{-(t+i\omega/2\sqrt{a})^2} dt - \int_0^{\omega/2\sqrt{a}} e^{-(R+it)^2} dt. \end{aligned} \quad (6.1.19)$$

Next we observe that

$$\begin{aligned} \left| \int_0^{\omega/2\sqrt{a}} e^{-(\pm R+it)^2} dt \right| &\leq \int_0^{\omega/2\sqrt{a}} |e^{-R^2} e^{\pm 2iRt} e^{-t^2}| dt \\ &\leq e^{-R^2} \int_0^{\omega/2\sqrt{a}} dt = e^{-R^2} \omega/2\sqrt{a} \rightarrow 0 \text{ as } R \rightarrow \infty \end{aligned}$$

where we used that  $|e^{\pm 2iRt}| = 1$  and  $|e^{-t^2}| \leq 1$ . Passing to the limit in (6.1.19) we obtain

$$\int_{-\infty}^{\infty} e^{-(t+i\omega/2\sqrt{a})^2} dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

Thus

$$\mathcal{F}[e^{-ax^2}](\omega) = \frac{e^{-\omega^2/4a}}{2\pi\sqrt{a}} \int_{-\infty}^{\infty} e^{-(t+i\omega/2\sqrt{a})^2} dt = \frac{e^{-\omega^2/4a}}{\sqrt{2a}}, \quad (6.1.20)$$

and also

$$\mathcal{F}^{-1}[e^{-a\omega^2}](x) = \frac{1}{\sqrt{2a}} e^{-x^2/4a}. \quad (6.1.21)$$

After this preliminaries let us tackle the problem

$$\begin{aligned} u_t &= Du_{xx} & -\infty < x < +\infty, \quad 0 < t < +\infty, \\ u(x, 0) &= \phi(x). \end{aligned} \quad (6.1.22)$$



We apply the Fourier transform to both equations and, using

$$\mathcal{F}[u_t](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x, t) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x} dx = \frac{\partial}{\partial t} \mathcal{F}[u](\omega),$$

we obtain the following initial value problem for a first order ordinary differential equation with a parameter

$$\begin{aligned} \hat{u}_t + D\omega^2 \hat{u} &= 0 & -\infty < \omega < +\infty, 0 < t < +\infty, \\ \hat{u}(\omega, 0) &= \hat{\phi}(\omega). \end{aligned} \quad (6.1.23)$$

Immediately we obtain the solution in the form

$$\hat{u}(\omega, t) = \hat{\phi}(\omega) e^{-\omega^2 D t}.$$

The solution of the original problem (6.1.22) is given by

$$u = \mathcal{F}^{-1}[\hat{u}]$$

that is, by (6.1.18), (6.1.21) and (6.1.16)

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1}[\hat{\phi}(\omega) e^{-\omega^2 D t}] = \sqrt{2\pi} \mathcal{F}^{-1}[\hat{\phi}] * \mathcal{F}^{-1}[e^{-\omega^2 D t}] \\ &= \frac{1}{2\sqrt{\pi D t}} \phi * e^{-x^2/4Dt} \\ &= \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4Dt} \phi(y) dy. \end{aligned} \quad (6.1.24)$$

### 1.3 Theorem that the constructed solution is the genuine solution of the initial value problem for the diffusion equation

For any bounded and continuous function  $\phi$  we define

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{+\infty} S(x-y, t) \phi(y) dy \\ &= \frac{1}{\sqrt{4D\pi t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4Dt} \phi(y) dy, \quad -\infty < x < +\infty, t > 0. \end{aligned} \quad (6.1.25)$$

As discussed above, we expect such defined  $u$  to be the solution of the problem (6.1.22). However, there are two main points to check. First is whether  $u$  is a differentiable function satisfying the diffusion equation (we tacitly accepted this writing (6.1.2), but there was no real justification of this fact). Second, does  $u$  satisfy the initial condition? Note that this point is far from trivial - in the formula (6.1.25) there appears the factor  $1/t$  so that simple substitution  $t = 0$  to obtain the initial value  $u(x, 0)$  does not make any sense.

Before we formulate and prove the relevant theorem we prove an important property of the source function.

**Lemma 1.1** *For any  $x \in \mathbb{R}, t > 0$  we have*

$$\int_{-\infty}^{\infty} S(x-y, t) dy = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4Dt} dy = 1. \quad (6.1.26)$$

**Proof.** Using the change of variables  $p = (x - y)/\sqrt{4Dt}$  with  $dp = -\frac{dy}{\sqrt{4Dt}}$  we obtain

$$\int_{-\infty}^{\infty} S(x - y, t) dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp$$

The integral in this formula is not an elementary integral - we cannot simply find the indefinite integral and pass to the limit with limits of integration. First we note that this integral exists by the Dominated Convergence Theorem of Calculus. To find its value we proceed as follows. Let

$$I = \int_{-\infty}^{\infty} e^{-p^2} dp,$$

then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-p^2} dp \times \int_{-\infty}^{\infty} e^{-s^2} ds = \int_{\mathbb{R}^2} e^{-(p^2+s^2)} dp ds \\ &= \int_0^{\infty} \left( \int_0^{2\pi} e^{-r^2} r d\alpha \right) dr = 2\pi \int_0^{\infty} e^{-r^2} r dr = \pi \int_0^{\infty} e^{-z} dz = \pi \left( 1 - \lim_{z \rightarrow \infty} e^{-z} \right) \\ &= \pi. \end{aligned}$$

Thus  $I = \sqrt{\pi}$  and formula (6.1.26) is proved. ■

**Theorem 1.1** Let  $\phi$  be a bounded continuous function on  $\mathbb{R}$ . Then the formula

$$u(x, t) = \frac{1}{\sqrt{4D\pi t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4Dt} \phi(y) dy, \quad t > 0, x \in \mathbb{R} \quad (6.1.27)$$

defines an infinitely differentiable function which satisfies the diffusion equation  $u_t = Du_{xx}$  for  $-\infty < x < +\infty, t > 0$  and the initial condition of (6.1.22) in the following sense

$$\lim_{t \rightarrow 0^+} u(x, t) = \phi(x). \quad (6.1.28)$$

The proof of this theorem is omitted.

**Remark 1.2** Since  $S(x, 0) \neq 0$  only for  $x = 0$  and  $S(x, t) > 0$  for any  $t > 0$  and  $-\infty < x < +\infty$ , it means that the information of the initial state of the system propagates at infinite speed. This contradicts the Special Theory of Relativity and hence should be discarded by Physics. Surprisingly, the calculations utilizing the solution to the diffusion equation yield results which agree very well with experiments. To explain this seemingly contradictory statements we note that the diffusion equation itself is only an approximate model for a real system and therefore its solution can be only considered as approximate description of the evolution of this system. As we saw above, for small times and large  $x$  the function  $S$  is very close to zero and therefore it can be considered to be equal to zero for numerical purposes.

**Remark 1.3** In Theorem 1.1 we proved that the solution of the diffusion equation is infinitely differentiable both in  $t$  and  $x$  for any  $t > 0$ . Another way of looking at this property is that the diffusion equation is smoothing, that is, even if the initial value is rugged, the solution becomes immediately smooth. To illustrate this point, let us consider the initial value problem

$$u_t = u_{xx}, \quad u(x, 0) = \phi(x),$$

where  $\phi(x) = 1 - |x|$  for  $|x| < 1$  and  $\phi(x) = 0$  for  $|x| > 1$ . The solution is then given by

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \left( \int_{-1}^1 e^{-(x-y)^2/4t} dy + \int_{-1}^0 e^{-(x-y)^2/4t} y dy - \int_0^1 e^{-(x-y)^2/4t} y dy \right). \quad (6.1.29)$$

These integrals cannot be evaluated in elementary way, however, they can be written in terms of the so-called *error function*

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds, \quad (6.1.30)$$

which is commonly used in statistics and applied sciences and which is therefore available on most scientific calculators. It is easy to check that

$$\int_a^b e^{-s^2} ds = \frac{\sqrt{\pi}}{2} (Erf(b) - Erf(a)).$$

Changing variables and using this formula we can rewrite (6.1.29) as

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{\pi}} \left( \int_{(1-x)/2\sqrt{t}}^{(-1-x)/2\sqrt{t}} e^{-p^2} dp + \int_{-x/2\sqrt{t}}^{(-1-x)/2\sqrt{t}} e^{-p^2} (x + 2p\sqrt{t}) dp - \int_{(1-x)/2\sqrt{t}}^{-x/2\sqrt{t}} e^{-p^2} (x + 2p\sqrt{t}) dp \right) \\ &= \frac{1}{2} \left( (1-x) Erf\left(\frac{1-x}{2\sqrt{t}}\right) + (1+x) Erf\left(\frac{1+x}{2\sqrt{t}}\right) - 2x Erf\left(\frac{x}{2\sqrt{t}}\right) \right) \\ &\quad + \sqrt{\frac{t}{\pi}} \left( e^{-(1+x)^2/4t} + e^{-(1-x)^2/4t} - 2e^{-x^2/4t} \right). \end{aligned} \quad (6.1.31)$$

The graph of the solution is presented on the figures below.

*Fig 23. 3-dimensional visualization of the diffusion with the tent-shaped initial value.*

*Fig 24. Evolution of the density - subsequent shapes are presented on the same graph.*

*Fig 25. Evolution of the density - snapshots of the solution for times between  $t = 0.00001$  to  $t = 0.6$ .*

The next three figures present a variation of the same example. This time we consider the saw-like initial density and observe an immediate smoothing of its shape.

*Fig 26. 3-dimensional visualization of the diffusion with the saw-shaped initial value.*

*Fig 27. Evolution of the density - subsequent shapes are presented on the same graph.*

*Fig 28. Evolution of the density - snapshots of the solution for times between  $t = 0.00001$  to  $t = 0.6$ .*

## 2 Diffusion in several dimensions

Unlike for the wave equation, the solution to the initial value problem for the three dimensional diffusion equation:

$$\begin{aligned} u_t(\mathbf{r}, t) &= D\Delta u(\mathbf{r}, t), & \mathbf{r} \in \mathbb{R}^3, t > 0 \\ u(\mathbf{r}, 0) &= \phi(\mathbf{r}), \end{aligned} \quad (6.2.1)$$

can be derived from the one dimensional ones in a relatively easy way. Let  $S$  be a source function given by Eq. (6.1.12). We define

$$S_3(x, y, z, t) = S(x, t)S(y, t)S(z, t), \quad (6.2.2)$$

or explicitly

$$\begin{aligned} S_3(x, y, z, t) &= \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt} \times \frac{1}{\sqrt{4\pi Dt}} e^{-y^2/4Dt} \times \frac{1}{\sqrt{4\pi Dt}} e^{-z^2/4Dt} \\ &= \frac{1}{(4\pi Dt)^{3/2}} e^{-(x^2+y^2+z^2)/4Dt} = \frac{1}{(4\pi Dt)^{3/2}} e^{-\|\mathbf{r}\|^2/4Dt}. \end{aligned}$$

Due to the fact that  $S$  solves the diffusion equation:  $S_t(x, t) = DS_{xx}(x, t)$  we obtain

$$\begin{aligned} \frac{\partial S_3(x, y, z, t)}{\partial t} &= S_t(x, t)S(y, t)S(z, t) + S(x, t)S_t(y, t)S(z, t) + S(x, t)S(y, t)S_t(z, t) \\ &= D(S_{xx}(x, t)S(y, t)S(z, t) + S(x, t)S_{yy}(y, t)S(z, t) + S(x, t)S(y, t)S_{zz}(z, t)) \\ &= D\left(\frac{\partial^2 S_3(x, y, z, t)}{\partial x^2} + \frac{\partial^2 S_3(x, y, z, t)}{\partial y^2} + \frac{\partial^2 S_3(x, y, z, t)}{\partial z^2}\right) \\ &= D\Delta S_3(x, y, z, t) \end{aligned}$$

so that  $S_3$  is a solution of the three dimensional diffusion equation. Similarly, from (6.1.13) we have also

$$\begin{aligned} \int_{\mathbb{R}^3} S_3(\mathbf{r}, t) d\mathbf{r} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(x, t)S(y, t)S(z, t) dx dy dz \\ &= \int_{-\infty}^{\infty} S(x, t) dx \times \int_{-\infty}^{\infty} S(y, t) dy \times \int_{-\infty}^{\infty} S(z, t) dz = 1 \end{aligned} \quad (6.2.3)$$

and hence, exactly as in one dimension, it can be proved that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^3} S_3(\mathbf{r} - \mathbf{w}) \phi(\mathbf{w}) d\mathbf{w} = \phi(\mathbf{r}).$$

Therefore the unique bounded solution to the problem (6.2.1) is given by the formula

$$u(\mathbf{r}, t) = \frac{1}{(4\pi Dt)^{3/2}} \int_{\mathbb{R}^3} e^{-(\mathbf{r}-\mathbf{w})^2/4Dt} \phi(\mathbf{w}) d\mathbf{w}. \quad (6.2.4)$$

## 3 Black-Scholes formulae

In this section we shall derive the Black-Scholes formulas for option pricing. Let us recall that we are looking for the solution of the following initial-boundary value problem

$$\begin{aligned} \frac{\partial V}{\partial t} &= -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r \left( V - S \frac{\partial V}{\partial S} \right), \\ V(S, T) &= \max\{S - E, 0\}, \quad \text{for all } S > 0, \\ V(0, t) &= 0, \quad \text{for all } t \leq T, \\ \lim_{S \rightarrow +\infty} V(S, t)/S &= 1, \quad \text{for all } t \leq T. \end{aligned} \quad (6.3.5)$$

Using the change of variables (3.9.16)

$$\begin{aligned}x(S, t) &= \ln S + \left(r - \frac{1}{2}\sigma^2\right)(T - t), \\y(t) &= \frac{1}{2}\sigma^2(T - t),\end{aligned}$$

with the inverse

$$\begin{aligned}T - t &= \frac{2}{\sigma^2}y, \\S &= e^{x - \left(\frac{2r}{\sigma^2} - 1\right)y},\end{aligned}$$

we reduced the problem (6.3.5) to the following initial-boundary value problem for the diffusion equation:

$$\begin{aligned}G_y &= G_{xx}, \quad \text{for } -\infty < x < \infty, y > 0 \\G(x, 0) &= \max\{e^x - E, 0\}, \quad \text{for } -\infty < x < \infty, \\ \lim_{x \rightarrow -\infty} G(x, y) &= 0, \quad \text{for any } y \geq 0, \\ \lim_{x \rightarrow -\infty} e^{-(x+y)} G(x, y) &= 1, \quad \text{for any } y \geq 0.\end{aligned}\tag{6.3.6}$$

So, if we have the solution to the diffusion problem,  $G(x, y)$ , then the solution to the Black-Scholes problem is given by

$$V(S, t) = e^{-r(T-t)} G\left(\ln S + \left(r - \frac{1}{2}\sigma^2\right)(T - t), \frac{1}{2}\sigma^2(T - t)\right)\tag{6.3.7}$$

From the previous sections we know that the solution of the pure initial value problem with the initial condition  $\phi$  is given by

$$u(x, t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4Dt} \phi(y) dy.$$

In our case  $D = 1$  and since  $e^x$  is an increasing function, the initial value takes the simpler form

$$\phi(x) = \begin{cases} e^x - E & \text{for } x > \ln E \\ 0 & \text{for } x \leq \ln E. \end{cases}$$

Thus, we obtain

$$G(x, y) = \frac{1}{2\sqrt{\pi y}} \int_{\ln E}^{\infty} e^{-\frac{(x-s)^2}{4y}} (e^s - E) ds.\tag{6.3.8}$$

This solution is closely related to the cumulative distribution function for a normal random variable and it is sensible to express it in terms of the distribution function of standardised normal random variable defined as

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{p^2}{2}} dp.$$

To write  $G$  (and subsequently  $V$ ) in terms of  $N$  we split the integral (10.2.1) and evaluate them using to subsequent changes of variables.

$$\begin{aligned}G(x, y) &= \frac{1}{2\sqrt{\pi y}} \left( \int_{\ln E}^{\infty} e^{-\frac{(x-s)^2}{4y}} e^s ds - E \int_{\ln E}^{\infty} e^{-\frac{(x-s)^2}{4y}} ds \right) \\ &= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\frac{x - \ln E}{\sqrt{2y}}} e^{-\frac{p^2}{2}} e^{x - \sqrt{2y}p} dp - E \int_{-\infty}^{\frac{x - \ln E}{\sqrt{2y}}} e^{-\frac{p^2}{2}} dp \right)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \left( e^{x+y} \int_{-\infty}^{\frac{x-\ln E}{\sqrt{2y}}} e^{-\frac{(p+\sqrt{2y})^2}{2}} dp - E \int_{-\infty}^{\frac{x-\ln E}{\sqrt{2y}}} e^{-\frac{p^2}{2}} dp \right) \\
&= \frac{1}{\sqrt{2\pi}} \left( e^{x+y} \int_{-\infty}^{\frac{x-\ln E+2y}{\sqrt{2y}}} e^{-\frac{p^2}{2}} dp - E \int_{-\infty}^{\frac{x-\ln E}{\sqrt{2y}}} e^{-\frac{p^2}{2}} dp \right) \\
&= e^{x+y} N\left(\frac{x-\ln E+2y}{\sqrt{2y}}\right) - EN\left(\frac{x-\ln E}{\sqrt{2y}}\right).
\end{aligned}$$

Before we write down the Black-Scholes formula, let us note that despite the fact that we have derived  $G$  as the solution of the initial value problem, luckily it also satisfies the boundary conditions. To prove it, let us first observe that

$$\begin{aligned}
\lim_{x \rightarrow -\infty} N(x) &= 0, \\
\lim_{x \rightarrow +\infty} N(x) &= 1.
\end{aligned}$$

Since  $\lim_{x \rightarrow -\infty} e^{x+y} = 0$  for any fixed  $y$ , we immediately see that

$$\lim_{x \rightarrow -\infty} G(x, y) = \lim_{x \rightarrow -\infty} \left( e^{x+y} N\left(\frac{x-\ln E+2y}{\sqrt{2y}}\right) - EN\left(\frac{x-\ln E}{\sqrt{2y}}\right) \right) = 0$$

and

$$\lim_{x \rightarrow \infty} e^{-(x+y)} G(x, y) = \lim_{x \rightarrow \infty} \left( N\left(\frac{x-\ln E+2y}{\sqrt{2y}}\right) - e^{-(x+y)} EN\left(\frac{x-\ln E}{\sqrt{2y}}\right) \right) = 1,$$

so that  $G$  is the function which we have been looking for.

To derive explicit formulas for option pricing we must change back the variables  $x$  and  $y$  into  $S$  and  $t$ . We get

$$\frac{x-\ln E}{\sqrt{2y}} = \frac{\ln S/E + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{x-\ln E+2y}{\sqrt{2y}} = \frac{\ln S/E + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Since

$$e^{x+y} = e^{\ln S + r(T-t)} = Se^{r(T-t)},$$

we obtain explicitly from (6.3.7)

$$\begin{aligned}
V(S, t) &= SN\left(\frac{\ln S/E + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - Ee^{-r(T-t)} N\left(\frac{\ln S/E + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned} \tag{6.3.9}$$

The figures below show the graphs of the solution.



*Fig 29. 3-dimensional visualization of the solution to the Black-Scholes equation giving the pricing of European call options.*

*Fig 30. The price of a European call option as a function of the price of the underlying share as time approaches the expiry time. The bottom line gives the pay-off.*

## 4 Miscellaneous examples

As we know, the solution to the initial value problem for the diffusion equation is given by the integral

$$u(x, t) = \frac{1}{\sqrt{4D\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4Dt} \phi(y) dy,$$

where  $\phi$  is the stipulated initial value of the solution. Unfortunately, this integral can be evaluated explicitly only in very few cases. In this section we shall present some such cases and discuss several other situations when one can obtain the solution to the problem, not necessarily resorting the integral formula.

### Example 4.1 Matching the known solution

Find the solution of the problem

$$u_t = 2u_{xx}, \quad u(x, 0) = e^{-x^2+x}.$$

To solve this equation we shall use the fact that if  $v(x, t)$  is a solution to the diffusion equation, then for any constants  $a$ ,  $x_0$  and  $t_0$  the function  $u(x, t) = av(x + x_0, t + t_0)$  is also a solution. We know that

$$S(x, t) = \frac{1}{\sqrt{8\pi t}} e^{-x^2/8t}$$

is a solution to the equation above, and since the initial value has a form similar to  $S$ , we shall try and find constants such that the function

$$u(x, t) = aS(x + x_0, t + t_0) = \frac{a}{\sqrt{8\pi(t + t_0)}} e^{-(x+x_0)^2/8(t+t_0)}$$

satisfies the initial condition. We must have

$$e^{-x^2+x} = \frac{a}{\sqrt{8\pi t_0}} e^{-(x+x_0)^2/8t_0} = \frac{a}{\sqrt{8\pi t_0}} e^{-(x^2+2xx_0+x_0^2)/8t_0}$$

Comparing both sides we see that

$$\begin{aligned} 1 &= 1/8t_0, \\ 1 &= -x_0/4t_0, \\ 1 &= \frac{ae^{-x_0^2/8t_0}}{\sqrt{8\pi t_0}}. \end{aligned}$$

Therefore  $t_0 = 1/8$ ,  $x_0 = -1/2$  and  $a = e^{1/4}\sqrt{\pi}$  and the solution is given by

$$u(x, t) = \frac{e^{1/4}}{\sqrt{8t+1}} e^{-\frac{(x-1/2)^2}{8t+1}}.$$

Note that this method can be used only for the initial data which have the particular form

$$\phi(x) = ae^{-bx^2+cx}$$

where  $a, b, c$  are constants with  $b > 0$ .

A variation of this method can be used for a little more general initial conditions. For example, to solve

$$u_t = u_{xx}, \quad u(x, 0) = x^2 e^{-x^2} \tag{6.4.1}$$

we may use the fact that

$$x^2 e^{-x^2} = \frac{1}{4} \frac{d^2}{dx^2} e^{-x^2} + \frac{1}{2} e^{-x^2}$$

Using the approach described above, we obtain that the solution to the problem

$$u_t = u_{xx}, \quad u(x, 0) = e^{-x^2}$$

is given by

$$u_1(x, t) = \frac{1}{\sqrt{4t+1}} e^{-x^2/(4t+1)}.$$

Since we know that the derivative of a solution is a solution and that sum of two solutions is a solution, we obtain that the solution of the problem (6.4.1) is given by

$$\begin{aligned} u(x, t) &= \frac{1}{4} \frac{d^2 u_1(x, t)}{dx^2} + \frac{1}{2} u_1(x, t) \\ &= -\frac{1}{2(4t+1)^{3/2}} e^{-x^2/(4t+1)} + \frac{x^2}{(4t+1)^{5/2}} e^{-x^2/(4t+1)} + \frac{1}{2(4t+1)^{3/2}} e^{-x^2/(4t+1)} \\ &= \frac{x^2}{(4t+1)^{5/2}} e^{-x^2/(4t+1)} \end{aligned}$$

This method can be used for initial values of the form

$$\phi(x) = P_n(x) e^{-bx^2+cx}$$

where  $P_n$  is an arbitrary polynomial and  $a, b, c$  are as above.

**Example 4.2 Using the integral formula**

If the initial datum is a polynomial, an exponential function, sine or cosine, or a linear combination of them, then the integral in the representation formula can be evaluated explicitly and the solution is given as a combination of elementary functions. As an example we shall solve the problem

$$u_t = u_{xx}, \quad u(x, 0) = \cos^2 x.$$

First we observe that

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x,$$

thus we can split the problem into two simpler ones:

$$v_t = v_{xx}, \quad v(x, 0) = \cos 2x,$$

and

$$w_t = w_{xx}, \quad w(x, 0) = 1.$$

If we find  $v$  and  $w$ , then from the linearity (superposition principle) we obtain that

$$u(x, t) = \frac{1}{2}v + \frac{1}{2}w.$$

Let us start with finding  $v$ . Using the representation formula we obtain

$$v(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} \cos 2y dy = \operatorname{Re} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} e^{2iy} dy,$$

where  $\operatorname{Re}$  denotes the real part of the complex number. For the exponents, completing the square, we obtain,

$$\begin{aligned} -(x-y)^2/4t + 2iy &= -\frac{x^2 - 2xy + y^2 - 8iyt}{4t} = -\frac{(x-y+4it)^2 - 8ixt + 16t^2}{4t} \\ &= -\frac{(x-y+4it)^2}{4t} + 2ix - 4t, \end{aligned}$$

therefore

$$v(x, t) = \operatorname{Re} \frac{e^{2ix-4t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y+4it)^2/4t} dy.$$

The integral

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y+4it)^2/4t} dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(p+2i\sqrt{t})^2} dp$$

resembles the integral  $\int_{-\infty}^{\infty} e^{-p^2} dp$ , however, it is evaluated along the line in the complex plane. Using the methods of complex integration (see e.g. Subsection 1.2) we obtain that both integrals are equal with common value  $\sqrt{\pi}$ . Therefore

$$v(x, t) = \operatorname{Re} e^{2ix-4t} = e^{-4t} \cos 2x.$$

To find  $w$  we evaluate the integral

$$w(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} \cdot 1 dy = 1.$$

Therefore the solution to the original problem is given by

$$u(x, t) = \frac{1}{2} + \frac{1}{2} e^{-4t} \cos 2x.$$

**Example 4.3 Polynomial initial values**

To solve the initial value problem

$$u_t = Du_{xx}, \quad u(x, 0) = P_n(x),$$

where  $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$  is a polynomial of  $n$ -th degree we can use the method of the previous example but for higher order polynomials the evaluation of integrals becomes quite labourious. Here we present an alternative way of solving the problem.

Let us assume that  $u(x, t)$  is the solution; then its  $n + 1$  derivative

$$v(x, t) = \frac{d^{n+1}u(x, t)}{dx^{n+1}}$$

also is a solution to the diffusion equation satisfying the initial condition

$$v(x, 0) = \frac{d^{n+1}u(x, 0)}{dx^{n+1}} = \frac{d^{n+1}P_n(x)}{dx^{n+1}} = 0.$$

However, the only solution to the problem

$$v_t = Dv_{xx}, \quad v(x, 0) = 0,$$

is  $v(x, t) \equiv 0$ . Thus, the solution  $u$  of the original problem can be obtained by integrating the zero function  $n + 1$  times with respect to  $x$ , that is

$$u(x, t) = A_n(t)x^n + A_{n-1}(t)x^{n-1} + \dots + A_0,$$

where  $A_i(t)$  are as yet undetermined functions of  $t$  only, which must be found by inserting  $u$  to the equation, and comparing it to the initial value.

As an example we shall solve the initial value problem

$$u_t = Du_{xx}, \quad u(x, 0) = x^3 + x.$$

Following the approach outlined above we are looking for  $u$  in the form

$$u(x, t) = A_3(t)x^3 + A_2(t)x^2 + A_1(t)x + A_0(t).$$

Inserting such defined  $u$  into the equation we obtain

$$A'_3(t)x^3 + A'_2(t)x^2 + A'_1(t)x + A'_0(t) = 6DA_3(t)x + 2DA_2(t)$$

and comparing coefficients at the same powers of  $x$  we obtain

$$\begin{aligned} A'_3 &= 0, \\ A'_2 &= 0, \\ A'_1 &= 6DA_3, \\ A'_0 &= 2DA_2. \end{aligned}$$

Next, from the initial condition we obtain

$$A_3(0)x^3 + A_2(0)x^2 + A_1(0)x + A_0(0) = x^3 + x$$

which yields

$$\begin{aligned} A_3(0) &= 1, \\ A_2(0) &= 0, \\ A_1(0) &= 1, \\ A_0(0) &= 0. \end{aligned}$$

Solving the above system of ordinary differential equations we obtain

$$\begin{aligned} A_3(t) &= 1, \\ A_2(t) &= 0, \\ A_1(t) &= 6Dt + 1, \\ A'_0 &= 0. \end{aligned}$$

Thus, the solution is given by

$$u(x, t) = x^3 + (6Dt + 1)x.$$

#### Example 4.4 Drift-diffusion equation

Let us consider the general parabolic equation in two variables (drift-diffusion equation)

$$u_t = Au + Bu_x + Cu_{xx}.$$

This equation can be reduced to the basic diffusion equation by introducing the new unknown function  $v$  according to the formula

$$u(x, t) = e^{ax+bt}v(x, t),$$

where  $a$  and  $b$  are coefficients to be determined. Differentiating, we obtain

$$\begin{aligned} u_t &= be^{ax+bt}v(x, t) + e^{ax+bt}v_t(x, t), \\ u_x &= ae^{ax+bt}v(x, t) + e^{ax+bt}v_x(x, t), \\ u_{xx} &= a^2e^{ax+bt}v(x, t) + 2ae^{ax+bt}v_x(x, t) + e^{ax+bt}v_{xx}(x, t) \end{aligned}$$

Inserting the above expressions into the equation, collecting terms and dividing by  $e^{ax+bt}$  we arrive at

$$v_t = (A + Ba + Ca^2 - b)v + (B + 2Ca)v_x + Cv_{xx}.$$

From the above equation we see that taking

$$a = -\frac{B}{2C}, \quad b = A - \frac{B^2}{4C}$$

will make the coefficients multiplying  $v$  and  $v_x$  equal to zero, so that  $v$  will be the solution to

$$v_t = Cv_{xx}.$$

To illustrate this technique let us consider the initial-value problem

$$u_t = 2u_x + u_{xx}, \quad u(x, 0) = x^2.$$

Following the above considerations we find  $a = b = -1$  so that

$$u(x, t) = e^{-x-t}v(x, t).$$

Consequently,  $v$  solves the following initial-value problem

$$v_t = v_{xx}, \quad v(x, 0) = x^2e^x.$$

Using the integral formula we obtain

$$v(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} y^2 e^y dy$$

For the exponents, completing the square, we obtain,

$$\begin{aligned} -(x-y)^2/4t + y &= -\frac{x^2 - 2xy + y^2 + 4yt}{4t} = -\frac{(x-y+2t)^2 - 4xt - 4t^2}{4t} \\ &= -\frac{(x-y+2t)^2}{4t} + x + t, \end{aligned}$$

therefore

$$v(x, t) = \frac{e^{x+t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y+2t)^2/4t} y^2 dy.$$

Changing the variable according to  $p = -(x-y+2t)/2\sqrt{t}$  so that  $y = x + 2t + 2p\sqrt{t}$  we obtain

$$\begin{aligned} v(x, t) &= \frac{e^{x+t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (x + 2t + 2p\sqrt{t})^2 dp \\ &= \frac{(x + 2t)^2 e^{x+t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp + \frac{4(x + 2t)\sqrt{t} e^{x+t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p dp \\ &\quad + \frac{4t e^{x+t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^2 dp \end{aligned}$$

The first integral in the above is equal to  $\sqrt{\pi}$ . The second is zero, as  $pe^{-p^2}$  is an odd function. The last one can be evaluated by parts as follows

$$\int_{-\infty}^{\infty} e^{-p^2} p^2 dp = -\frac{1}{2} e^{-p^2} p \Big|_{-\infty}^{+\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-p^2} dp = \frac{\sqrt{\pi}}{2}.$$

Putting these together we obtain

$$v(x, t) = \frac{e^{x+t}}{\sqrt{\pi}} \left( \sqrt{\pi} (x + 2t)^2 + 4t \frac{\sqrt{\pi}}{2} \right) = e^{x+t} (2t + (x + 2t)^2),$$

thus we finally obtain

$$u(x, t) = (2t + (x + 2t)^2).$$

#### Example 4.5 Black-Scholes formula for put options

In the main body of the course we have focused on the European call options, that is, contracts allowing the holder to buy a share at the prescribed expiry time  $T$  at a prescribed exercise price  $E$ . A somewhat complementary contract which allows the holder to sell a share at a prescribed price at a prescribed time is called the put option. The equation governing the evolution of the put option price is the same Black-Scholes equation (recall that in the process of deriving this equation we have never used the fact that we have call options in the portfolio), that is, the put option price satisfies

$$\frac{\partial V}{\partial t} = -\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r \left( V - S \frac{\partial V}{\partial S} \right).$$

The difference appears in side conditions. Let us consider the pay-off at the expiry time. If the value of the share  $S$  is smaller than the exercise price  $E$ , then clearly it is worthwhile to sell the share at  $E$  and get the profit of  $E - S$ . On the other hand, if  $S > E$ , then it is better to sell the share elsewhere, so the option is worthless. Thus we arrived at the final condition

$$V(S, T) = \max\{E - S, 0\}.$$

As far as the boundary conditions are concerned, we note that if at some stage  $S$  becomes zero, then it will stay at this level and then we shall exercise the option getting certain profit of  $E$  at time  $T$ . However, our basic assumption is that we operate in an arbitrage-free market and therefore this profit must be equal to that obtained by depositing the amount of  $V(0, t)$  in a bank at time  $t$ . Assuming constant interest rate  $r$  we obtain that

$$E = V(0, t)e^{r(T-t)},$$

so that we obtain the boundary condition at  $S = 0$  to be

$$V(0, t) = Ee^{-r(T-t)}.$$

The other natural boundary is as  $S \rightarrow \infty$ . If  $S$  becomes very large, then it is unlikely that the option will be exercised, thus the value of the option is zero. Hence, the full initial-boundary value problem for the Black-Scholes equation for pricing put options is

$$\begin{aligned} \frac{\partial V}{\partial t} &= -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r \left( V - S \frac{\partial V}{\partial S} \right), \\ V(S, T) &= \max\{E - S, 0\}, \quad \text{for all } S > 0, \\ V(0, t) &= Ee^{-r(T-t)}, \quad \text{for all } t \leq T, \\ \lim_{S \rightarrow +\infty} V(S, t) &= 0, \quad \text{for all } t \leq T. \end{aligned} \quad (6.4.2)$$

Using the same change of variables as for call options (the equation is the same) we obtain the relation between  $V$  and the solution  $G$  to the diffusion equation

$$V(S, t) = e^{-r(T-t)} G \left( \ln S + \left( r - \frac{1}{2}\sigma^2 \right) (T-t), \frac{1}{2}\sigma^2 (T-t) \right) \quad (6.4.3)$$

Clearly, if  $t = T$ , then  $V(S, T) = G(\ln S, 0)$ , that is the initial value for  $G$  is

$$G(x, 0) = V(e^x, 0) = \max\{E - e^x, 0\}.$$

As  $S$  approaches 0, then  $x = \ln S + \left( r - \frac{1}{2}\sigma^2 \right) (T-t)$  approaches  $-\infty$  and we obtain

$$Ee^{-r(T-t)} = \lim_{S \rightarrow 0} V(S, t) = \lim_{x \rightarrow -\infty} e^{-r(T-t)} G(x, y)$$

so that one boundary condition for  $G$  reads:

$$\lim_{x \rightarrow -\infty} G(x, y) = E.$$

On the other hand, as  $S \rightarrow \infty$ , so does  $x$ , and the last boundary condition for  $G$  takes the form

$$\lim_{x \rightarrow +\infty} G(x, y) = 0.$$

Summarizing, to obtain the solution for the Black-Scholes equation via formula (6.4.3) we must solve the initial-boundary value problem

$$\begin{aligned} G_y &= G_{xx}, \quad \text{for } -\infty < x < \infty, y > 0 \\ G(x, 0) &= \max\{E - e^x, 0\}, \quad \text{for } -\infty < x < \infty, \\ \lim_{x \rightarrow -\infty} G(x, y) &= E, \quad \text{for any } y \geq 0, \\ \lim_{x \rightarrow \infty} G(x, y) &= 0, \quad \text{for any } y \geq 0. \end{aligned} \quad (6.4.4)$$

The following calculation are almost the same as in Section 3. The initial condition can be written as

$$\phi(x) = \begin{cases} E - e^x & \text{for } x < \ln E \\ 0 & \text{for } x \geq \ln E, \end{cases}$$

hence we obtain

$$G(x, y) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{\ln E} e^{-\frac{(x-s)^2}{4y}} (E - e^s) ds. \quad (6.4.5)$$

As before we express this solution in terms of the distribution function of standardised normal random variable defined as

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{p^2}{2}} dp.$$

We split the integral (6.4.5) and performing subsequently two changes of variables we get

$$\begin{aligned} G(x, y) &= \frac{1}{2\sqrt{\pi y}} \left( E \int_{-\infty}^{\ln E} e^{-\frac{(x-s)^2}{4y}} ds - \int_{-\infty}^{\ln E} e^{-\frac{(x-s)^2}{4y}} e^s ds \right) \\ &= \frac{1}{\sqrt{2\pi}} \left( E \int_{-\infty}^{\frac{\ln E - x}{\sqrt{2y}}} e^{-\frac{p^2}{2}} dp - \int_{-\infty}^{\frac{\ln E - x}{\sqrt{2y}}} e^{-\frac{p^2}{2}} e^{x - \sqrt{2y}p} dp \right) \\ &= \frac{1}{\sqrt{2\pi}} \left( E \int_{-\infty}^{\frac{\ln E - x}{\sqrt{2y}}} e^{-\frac{p^2}{2}} dp - e^{x+y} \int_{-\infty}^{\frac{\ln E - x}{\sqrt{2y}}} e^{-\frac{(p - \sqrt{2y})^2}{2}} dp \right) \\ &= \frac{1}{\sqrt{2\pi}} \left( E \int_{-\infty}^{\frac{\ln E - x}{\sqrt{2y}}} e^{-\frac{p^2}{2}} dp - e^{x+y} \int_{-\infty}^{\frac{\ln E - x - 2y}{\sqrt{2y}}} e^{-\frac{p^2}{2}} dp \right) \\ &= EN \left( \frac{\ln E - x}{\sqrt{2y}} \right) - e^{x+y} N \left( \frac{\ln E - x - 2y}{\sqrt{2y}} \right). \end{aligned}$$

As before let us note that despite the fact that we derived  $G$  as the solution of the initial value problem, luckily it also satisfies the boundary conditions. To prove it, let us recall that

$$\begin{aligned} \lim_{x \rightarrow -\infty} N(x) &= 0, \\ \lim_{x \rightarrow +\infty} N(x) &= 1. \end{aligned}$$

Since  $\lim_{x \rightarrow -\infty} e^{x+y} = 0$  for any fixed  $y$ , we immediately see that

$$\lim_{x \rightarrow -\infty} G(x, y) = \lim_{x \rightarrow -\infty} \left( EN \left( \frac{\ln E - x}{\sqrt{2y}} \right) - e^{x+y} N \left( \frac{\ln E - x - 2y}{\sqrt{2y}} \right) \right) = E,$$

and

$$\lim_{x \rightarrow \infty} e^{-(x+y)} G(x, y) = \lim_{x \rightarrow \infty} \left( e^{-(x+y)} EN \left( \frac{\ln E - x}{\sqrt{2y}} \right) - N \left( \frac{\ln E - x - 2y}{\sqrt{2y}} \right) \right) = 0,$$

so that  $G$  is the function which we have been looking for.

To derive explicit formulas for option pricing we must change back the variables  $x$  and  $y$  into  $S$  and  $t$ . As for call options we get

$$\frac{\ln E - x}{\sqrt{2y}} = - \frac{\ln S/E + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

and

$$\frac{\ln E - x - 2y}{\sqrt{2y}} = - \frac{\ln S/E + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$



Since

$$e^{x+y} = e^{\ln S + r(T-t)} = Se^{r(T-t)},$$

we obtain explicitly from (6.4.3)

$$\begin{aligned} V(S, t) = & Ee^{-r(T-t)} N\left(-\frac{\ln S/E + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & - SN\left(-\frac{\ln S/E + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned} \quad (6.4.6)$$

Next we derive the formula relating call and put options. To distinguish them we denote by  $V_c$  and  $V_p$  the price of call and put options, respectively.

Let us first observe that since the function  $e^{-p^2/2}$  is even and

$$\lim_{x \rightarrow \infty} N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{p^2}{2}} dp = 1,$$

we have

$$N(-x) = 1 - N(x). \quad (6.4.7)$$

Denoting

$$\alpha = \frac{\ln S/E + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\beta = \frac{\ln S/E + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

we see that the price formula for call options (6.3.9) can be written as

$$V_c(S, t) = SN(\beta) - Ee^{-r(T-t)}N(\alpha),$$

whereas for put options we have from (6.4.6)

$$V_p(S, t) = Ee^{-r(T-t)}N(-\alpha) - SN(-\beta).$$

Subtracting we obtain using (6.4.7)

$$V_c(S, t) - V_p(S, t) = SN(\beta) - Ee^{-r(T-t)}N(\alpha) - Ee^{-r(T-t)}(1 - N(\alpha)) + S(1 - N(\beta)) = S - Ee^{-r(T-t)}.$$

The derived formula

$$S - V_c(S, t) + V_p(S, t) = Ee^{-r(T-t)} \quad (6.4.8)$$

is called *put-call parity* and expresses the relationship between the underlying asset and its options.



# Lecture 8

## Comparison of wave and diffusion processes

In this lecture we summarize and compare the fundamental properties of the solutions to the wave and diffusion equations in infinite domain.

Property	Waves	Diffusion
Speed of propagation	Finite ( $\leq c$ )	Infinite
Singularities	Transported along characteristics	Lost immediately
Well-posedness for $t > 0$	Yes	Yes (for bounded initial values)
Well-posedness for $t < 0$	Yes	No
Maximum principle	No	Yes
Behaviour as $t \rightarrow +\infty$	Energy is constant	Solution decays to 0 $\int_{-\infty}^{+\infty} u(x, t) dx$ is constant

Most of these properties have been already discussed. We shall go through them once again providing proofs when necessary.

Speed of propagation for the wave equation was discussed in Section 4.2. Amongst others we proved there that for one and two dimensional diffusion equations any signal propagates at a speed less or equal than  $c$  whilst for the three dimensional equations the propagation speed is equal exactly to  $c$ .

For the diffusion equations we proved that the propagation speed is infinite in Lectures 6 and 7.

If the initial data for the wave equation have a discontinuity or any other type of singularity then it will be transported along characteristics - this follows from d'Alembert's formula in one dimension and Kirchhoff's formula in two and three dimensions.

For the diffusion equation, however, the solution is smooth for any  $t > 0$ , as follows from the representation (6.1.25) of the solution and therefore, whatever is the shape of the initial datum, this shape is smoothed down immediately.

Let us recall that the problem is well-posed (in some class of initial and/or boundary data) if (see Section 2.8:

- (a) there exists a solution for any data,
- (b) the solution is unique,

(c) the solution depends on the data in a continuous (stable) manner.

Both diffusion and wave equations satisfy for  $t > 0$  points (a) and (b) of the definition. Stability of the solution to the diffusion equation was proved in Section 1 and for the wave equation it follows directly from d'Alembert's and Kirchhoff's formulae.

The situation changes dramatically for  $t < 0$ . The wave equation is still well-posed as can be seen when we change the arrow of time  $t' = -t$ ; the wave equation remains the same and therefore all results proved for the "forward" wave equation remain valid for the "backward" one.

On the other hand when we change the arrow of time in the diffusion equation we obtain

$$u_{t'} = -Du_{xx} \quad (8.4.1)$$

which has completely different properties from the forward diffusion equation. To see this we solve the equation (8.4.1) for a reasonable initial datum:

$$u(x, 0) = S(x, 1) = \frac{1}{\sqrt{4D\pi}} e^{-x^2/4D}$$

where  $S$  is the source function. From the properties of the source function it follows that

$$S(x, -t + 1) = \frac{1}{\sqrt{4D\pi(1-t)}} e^{-x^2/4D(1-t)}$$

is a solution to (8.4.1) which satisfies the initial condition. However, this solution blows up at  $t = 1$  ( $S(0, 1) = +\infty$ ) and cannot be continued beyond  $t = 1$ .

The Maximum Principle for the diffusion equation was proved in Section 1. That there is no Maximum Principle for the wave equation can be seen from the picture of any solution.

The conservation of energy for the wave equation was proved in Section 1 of Lecture 5. If we consider now the solution to the diffusion equation given by Eq. (6.1.25) we see that under assumption that  $\phi$  is integrable we have

$$\begin{aligned} \int_{-\infty}^{+\infty} u(x, t) dx &= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} S(x-y, t) \phi(y) dy \right) dx \\ &= \int_{-\infty}^{+\infty} \phi(y) \left( \int_{-\infty}^{+\infty} S(x-y, t) dx \right) dy = \int_{-\infty}^{+\infty} \phi(y) dy, \end{aligned}$$

where we used the property (6.1.13) of the source function. Physically this means that the number of particles is conserved in the process of diffusion.

On the other hand, again for integrable  $\phi$  we obtain that for any fixed  $x$ ,

$$\begin{aligned} \lim_{t \rightarrow +\infty} u(x, t) &= \lim_{t \rightarrow +\infty} \int_{-\infty}^{+\infty} S(x-y) \phi(y) dy \\ &= \int_{-\infty}^{+\infty} \lim_{t \rightarrow +\infty} \left( \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4D\pi t}} \right) \phi(y) dy = 0, \end{aligned}$$

which follows from the fact that

$$\frac{e^{-(x-y)^2/4kt}}{\sqrt{4D\pi t}} \leq \frac{1}{\sqrt{4D\pi t}}.$$

This property expresses fact that if we have a finite amount of a substance ( $\phi$  integrable) in an infinite region, then the concentration of the substance at each point is decaying to zero as the substance is spreading to fill the whole region.

# Lecture 7

## Separation of variables for initial-boundary value problems

### 1 Types of boundary conditions

Up to now we have been dealing mainly with problems in the whole space (on the whole open line). This is clearly an idealization of the real world (the Universe itself is not infinite), however, it allowed us to pinpoint and compare basic features of the most important time-dependent physical processes: wave propagation and diffusion.

In this lecture we focus ourselves on problems which are posed in finite, bounded regions. Then, as we expect from physics, to get a well-posed problem we have to supplement the equation and initial data with the so-called boundary conditions, that is, we assume that the behaviour of the solution on the boundary of the region is known.

Let us consider one dimensional problems on the interval  $[0, l]$ . For both diffusion and wave equation we have three classical types of boundary conditions:

$$\begin{array}{ll} \text{Dirichlet conditions} & u(0, t) = u(l, t) = 0, \\ \text{Neumann conditions} & u_x(0, t) = u_x(l, t) = 0, \\ \text{Robin conditions} & u_x(0, t) - a_1 u(0, t) = u_x(l, t) + a_2 u(l, t) = 0. \end{array}$$

We can also consider mixed boundary conditions by imposing e.g., the Dirichlet condition at  $x = 0$  and the Neumann at  $x = l$ .

The conditions listed above are referred to as the homogeneous boundary conditions on the account of 0 appearing at the right-hand side; if we replace 0 by any arbitrary, but known, function of  $t$ , then we will have nonhomogeneous boundary conditions.

The boundary conditions have the following physical interpretation (at the end point  $x = 0$ ).

Heat equation

(temperature distribution in a finite rod)

Dirichlet condition	the end is kept at temperature $0^\circ$
Neumann condition	the end is insulated
Robin condition	radiation of energy for $a_1 > 0$ absorption of energy for $a_1 < 0$

Diffusion equation

Dirichlet condition	the concentration at the end is 0
Neumann condition	the end is not permeable
Robin condition	the boundary is partly permeable the concentration outside is 0

Wave equation  
(transverse vibration of a string)

---

Dirichlet condition	the end is fixed
Neumann condition	the end can freely move
Robin condition	the end is elastically fixed

## 2 Simple separation of variables

In this section we present basic ideas of the method of separation of variables for solving initial-boundary value problems. As an example we take the wave equation with homogeneous Dirichlet conditions:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \quad t > 0, \quad (7.2.1)$$

$$u(0, t) = u(l, t) = 0 \quad (7.2.2)$$

and the standard initial conditions:

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \quad (7.2.3)$$

Firstly, we look for all possible solutions in the form

$$u(x, t) = X(x)T(t). \quad (7.2.4)$$

We require that  $X(x)$  satisfies the boundary conditions  $X(0) = X(l) = 0$  corresponding to (7.2.2).

Inserting (7.2.4) into (7.2.1) we obtain

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \quad (7.2.5)$$

This equation defines a quantity  $\lambda$ , which must be constant.

Equation (7.2.5) is equivalent to the set of two ordinary differential equations:

$$\begin{aligned} X'' + \lambda X &= 0, \\ X(0) = X(l) &= 0. \end{aligned} \quad (7.2.6)$$

and

$$T'' + \lambda T = 0. \quad (7.2.7)$$

General solution to the equation in (7.2.6) is given (in a complex form) by

$$X_\lambda(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \quad (7.2.8)$$

and, by requiring the boundary conditions to be satisfied, we obtain

$$C_1 + C_2 = 0, \quad C_1 e^{\sqrt{-\lambda}l} + C_2 e^{-\sqrt{-\lambda}l} = 0. \quad (7.2.9)$$

Thus  $e^{2\sqrt{-\lambda}l} = 1$  which yields  $\operatorname{Re} \sqrt{-\lambda} = 0$  and  $\operatorname{Im} \sqrt{-\lambda} = \pi n/l$ . Finally,

$$\lambda = \frac{n^2 \pi^2}{l^2} \quad (7.2.10)$$

for integer  $n \neq 0$  gives all possible solutions to the problem (7.2.6). Note, that  $n = 0$  must be discarded, since then the solution to the equation in (7.2.6) is  $X(x) = Ax + B$  which doesn't satisfy the boundary conditions unless  $A = B = 0$ .

Numbers in (7.2.10) are called the eigenvalues of the problem.

The function in (7.2.8) can be written in real form as

$$X_n(x) = \bar{C}_1 \cos \frac{n\pi x}{l} + \bar{C}_2 \sin \frac{n\pi x}{l},$$

where  $\bar{C}_1 = C_1 + C_2$  and  $\bar{C}_2 = C_2 - C_1$ . By (7.2.9) we have  $\bar{C}_1 = 0$ , thus the only solutions to (7.2.6) are

$$X_n(x) = \sin \frac{n\pi x}{l}. \quad (7.2.11)$$

These functions are called the eigenfunctions of the problem.

Having determined all  $\lambda$  for which (7.2.6) has a non-zero solution we can solve (7.2.7). We obtain

$$T_n(t) = A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}$$

and all possible solutions to the problem (7.2.2)-(7.2.2) in the form (7.2.4) are given by

$$u_n(x, t) = \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}. \quad (7.2.12)$$

Since the wave equation is linear, any finite linear combination of functions of the form (7.2.12):

$$u(x, t) = \sum_n \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad (7.2.13)$$

is again a solution to (7.2.2) which satisfies the initial conditions. However, such a solution satisfies very special initial conditions:

$$\begin{aligned} u(x, 0) &= \phi(x) = \sum_n A_n \sin \frac{n\pi x}{l} \\ u_t(x, 0) &= \psi(x) = \sum_n \frac{n\pi c}{l} B_n \cos \frac{n\pi x}{l}, \end{aligned} \quad (7.2.14)$$

hence the method developed so far gives the solution to (7.2.2)-(7.2.3) only for a relatively small set of initial conditions.

The next step in the development of the method is much more delicate. We know that we have at our disposal an infinite number of solutions of the form (7.2.11). What if we replace the finite sums in (7.2.14) by infinite series

$$\begin{aligned} u(x, 0) &= \phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \\ u_t(x, 0) &= \psi(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos \frac{n\pi x}{l} \end{aligned} \quad (7.2.15)$$

Such series are called the sine and cosine Fourier series, respectively. The answer to this question, which will be discussed in more details in the next lecture, is that almost all reasonable functions can be expanded in the infinite series of the form (7.2.14). Thus, provided we can find coefficients  $A_n$  and  $B_n$ , it is reasonable to expect that the function

$$u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}. \quad (7.2.16)$$

is the solution to (7.2.2)-(7.2.3).

However, at least two questions remain open at this moment. Firstly, the function  $u$  may be not defined since the infinite series on the right-hand side may be not convergent. Secondly, even if the function  $u$  is defined, it may be not differentiable and therefore it may not be the solution of the wave equation. Both these questions will be addressed in the next lecture.





# Lecture 8

## Basic properties of Fourier series

In the previous lecture we derived the formula (7.2.16) which we expect should give a solution to the initial-boundary value problem (7.2.2)-(7.2.3). Three questions, however, have to be addressed before we can claim that such defined  $u$  is a genuine solution to this problem. These are:

1. What functions  $\psi$  and  $\phi$  can be represented in by Fourier series (7.2.15)?
2. How can the coefficients  $A_n$  and  $B_n$  be determined?
3. Under what conditions the function  $u$  defined by (7.2.16) is two times differentiable with respect to both  $t$  and  $x$ ?

### 1 Coefficients of Fourier series

We start with the answer to the second question. Let us recall that we encountered two types of Fourier series on  $(0, l)$ : the sine Fourier series

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad (8.1.1)$$

and the cosine Fourier series

$$\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{l}. \quad (8.1.2)$$

Assuming that this two Fourier series converge to their respective limits, how can we find coefficients  $a_n$  and  $b_n$ ?

The basic observation which leads to this is that for  $n, m = 0, 1, \dots$  the following relations hold:

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} dx = \int_0^l \cos \frac{m\pi x}{l} \cos \frac{n\pi x}{l} dx = \begin{cases} 0 & m \neq n \\ \frac{l}{2} & m = n. \end{cases} \quad (8.1.3)$$

The functions satisfying (8.1.3) are called orthogonal and  $\{\sin \frac{n\pi x}{l}\}_{n=0}^{\infty}$  and  $\{\cos \frac{n\pi x}{l}\}_{n=0}^{\infty}$  are called orthogonal sets of functions. We shall see later other sets of orthogonal functions which occur as eigenfunctions in various boundary value problems.

Relations (8.1.3) are crucial in obtaining coefficients of the Fourier series. We concentrate on the sine Fourier series (8.1.1). Let us forget for a moment that the series is infinite, multiply both sides of (8.1.1) by  $\sin \frac{m\pi x}{l}$  with fixed  $m$  and integrate over  $(0, l)$ . We obtain

$$\int_0^l \phi(x) \sin \frac{m\pi x}{l} dx = \int_0^l \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \int_0^l a_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx \\
&= a_m \frac{l}{2},
\end{aligned}$$

where we used (8.1.3) to get to the last line.

We can proceed in the same way for the cosine Fourier series. The formulae for coefficients for both series are given below.

$$\text{Sine Fourier series: } a_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx$$

$$\text{Cosine Fourier series: } b_n = \frac{2}{l} \int_0^l \phi(x) \cos \frac{n\pi x}{l} dx$$

**Example 1.1** Let  $\phi(x) = x$ . Coefficients of its sine Fourier series over  $(0, l)$  are

$$a_n = (-1)^{n+1} \frac{2l}{n\pi},$$

thus

$$x = \frac{2l}{\pi} \left( \sin \frac{\pi x}{l} - \frac{1}{2} \sin \frac{2\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} - \dots \right).$$

Coefficients of its cosine Fourier series are

$$b_n = \begin{cases} \frac{-4l}{n^2\pi^2} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even} \end{cases}$$

and therefore

$$x = \frac{l}{2} - \frac{4l}{\pi^2} \left( \cos \frac{\pi x}{l} + \frac{1}{9} \cos \frac{3\pi x}{l} + \frac{1}{25} \cos \frac{5\pi x}{l} + \dots \right).$$

## 2 Convergence of Fourier series

From the formulae for coefficients of Fourier series we see that we can build a Fourier series for any integrable function  $\phi$ . However, the question whether such a Fourier series converges to this function and in which way is very difficult. Therefore we shall only briefly sketch the basic ideas.

Firstly, we shall recall some notions of convergence of infinite series. We say that an infinite series  $\sum_{n=1}^{\infty} f_n(x)$  converges pointwise in  $(0, l)$  to a function  $f$  if for each  $x \in (0, l)$

$$\lim_{N \rightarrow +\infty} \left| f(x) - \sum_{n=1}^N f_n(x) \right| = 0.$$

We say that this series converges to  $f(x)$  uniformly in  $(0, l)$  if

$$\lim_{N \rightarrow +\infty} \max_{0 < x < l} \left| f(x) - \sum_{n=1}^N f_n(x) \right| = 0.$$

Finally, we say that the convergence is in the mean-square sense if

$$\lim_{N \rightarrow +\infty} \int_0^l \left| f(x) - \sum_{n=1}^N f_n(x) \right|^2 dx = 0.$$

Note that the uniform convergence is stronger than both pointwise and mean-square convergence. It is also important to remember that a series may converge in a mean-square sense and not converge pointwise.

The main reason why we introduced the notion of the mean-square convergence is that the Fourier series are easiest to handle in this setting. In fact, we have the following theorem:

**Theorem 2.1** *If  $f$  is such that  $\int_0^l |f(x)|^2 dx < \infty$ , then both sine and cosine Fourier series converge to  $f$  in the mean square sense. Moreover, if  $\{a_n\}_{n=0}^\infty$  are coefficients of the expansion of  $f$  in either series, then the following relation, called the Parseval relation, holds*

$$\sum_{n=0}^{\infty} |a_n|^2 = \frac{2}{l} \int_0^l |f(x)|^2 dx \quad (8.2.4)$$

This result is, however, not immediately useful for us since, as we mentioned, such a series may not be pointwise convergent.

The most interesting for us is the uniform convergence. Firstly, since the uniform convergence is stronger than the mean square convergence we know from Theorem 2.1 that if the Fourier series of some function  $f$  converges uniformly, then it must converge uniformly to  $f$ .

Another important property of the uniform convergence may be stated as follows:

**Theorem 2.2** *Let  $f(x) = \sum_{n=1}^{\infty} f_n(x)$  pointwise and the series  $\sum_{n=1}^{\infty} f'_n(x)$  converges uniformly, then  $f$  is differentiable and  $f'(x) = \sum_{n=1}^{\infty} f'_n(x)$ .*

Due to this property we will be able to determine whether the function  $u$  given by (7.2.16) is a genuine solution of the wave equation. If the series obtained by differentiating all terms of the expansion (7.2.16) twice with respect to both  $x$  and  $t$  are uniformly convergent, then  $u$  is twice differentiable and satisfies the wave equation (7.2.2).

We complete this survey by a useful test for uniform convergence.

**Theorem 2.3** (Weierstrass test) *If  $\max_{0 < x < l} |f_n(x)| \leq a_n$  and  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly to some  $f$  on  $[0, l]$ .*

We use the Weierstrass test to prove the following theorem:

**Theorem 2.4** *If  $f$  is twice differentiable in  $[0, l]$  and*

(a)  $f(0) = f(l) = 0$ , then its sine Fourier series is uniformly convergent to  $f$  on  $[0, l]$ ,

(b)  $f'(0) = f'(l) = 0$ , then its cosine Fourier series is uniformly convergent on  $[0, l]$ .

**Proof.** We prove only (a), the proof of (b) being analogous. From the coefficient formula for  $a_n$  we get differentiating by parts twice:

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx = \frac{2}{n\pi} \left( -\cos \frac{n\pi x}{l} f(x) \Big|_0^l + \frac{2}{l} \int_0^l f'(x) \cos \frac{n\pi x}{l} dx \right)$$

$$\begin{aligned}
&= \frac{2l}{n^2\pi^2} \left( \sin \frac{n\pi x}{l} f'(x) \Big|_0^l - \frac{2}{l} \int_0^l f''(x) \sin \frac{n\pi x}{l} dx \right) \\
&= -\frac{2l}{n^2\pi^2} \frac{2}{l} \int_0^l f''(x) \sin \frac{n\pi x}{l} dx,
\end{aligned}$$

hence  $|a_n| \leq \frac{\text{const}}{n^2}$  and by the Weierstrass test the series is uniformly convergent. That the convergence is to  $f$  follows from the comment after Theorem 2.1.

From this theorem we can derive conditions on  $\phi$  and  $\psi$  which would ensure that the solution  $u$  is twice differentiable. From Theorem 2.2 we know that  $u_{xx}$  and  $u_{tt}$ , if they exist, must be of the form:

$$u_{tt}(t) = -\sum_{n=1}^{\infty} \frac{n^2\pi^2 c^2}{l^2} \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad (8.2.5)$$

and

$$u_{xx}(x, t) = -\sum_{n=1}^{\infty} \frac{\pi^2 n^2}{l^2} \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}. \quad (8.2.6)$$

Let

$$\begin{aligned}
\phi(x) &= \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \\
\psi(x) &= \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \cos \frac{n\pi x}{l}.
\end{aligned} \quad (8.2.7)$$

Comparing with (7.2.15) we see that  $A_n = a_n$  and  $B_n = lb_n/n\pi c$  and if

$$|a_n| \leq \frac{\text{const}}{n^4} \quad |b_n| \leq \frac{\text{const}}{n^3}, \quad (8.2.8)$$

then  $u_{tt}$  and  $u_{xx}$  are well-defined by the Weierstrass theorem.

Proceeding as in the proof of Theorem 2.4 we can express conditions (8.2.8) in terms of regularity of  $\phi$  and  $\psi$ .

**Theorem 2.5** *If  $\phi$  is four times differentiable with  $\phi(0) = \phi(l) = \phi''(0) = \phi''(l) = 0$  and  $\psi$  is three times differentiable with  $\psi'(0) = \psi'(l)$ , then  $u$  given by (7.2.16) is twice differentiable and therefore it is a genuine solution of the problem (7.2.2)-(7.2.3).*

We must note, however, that the assumptions of this theorem are quite restrictive and in fact the formula (7.2.16) gives a solution (in somewhat weaker sense) for much larger class of initial data. For example, if  $|a_n| \leq \frac{\text{const}}{n^{2+\delta}}$  and  $|b_n| \leq \frac{\text{const}}{n^{1+\delta}}$ , then the series (8.2.5) and (8.2.6) converge in the mean-square sense and for all practical purposes we may treat  $u$  as a solution to (7.2.2)-(7.2.3).

## Lecture 9

# Application of separation of variables to transistor theory

Let us consider a  $p-n-p$  transistor with exponentially graded base and denote by  $p(x, t)$  the concentration of excess holes (positive charge carriers) at time  $t$  and position  $x$  in the base of the transistor which is assumed to occupy the interval  $[0, l]$ . With a good accuracy  $p$  is the solution of the following initial-boundary value problem:

$$p_t = D(p_{xx} - \frac{\eta}{l} p_x), \quad 0 < x < l, \quad t > 0, \quad (9.0.1)$$

$$\begin{aligned} p(0, t) &= p(x, t) = 0, \\ p(x, 0) &= \frac{Kl}{D} \left( \frac{1 - e^{-\eta(1-x/l)}}{\eta} \right) \end{aligned} \quad (9.0.2)$$

where  $K$ ,  $\eta$  and  $D$  are constants. This is not the pure diffusion equation but can be reduced to such by the following substitution:

$$p(x, t) = \exp \left( \frac{\eta}{2l} x - \frac{D\eta^2}{4l^2} t \right) u(x, t). \quad (9.0.3)$$

Inserting (9.0.3) into (9.0.1) we see that  $u$  satisfies the pure diffusion equation:

$$u_t = Du_{xx}. \quad (9.0.4)$$

The boundary conditions remain unchanged but the initial condition for  $u$  takes the form:

$$u(0, x) = \frac{Kl}{D\eta} e^{-\frac{\eta x}{2l}} \left( 1 - e^{-\eta(1-x/l)} \right). \quad (9.0.5)$$

Following the approach, presented in Lecture 8 for the wave equation, we obtain in this case the separated equations in the form:

$$\begin{aligned} X'' + \lambda X &= 0, \\ X(0) = X(l) &= 0, \end{aligned} \quad (9.0.6)$$

and

$$T' + \lambda DT = 0. \quad (9.0.7)$$

The first equation was solved in Section 2 of Lecture 8 giving the solution of the form (see (7.2.11)):

$$X_n(x) = \sin \frac{n\pi x}{l}.$$

The solution of the associated equation (9.0.7) is this time given by exponential functions:

$$T_n(t) = A_n \exp \left( -\frac{n^2 \pi^2 D}{l^2} t \right). \quad (9.0.8)$$

The solution  $u$  is thus given by the series

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 D}{l^2} t}, \quad (9.0.9)$$

where the constants  $A_n$  are the Fourier coefficients of the function  $u(0, t)$  defined by (9.0.5). Thus

$$A_n = \frac{2K}{D\eta} \int_0^l \left( e^{-\frac{\eta x}{2l}} - e^{-\eta} e^{\frac{\eta x}{2l}} \right) dx.$$

Using the formula

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos ax) + C$$

we obtain finally

$$A_n = \frac{2\pi Kl}{D} \frac{1 - e^{-\eta}}{\eta} \frac{n}{n^2 \pi^2 + \eta^2/4} \quad (9.0.10)$$

and the solution  $p$  is of the form

$$p(x, t) = \frac{2\pi Kl}{D} \frac{1 - e^{-\eta}}{\eta} e^{\frac{\eta x}{2l}} \sum_{n=1}^{\infty} \frac{n}{n^2 \pi^2 + \eta^2/4} \sin \frac{n\pi x}{l} e^{-\frac{D}{l^2} (n^2 \pi^2 + \eta^2/4) t}. \quad (9.0.11)$$

This is again a formal solution and it can be seen that it may not converge at  $t = 0$ . In fact, from the initial conditions for  $p$  we have  $p(0, 0) = \frac{Kl}{D} \left( \frac{1 - e^{-\eta}}{\eta} \right) \neq 0$  and the boundary conditions yield  $p(0, 0) = 0$ . Hence, the conditions of the problem force the discontinuity of the solution and we cannot expect the series to converge uniformly in the whole domain. However, for any  $t > 0$  the series in (9.0.11) behaves like the series  $\sum \frac{e^{-n^2}}{n}$  which is clearly convergent. Similar estimates can be obtained for all necessary derivatives and thus (9.0.11) turns out to be the genuine solution to the original problem (9.0.1)-(9.0.2) for  $t > 0$ . This is in perfect agreement with the smoothing property of the diffusion equation which was discussed in details in Lecture 5.

We shall demonstrate the usefulness of the obtained solution for calculating quantities of physical interest. For fixed  $x$  the function  $p(x, t)$  describes the manner in which the concentration of holes at the position  $x$  collapses with time as  $t \rightarrow \infty$ . The collapse gives rise to a current  $I(t)$ , called the emitter discharge current, which is given by

$$I(t) = \frac{I_p D}{K} p_x(0, t), \quad t > 0,$$

where  $I_p$  is a constant.

As was explained above, the presence of the exponential time factors in (9.0.11) allows to compute the derivative of  $p$  by differentiation of the series term by term. This gives

$$I(t) = 2\pi^2 I_p \frac{1 - e^{-\eta}}{\eta} \sum_{n=1}^{\infty} \frac{n^2}{n^2 \pi^2 + \eta^2/4} e^{-\frac{D}{l^2} (n^2 \pi^2 + \eta^2/4) t}. \quad (9.0.12)$$

A physical quantity of interest in transistor design is the reclaimable charge  $Q$  defined by the improper integral

$$Q = \int_{0^+}^{\infty} I(t) dt.$$

We substitute here  $I(t)$  as given by (9.0.12) and integrate termwise getting

$$Q = \frac{2I_p l^2}{D} \frac{1 - e^{-\eta}}{\eta} \sum_{n=1}^{\infty} \left( \frac{n\pi}{n^2 \pi^2 + \eta^2/4} \right)^2. \quad (9.0.13)$$

The series appearing in (9.0.13) can be expressed in a closed form. In fact, from (9.0.10) we see that the terms of the series in (9.0.13) are squares of the Fourier sine coefficients of the function

$$\psi(x) = \frac{D}{2Kl} \frac{\eta}{1 - e^{-\eta}} u(x, 0)$$

and hence, applying the Parseval relation (8.2.4), we obtain

$$\sum_{n=1}^{\infty} \left( \frac{n\pi}{n^2\pi^2 + \eta^2/4} \right)^2 = \frac{2}{l} \int_0^l |\psi(x)|^2 dx = \frac{1}{2} \frac{\frac{\sinh \eta}{\eta} - 1}{\cosh \eta - 1}.$$

Finally, we obtain  $Q$  in the form

$$Q = \frac{I_p l^2}{D} \frac{1 - e^{-\eta}}{\eta} \frac{\frac{\sinh \eta}{\eta} - 1}{\cosh \eta - 1}.$$





# Lecture 9

## Wave and diffusion equations with sources-Duhamel's formulas

So far we have discussed the initial value problems for homogeneous diffusion and wave equations, that is, for

$$\begin{aligned}u_t &= Du_{xx}, \\u(x, 0) &= \phi(x)\end{aligned}\tag{9.0.1}$$

and

$$\begin{aligned}u_{tt} &= c^2 u_{xx}, \\u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x),\end{aligned}\tag{9.0.2}$$

respectively. The aim of the present lecture is to show that if the solutions to these problems are known, then from them we can build solutions to nonhomogeneous problems:

$$\begin{aligned}u_t &= Du_{xx} + f(t, x), \\u(x, 0) &= \phi(x),\end{aligned}\tag{9.0.3}$$

and

$$\begin{aligned}u_{tt} &= c^2 u_{xx} + f(t, x), \\u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x),\end{aligned}\tag{9.0.4}$$

respectively. This is called the *Duhamel principle*.

Our approach will be through an analogy with the nonhomogeneous ordinary differential equations, that is why we shall start by recalling the suitable formulas.

### 1 Nonhomogeneous ordinary differential equations

Let us consider the following Cauchy problem for the ordinary first order linear differential equation:

$$\frac{du}{dt} + Au = f, \quad u(0) = u_0.\tag{9.1.1}$$

The solution of this problem can be obtained by the variation of constants formula. We first find the general solution to the homogeneous equation:

$$\frac{du}{dt} + Au = 0,$$

getting  $u(t) = Ce^{-At}$  and then we look for the solution of the nonhomogeneous problem in the form  $u(t) = C(t)e^{-At}$ . Differentiating, substituting to (9.1.1) and simplifying we get

$$\frac{dC}{dt} = e^{At}f(t),$$

that is  $C(t) = \int_0^t e^{As}f(s)ds + C_0$ , where  $C_0$  is an arbitrary constant. Thus, the general solution of the equation is given by

$$u(t) = C_0e^{-At} + e^{-At} \int_0^t e^{As}f(s)ds \quad (9.1.2)$$

and the solution to the initial value problem (9.1.1) can be written as

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}f(s)ds. \quad (9.1.3)$$

Let us have a closer look at the structure of this solution. It consists of the term  $\bar{u}(t) = e^{-At}u_0$  which solves the problem with  $f \equiv 0$  and the proper initial data, and  $\tilde{u}(t) = \int_0^t e^{-A(t-s)}f(s)ds$  which solves the problem with the initial data equal to zero and the proper  $f$ . Note that the function  $e^{-At}$  is involved in both terms and therefore the knowledge of this function (which depends only on the differential operator and not on the particular data) is sufficient to construct all possible solutions to (9.1.1).

It turns out that it is convenient to introduce the so-called solution operator  $\mathcal{S}(t)$  which acts on initial states producing the solutions to the differential equations. In other words, for any given  $u_0$ , the function  $u(t) = \mathcal{S}(t)u_0$  is the solution to the problem (9.1.1). Of course, in our case this operator is simply the multiplication by  $e^{-At}$ :

$$u(t) = \mathcal{S}(t)u_0 = e^{-At} \cdot u_0, \quad (9.1.4)$$

but this convention will be useful when we return to partial differential equations. Using the solution operator we can write (9.1.3) in the following compact form

$$u(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-s)f(s)ds. \quad (9.1.5)$$

Note that the integrand  $v(t) = \mathcal{S}(t-s)f(s)$  is, for each fixed  $s$ , the solution to the equation  $u' + Au = 0$  satisfying the "initial" condition  $v(s) = f(s)$ .

Let us turn our attention to the second order initial value problem

$$\begin{aligned} \frac{d^2u}{dt^2} &= -A^2u + f, \\ u(0) &= u_0, \quad \frac{du}{dt}(0) = u_1. \end{aligned} \quad (9.1.6)$$

As for the first order equation, we first find the general solution to the homogeneous equation

$$\frac{d^2u}{dt^2} + A^2u = 0.$$

The characteristic equation is  $r^2 = -A^2$  with two imaginary roots  $r_{1,2} = \pm iA$ , where  $i$  is the imaginary unit. Thus, the general solution is given by

$$u(t) = \bar{C}_1e^{-iAt} + \bar{C}_2e^{iAt}. \quad (9.1.7)$$

Since, however, we are interested in real valued solution, we note that

$$\begin{aligned} u(t) &= \bar{C}_1 e^{-iAt} + \bar{C}_2 e^{iAt} = \bar{C}_1 (\cos At - i \sin At) + \bar{C}_2 (\cos At + i \sin At) \\ &= (\bar{C}_1 + \bar{C}_2) \cos At + i(-\bar{C}_1 + \bar{C}_2) \sin At \\ &= C_1 \cos At + C_2 \sin At \end{aligned}$$

where  $C_1$  and  $C_2$  are real constants.

To obtain a solution to the nonhomogeneous equation we use the same approach as for the first order equation, that is, we shall be looking for the solution in the form

$$u(t) = C_1(t) \cos At + C_2(t) \sin At,$$

where  $C_1$  and  $C_2$  are two functions to be determined. Differentiating we obtain

$$\frac{du}{dt} = \frac{dC_1}{dt}(t) \cos At + \frac{dC_2}{dt}(t) \sin At - AC_1(t) \sin At + AC_2(t) \cos At$$

and before we proceed any further we require

$$\frac{dC_1}{dt}(t) \cos At + \frac{dC_2}{dt}(t) \sin At = 0. \quad (9.1.8)$$

This requirement will result in the absence of the second order derivatives of  $C_1$  and  $C_2$  in the second order derivative of  $u$ . Indeed, evaluating the second order derivative and taking into account (9.1.8) we obtain

$$\frac{d^2 u}{dt^2} = -\frac{C_1}{dt}(t)A \sin At + \frac{dC_2}{dt}(t)A \cos At - A^2(C_1(t) \cos At + C_2(t) \sin At).$$

Substituting the above equation into (9.1.6) we obtain the second equation for  $C_1, C_2$ :

$$-\frac{dC_1}{dt}(t)A \sin At + \frac{dC_2}{dt}(t)A \cos At = f(t) \quad (9.1.9)$$

so that we have the system

$$\begin{aligned} \frac{dC_1}{dt}(t) \cos At + \frac{dC_2}{dt}(t) \sin At &= 0 \\ -\frac{dC_1}{dt}(t)A \sin At + \frac{dC_2}{dt}(t)A \cos At &= f(t) \end{aligned} \quad (9.1.10)$$

The determinant of this system is given by  $A(\cos At \cos At + \sin At \sin At) = A$ , thus we obtain

$$\begin{aligned} \frac{dC_1}{dt}(t) &= -A^{-1}f(t) \sin At \\ \frac{dC_2}{dt}(t) &= A^{-1}f(t) \cos At \end{aligned}$$

Integrating, we obtain

$$\begin{aligned} C_1(t) &= -A^{-1} \int_0^t f(s) \sin(As) ds + c_1 \\ C_2(t) &= A^{-1} \int_0^t f(s) \cos(As) ds + c_2 \end{aligned}$$

where  $c_1$  and  $c_2$  are constants. Thus, the general solution to (9.1.6) is given by

$$\begin{aligned}
 u(t) &= c_1 \cos At + c_2 \sin At - A^{-1} \cos At \int_0^t f(s) \sin(As) ds + A^{-1} \sin At \int_0^t f(s) \cos(As) ds \\
 &= c_1 \cos At + c_2 \sin At + A^{-1} \int_0^t (\sin At \cos As - \sin As \cos At) f(s) ds \\
 &= c_1 \cos At + c_2 \sin At + \int_0^t \sin A(t-s) f(s) ds.
 \end{aligned} \tag{9.1.11}$$

To determine the solution of the initial value problem, we have

$$u(0) = c_1 = u_0$$

and differentiating

$$\frac{du}{dt}(t) = -Ac_1 \sin At + Ac_2 \cos At + A^{-1} \sin A(t-t)f(t) + \int_0^t \cos A(t-s)f(s)ds$$

thus

$$\frac{du}{dt}(0) = Ac_2 = u_1.$$

Finally, the solution to the initial value problem is given by

$$u(t) = u_0 \cos At + A^{-1}u_1 \sin At + A^{-1} \int_0^t (\sin A(t-s))f(s)ds. \tag{9.1.12}$$

Here, similarly to the first order case, we see that the solution can be written in terms of a single solution operator  $\mathcal{S}(t)$  defined for  $v \in \mathbb{R}$  as

$$\mathcal{S}(t)v = (A^{-1} \sin At) \cdot v.$$

In fact, we have

$$u(t) = \frac{d}{dt} \mathcal{S}(t) \cdot u_0 + \mathcal{S}(t) \cdot u_1 + \int_0^t \mathcal{S}(t-s) \cdot f(s) ds \tag{9.1.13}$$

and we see that we know all the solution of the nonhomogeneous initial value problem (9.1.6) once we know the solution of one specific problem

$$\begin{aligned}
 \frac{d^2 u}{dt^2} + A^2 u &= 0, \\
 u(0) &= 0, \quad \frac{du}{dt}(0) = 1.
 \end{aligned}$$

## 2 Solution of the diffusion equation with a source

In this section we shall try to apply the considerations of the first part of the previous section to solve the following problem

$$\begin{aligned}
 u_t &= Du_{xx} + f(x, t) \\
 u(x, 0) &= u_0.
 \end{aligned} \tag{9.2.1}$$

We expect similar properties of the solution to the ordinary differential equation  $u_t + Au = f$ , where  $A$  is a number, and the solution to the partial differential equation  $u_t - Du_{xx} = f$ . Note that the expected similarity is not too far fetched - for example the Laplace transform establishes a kind of equivalence between derivation and multiplication.

Let us then observe that in both cases we have the solution operator, that is, a rule which having the initial value as an input, gives the solution to the relevant initial value problem as an output. For the ODE we have

$$u(t) = \mathcal{S}(t)u_0 = e^{-At} \cdot u_0$$

and for the diffusion equation we have

$$u(x, t) = \mathcal{S}(t)\phi(x) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4Dt} \phi(y) dy.$$

The technicalities are different but the idea is the same in both cases.

To continue with our analogy, we expect the function

$$v(x, t) = \int_0^t \mathcal{S}(t-s)f(x, s)ds$$

to be the solution to the problem

$$v_t = Dv_{xx} + f(x, t), \quad v(x, 0) = 0. \quad (9.2.2)$$

In this case the explicit expression of  $v$  is as follows:

$$v(x, t) = \int_0^t \left( \frac{1}{\sqrt{4\pi D(t-s)}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4D(t-s)} f(y, s) dy \right) ds \quad (9.2.3)$$

It follows that our construction is correct. To prove this we shall use the formula for differentiation of integrals:

$$\frac{d}{dt} \int_0^t g(t, s) ds = g(t, t) + \int_0^t g_t(t, s) ds, \quad (9.2.4)$$

which is valid if  $f$  and  $f_t$  are continuous.

In our case the integrand

$$G(x, t-s) = \frac{1}{\sqrt{4\pi D(t-s)}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4D(t-s)} f(y, s) dy$$

unfortunately is not as regular as required in the above formula so that the proof we provide is purely formal; however, the justification can be provided along similar lines as in the proof of Theorem 7.1.1 provided  $f$  is a sufficiently regular function (for example once differentiable in  $t$  and twice in  $x$ ).

Let us observe that in order to use Eq. (9.2.4) we have to calculate  $G$  at  $t = s$ . This is possible thanks to Theorem 7.1.1 and we obtain

$$G(x, 0) = \lim_{t \rightarrow s} G(x, t-s) = f(x, t).$$

Also, for any fixed  $s$ ,  $G(x, t-s)$  is the shift of the solution of the homogeneous equation with the initial value given by  $f(x, s)$ , therefore it is itself a solution of the same diffusion equation by Property d) of Lecture 7. Therefore

$$G_t(x, t-s) = DG_{xx}(x, t-s).$$

Using these two facts we have formally

$$\begin{aligned}
 v_t(x, t) &= G(x, 0) + \int_0^t G_t(x, t-s) ds \\
 &= f(x, t) + D \int_0^t G_{xx}(x, t-s) ds = f(x, t) + D \frac{\partial^2}{\partial x^2} \int_0^t G(x, t-s) ds \\
 &= Dv_{xx}(x, t) + f(x, t)
 \end{aligned}$$

and  $v$  is indeed a solution to the nonhomogeneous diffusion equation.

Next we have

$$v(x, 0) = \lim_{t \rightarrow 0^+} v(x, t) = \lim_{t \rightarrow 0^+} \int_0^t G(x, t-s) ds = 0,$$

where we used the fact that the integrals of a bounded function over intervals with length tending to 0 also tend to zero.

Thus the Duhamel formula (9.2.3) for the diffusion equation has been formally justified.

**Example 2.1** Solve the following initial value problem

$$\begin{aligned}
 u_t &= u_{xx} + xt, \quad \text{for } t > 0, -\infty < x < \infty, \\
 u(x, 0) &= 0, \quad \text{for } -\infty < x < \infty.
 \end{aligned}$$

We see that  $f(x, t) = xt$ , thus according to the formula (9.2.3) we obtain the solution in the form

$$u(x, t) = \int_0^t \left( \frac{1}{\sqrt{4\pi(t-s)}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4(t-s)} ys dy \right) ds.$$

Using the substitution  $y = 2p\sqrt{t-s} + x$  we obtain

$$\begin{aligned}
 u(x, t) &= \frac{1}{\pi} \int_0^t \left( s \int_{-\infty}^{\infty} e^{-p^2} (2p\sqrt{t-s} + x) dp \right) ds \\
 &= \frac{1}{\pi} \left( \int_0^t 2s\sqrt{t-s} \int_{-\infty}^{\infty} e^{-p^2} p dp + sx \int_{-\infty}^{\infty} e^{-p^2} dp \right) ds = \frac{xt^2}{2}
 \end{aligned}$$

where we used that  $\int_{-\infty}^{\infty} e^{-p^2} p dp = 0$  (as  $e^{-p^2} p$  is an odd function) and  $\int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$  by Lemma 7.1.1.

### 3 The wave equation with a source

Following the ideas of the previous section we shall construct a solution for the wave equation with a source

$$u_{tt} = c^2 u_{xx} + f(x, t)$$

using the analogy with the second order ordinary differential equation. Let us recall that in the latter case the solution has the following structure

$$u(t) = \frac{d}{dt} \mathcal{S}(t) \cdot u_0 + \mathcal{S}(t) \cdot u_1 + \int_0^t \mathcal{S}(t-s) \cdot f(s) ds$$

that is the solution operator acts on the initial value of the time derivative of the solution. Remembering d'Alembert's formula (1.6.19)

$$u(x, t) = \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

we expect that the solution operator should be given by

$$\mathcal{S}(t)\phi(x) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy.$$

To test this guess we check that the first term of d'Alembert's formula is given by the time derivative of  $\mathcal{S}$ . In fact

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{S}(t)\phi(x) &= \frac{\partial}{\partial t} \left( \frac{1}{2c} \int_{x-ct}^{x+ct} \phi(y) dy \right) \\ &= \frac{1}{2c} (\phi(x + ct)c - \phi(x - ct)(-c)) = \frac{1}{2} (\phi(x + ct) + \phi(x - ct)) \end{aligned}$$

and we see that we are on the right track. Thus, we expect the solution of the inhomogeneous equation with homogeneous initial data:

$$v_{tt} = c^2 v_{xx} + f(x, t), \quad v(x, 0) = v_t(x, 0) = 0$$

to be given by

$$\begin{aligned} v(x, t) &= \int_0^t S(t-s) f(s) ds = \frac{1}{2c} \int_0^t \left( \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy \right) ds \\ &= \int_R f(y, s) dy ds \end{aligned} \tag{9.3.1}$$

where  $R$  is the characteristic triangle with vertices at  $(x - ct, 0)$ ,  $(x + ct, 0)$  and  $(x, t)$ . In other words

$$R = \{(y, s); x - ct \leq y \leq x + ct, 0 \leq s \leq t\}.$$

The proof that the formula is correct is much easier than for the diffusion equation. In fact, if  $f$  is e.g. a differentiable function, then

$$F(x, t, s) = \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy$$

is also differentiable with respect to  $t$ , and the derivatives are given by

$$F_t(x, t, s) = c(f(x + c(t-s), s) + f(x - c(t-s), s)) \tag{9.3.2}$$

and

$$F_{tt}(x, t, s) = c^2(f_{,1}(x + c(t-s), s) + f_{,1}(x - c(t-s), s)),$$

where  $f_{,1}$  denotes the derivative with respect to the first variable. Exactly in the same way we obtain

$$F_{xx}(x, t, s) = f_{,1}(x + c(t-s), s) + f_{,1}(x - c(t-s), s)$$

and therefore for any  $s$

$$F_{tt}(x, t, s) = c^2 F_{xx}(x, t, s). \tag{9.3.3}$$

Hence we have

$$v_t(x, t) = \frac{1}{2c} \frac{\partial}{\partial t} \left( \int_0^t F(x, t, s) ds \right) = \frac{1}{2c} \left( F(x, t, t) + \int_0^t F_t(x, t, s) ds \right) = \frac{1}{2c} \int_0^t F_t(x, t, s) ds, \quad (9.3.4)$$

as  $F(x, t, t) = \int_x^x f(y, s) dy = 0$ . Differentiating once again we obtain by (9.3.2) and (9.3.3)

$$\begin{aligned} v_{tt}(x, t) &= \frac{1}{2c} \left( F_t(x, t, t) + \int_0^t F_{tt}(x, t, s) ds \right) = \frac{1}{2c} \left( 2cf(x, t) + c^2 \int_0^t F_{xx}(x, t, s) ds \right) \\ &= f(x, t) + \frac{c}{2} \int_0^t F_{xx}(x, t, s) ds \end{aligned}$$

On the other hand

$$v_{xx}(x, t) = \frac{1}{2c} \int_0^t F_{xx}(x, t, s) ds,$$

thus

$$v_{tt}(x, t) = f(x, t) + c^2 v_{xx}(x, t)$$

and the equation is satisfied.

Also, it follows that

$$v(x, 0) = \frac{1}{2c} \int_0^0 F(x, 0, s) ds = 0$$

and from (9.3.4)

$$v_t(x, 0) = \frac{1}{2c} \int_0^0 F_t(x, 0, s) ds = 0,$$

and therefore we proved the validity of the Duhamel formula for the wave equation.

**Example 3.1** Find the solution to the following initial problem

$$\begin{aligned} u_{tt} &= u_{xx} + x \sin t, \text{ for } -\infty < x < \infty, t > 0, \\ u(x, 0) &= u_t(x, 0) = 0, \text{ for } -\infty < x < \infty. \end{aligned}$$

We have  $f(x, t) = x \sin t$  and from Eq. (9.3.1) we obtain

$$\begin{aligned} u(x, t) &= \frac{1}{2} \int_0^t \left( \int_{x-c(t-s)}^{x+c(t-s)} y \sin s dy \right) ds \\ &= \frac{1}{2} \int_0^t \sin s \left( \frac{y^2}{2} \Big|_{x-(t-s)}^{x+(t-s)} \right) ds = \frac{1}{4} \int_0^t 4x(t-s) \sin s ds \\ &= tx \int_0^t \sin s ds - \int_0^t s \sin s ds \\ &= x(t - \sin t). \end{aligned}$$



## 4 Other applications

**Example 4.1** Let us consider the following initial-boundary value problem for the diffusion equation in the half-plane

$$\begin{aligned} u_t &= u_{xx} + f(t, x), & \text{for } t > 0, x > 0, \\ u(x, 0) &= \phi(x), & \text{for } x > 0, \\ u(0, t) &= h(t), & \text{for } t > 0. \end{aligned} \quad (9.4.1)$$

Let us note that here we have a combination of problems discussed before (nonhomogeneous equation, boundary value problem) with a new one—the boundary datum is not homogeneous, that is,  $u$  at the endpoint is equal to a non-zero function. Thanks to the linearity of the problem we will be able to reduce it in several steps to the known initial value problem. Firstly, we show how to convert this problem to the one with zero boundary data. We use the method of lifting which, in general, requires extending the boundary data to the whole domain where the problem is defined. In our particular problem this step is easy: such an extension can be taken as  $H(x, t) = h(t)$ . Next we define the new unknown function

$$U(x, t) = u(x, t) - H(x, t). \quad (9.4.2)$$

Let us find the problem solved by this function. We have  $U_t = u_t - h_t$ ,  $U_{xx} = u_{xx}$  and therefore  $U$  is a solution of the diffusion equation  $U_t - U_{xx} = u_t - h_t - u_{xx} = f - h_t$ . For the initial value, we see that  $U(x, 0) = u(x, 0) - h(0) = \phi(x) - h(0)$  and at the boundary  $x = 0$  we obtain  $U(0, t) = u(0, t) - h(t) = 0$ . Summarizing,  $U$  solves the following problem

$$\begin{aligned} U_t &= U_{xx} + f(t, x) - h_t(t), & \text{for } t > 0, x > 0, \\ U(x, 0) &= \phi(x) - h(0), & \text{for } x > 0, \\ U(0, t) &= 0, & \text{for } t > 0. \end{aligned} \quad (9.4.3)$$

Next, let us observe that if, say,  $\tilde{U}$  solves the initial value problem

$$\begin{aligned} \tilde{U}_t &= \tilde{U}_{xx} + \tilde{f}(t, x), & \text{for } t > 0, -\infty < x < \infty, \\ \tilde{U}(x, 0) &= \tilde{\phi}(x), & \text{for } -\infty < x < \infty, \end{aligned} \quad (9.4.4)$$

where both  $\tilde{f}$  and  $\tilde{\phi}$  are odd functions of  $x$ , then  $\tilde{U}$  is also an odd function of  $x$ . In fact, define  $v(x, t) = -\tilde{U}(y, t)$ , where  $y = -x$ . We have  $v_t = -\tilde{U}_t$ ,  $v_x = \tilde{U}_y$ ,  $v_{xx} = -\tilde{U}_{yy}$  so that

$$v_t - v_{xx} = -(\tilde{U}_t - \tilde{U}_{yy}) = -\tilde{f}(y, t) = -\tilde{f}(-x, t) = f(x, t),$$

where in the last step we used the fact that  $f$  is an odd function. Moreover,

$$v(x, 0) = -\tilde{U}(y, 0) = -\phi(y) = -\phi(-x) = \phi(x)$$

as also  $\phi$  is odd. Thus,  $v$  is the solution of exactly the same problem as  $\tilde{U}$  and by uniqueness,  $v(x, t) = \tilde{U}(x, t)$ . However, from the definition,  $v(x, t) = -\tilde{U}(-x, t)$ . Therefore  $\tilde{U}(x, t) = -\tilde{U}(-x, t)$  and  $\tilde{U}$  is an odd function of  $x$ .

Now, one of the properties of a (continuous) odd functions is that they are equal to zero at the origin. Thus,  $\tilde{U}(0, t) = 0$  for  $t > 0$  and we see that the boundary condition is automatically satisfied.

This observation suggests a way of solving (9.4.3)—we extend both  $f - h_t$  and  $\phi$  to the whole real line in an odd way and solve the corresponding initial value problem (9.4.4). The resulting solution, restricted to  $x > 0$  is then the solution of the problem (9.4.3) by the discussion above. Thus, let us define

$$\tilde{f}(x, t) = \begin{cases} f(x, t) - h_t(t) & \text{for } x > 0 \\ -f(-x, t) + h_t(t) & \text{for } x < 0 \end{cases}$$

and

$$\tilde{\phi}(x) = \begin{cases} \phi(x) - h(0) & \text{for } x > 0 \\ -\phi(-x) + h(0) & \text{for } x < 0 \end{cases}$$

Then the solution  $v$  of the problem (9.4.4) with such defined  $\tilde{f}$  and  $\tilde{\phi}$ , restricted to  $x > 0$ , is (at least formally) the solution of the problem (9.4.1). We used the word "formally" here as the construction above can lead to  $\tilde{f}$  and/or  $\tilde{\phi}$  being discontinuous functions and then the formulas for solutions proved above are not necessarily valid.

Note that if instead of the Dirichlet problem (9.4.1) we are required to solve the Neumann problem

$$\begin{aligned} u_t &= u_{xx} + f(t, x), \quad \text{for } t > 0, x > 0, \\ u(x, 0) &= \phi(x), \quad \text{for } x > 0, \\ u_x(0, t) &= h(t), \quad \text{for } t > 0. \end{aligned} \quad (9.4.5)$$

can be solved in a similar way with  $U(x, t) = u(x, t) - xh(t)$  and even extension of the data.

**Example 4.2** Solve the following boundary-initial value problem for the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + f(t, x), \quad \text{for } t > 0, x > 0, \\ u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x), \quad \text{for } x > 0, \\ u(0, t) &= h(t), \quad \text{for } t > 0. \end{aligned} \quad (9.4.6)$$

We could have used exactly the same method as above but we shall apply the results of Example 6.2 in Lecture 3 and split (9.4.6) into two easier problems:

$$\begin{aligned} v_{tt} &= c^2 v_{xx} + f(t, x), \quad \text{for } t > 0, x > 0, \\ v(x, 0) &= \phi(x), \quad v_t(x, 0) = \psi(x), \quad \text{for } x > 0, \\ v(0, t) &= 0, \quad \text{for } t > 0, \end{aligned} \quad (9.4.7)$$

and

$$\begin{aligned} w_{tt} &= c^2 w_{xx}, \quad \text{for } t > 0, x > 0, \\ w(x, 0) &= 0, \quad w_t(x, 0) = 0, \quad \text{for } x > 0, \\ w(0, t) &= h(t), \quad \text{for } t > 0. \end{aligned} \quad (9.4.8)$$

Clearly,  $u = v + w$  is the solution of the original problem (9.4.6).

Firstly, we verify directly that if  $f$  is an odd function, then the integral

$$v_0(x, t) = \frac{1}{2} \int_0^t \left( \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy \right) ds$$

satisfies

$$v_0(0, t) = \frac{1}{2} \int_0^t \left( \int_{-c(t-s)}^{c(t-s)} f(y, s) dy \right) ds = 0$$

as the interior integral is the integral of an odd function over a symmetric interval. Thus the formula

$$v(x, t) = \frac{1}{2}(\phi_{\text{odd}}(x + ct) + \phi_{\text{odd}}(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(s) ds + \frac{1}{2} \int_0^t \left( \int_{x-c(t-s)}^{x+c(t-s)} f_{\text{odd}}(y, s) dy \right) ds,$$

where  $\phi_{\text{odd}}, \psi_{\text{odd}}$  and  $f_{\text{odd}}$  denote odd extensions of  $\phi, \psi$  and  $f$ , respectively, gives the solution to the problem (9.4.7).

To solve (9.4.8) we use the fact that in the absence of sources the solution must be of the form

$$w(x, t) = m(x + ct) + n(x - ct), \quad (9.4.9)$$

for some functions  $m$  and  $n$ . Using the initial values we obtain  $m(s) + n(s) = 0$  and  $cm'(s) - cn'(s) = 0$  for  $s > 0$  which gives  $m(s) = -n(s) = k$  for some constant  $k$  and  $s > 0$ . From the boundary condition we obtain  $w(0, t) = m(ct) + n(-ct) = k + n(s) = h(-s/c)$  for  $s < 0$ . Now, the argument in  $m$  in (9.4.9) is always nonnegative, thus  $m(x + ct) = k$  for any  $x > 0, t > 0$ . On the other hand, the argument of  $n$  is positive for  $x > t$ , and there  $n(x - ct) = -k$  and it is negative for  $x < t$  which yields  $n(x - ct) = -k + h((ct - x)/c)$  in this region. Summarizing, the solution  $w$  is given by the following formula

$$w(x, t) = \begin{cases} 0 & \text{for } t < x \\ h\left(\frac{ct-x}{c}\right) & \text{for } t > x \end{cases}$$

Note that in order to have at least continuous solution we must make sure that the initial and boundary data are compatible in the common point of definition:  $t = 0, x = 0$ , that is, we must have  $u(0, 0) = h(0) = \psi(0)$  and  $u_t(0, 0) = h_t(0) = \psi'(0)$ . Otherwise, our formulas will define discontinuous functions along the characteristics passing through the origin.

To solve the Neumann problem

$$\begin{aligned} u_{tt} &= u_{xx} + f(t, x), & \text{for } t > 0, x > 0, \\ u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x), & \text{for } x > 0, \\ u_x(0, t) &= h(t), & \text{for } t > 0 \end{aligned} \tag{9.4.10}$$

we proceed analogously - the only change is that we have to take even extensions of  $\phi, \psi$  and  $f$ .



# Lecture 10

## The Laplace equation

In this lecture we shall discuss the boundary value problem for the last of the basic equations of classical mathematical physics -the Poisson equation:

$$\begin{aligned}\Delta u &= f \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u &= \phi, \quad \text{on } \partial\Omega,\end{aligned}\tag{10.0.1}$$

where  $\Delta u = u_{xx} + u_{yy}$  if  $n = 2$  and  $\Delta u = u_{xx} + u_{yy} + u_{zz}$  if  $n = 3$ .

Of special interest are solutions to (7.0.1) with  $f = 0$ , that is, the solutions to the Laplace equation. Any solution to the Laplace equation is called a *harmonic function*. Let us recall that both real and imaginary parts of any analytic function  $f(z)$ , where  $z = x + iy \in \mathbb{C}$  are harmonic functions of two variables  $x, y$ .

The main tools of our investigation are Green's integral identities which, by the way, were discovered exactly for this purpose and only later found their way into other branches of mathematics and physics. We start however with finding a special solution to the Laplace equation, called the fundamental solution, which plays the role similar to the source function  $S$  in the theory of the heat equation.

### 1 Fundamental solution to the Laplace equation

Let us consider the Laplace equation in the whole space  $\mathbb{R}^3$

$$\Delta u = 0.$$

We are interested in radially symmetric solutions to this equation, that is, solutions which depend only on the distance from the origin:  $u(x, y, z) = u(r)$ , where  $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$ . Following the procedure outlined in Section 6.3 we find that the Laplace equation for such functions can be transformed into the ordinary differential equation

$$\Delta u = u_{,rr} + \frac{2}{r}u_{,r} = 0,\tag{10.1.1}$$

which can be written as

$$(r^2 u_r)_r = 0.$$

This equation has the general solution

$$u(r) = c_1 r^{-1} + c_2.$$

The constant solution is not interesting; thus we shall focus on the particular solution to the Laplace equation defined as

$$\Phi_3(x, y, z) = \Phi_3(r) = \frac{1}{r} = \frac{1}{\|\mathbf{r}\|} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.\tag{10.1.2}$$

Similar procedure in two dimensions with  $\mathbf{r} = (x, y)$  leads to the equation

$$u_{rr} + \frac{1}{r}u_r = 0$$

which in turn can be written as

$$(ru_r)_r = 0,$$

yielding the general solution

$$u(r) = c_1 \log r + c_2,$$

where  $\log$  is the natural logarithm.

As before we drop the constant solution as not important and therefore in the sequel we will be using the following solution

$$\Phi_2 x, y, z = \Phi_2(r) = \log r = \log \|\mathbf{r}\| = \log(x^2 + y^2). \quad (10.1.3)$$

Note that strictly speaking neither function  $\Phi$  solves the Laplace in the whole space as both become infinite at  $\mathbf{r} = 0$ . This apparent drawback will, however, prove essential in constructing the solution for an arbitrary domain  $\Omega$  and function  $f$ .

## 2 Green's first identity and its applications

Let us start with the following identity for functions of one variable: for any two sufficiently regular functions  $u, v$  we have

$$(vu_x)_x = v_x u_x + vu_{xx}.$$

If  $u$  and  $v$  are functions of two or three variables, then this identity is valid for the remaining variables and therefore adding the identities for all the variables we obtain (let  $n = 3$  to fix attention)

$$\begin{aligned} \operatorname{div}(v\nabla u) &= (vu_x)_x + (vu_y)_y + (vu_z)_z = (v_x u_x + v_y u_y + v_z u_z) + (vu_{xx} + vu_{yy} + vu_{zz}) \\ &= \nabla v \cdot \nabla u + v\Delta u. \end{aligned}$$

If we integrate this identity and use the Gauss theorem

$$\int_{\Omega} \operatorname{div}(v\nabla u) d\mathbf{r} = \int_{\partial\Omega} v \frac{\partial u}{\partial n} dS$$

to the left-hand side term (recall that  $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$ ), then we obtain

$$\int_{\partial\Omega} v \frac{\partial u}{\partial n} dS = \int_{\Omega} \nabla v \cdot \nabla u d\mathbf{r} + \int_{\Omega} v \cdot \Delta u d\mathbf{r}. \quad (10.2.1)$$

This is *Green's first identity*. It is important to note that  $v$  and  $u$  are completely arbitrary (but of course sufficiently regular) functions.

We use (10.2.1) to prove the *mean value property* for harmonic functions.

**Theorem 2.1** *Let  $u$  be a harmonic function in a domain  $\Omega$ . Then for any ball  $B(\mathbf{r}_0, \rho) = \{\mathbf{r} \in \mathbb{R}^n; \|\mathbf{r} - \mathbf{r}_0\| \leq \rho\} \subset \Omega$  the integral of  $u$  over boundary of  $B$  (the sphere) is equal to the value of  $u$  at the centre  $\mathbf{r}_0$  multiplied by the area of the sphere  $4\pi\rho$ :*

$$u(\mathbf{r}_0) = \frac{1}{4\pi\rho^2} \int_{\partial B} u(\mathbf{r}) dS. \quad (10.2.2)$$

*The right-hand side of this formula is called the average of  $u$  over the sphere  $\partial B$ .*

**Proof.** First, let us note that the Laplace equation is invariant with respect to translations, hence it is enough to prove the theorem for  $\mathbf{r}_0 = \mathbf{0}$ . More precisely, suppose that we have proved the theorem at  $\mathbf{r}_0 = \mathbf{0}$  for an arbitrary harmonic function, and that we want to prove it for arbitrary  $\mathbf{r}_0$ . Let us define  $v(\mathbf{r}) = u(\mathbf{r}_0 - \mathbf{r})$ . Using the Chain Rule we see that  $v$  is again a harmonic function, thus the theorem is valid for  $v$  at  $\mathbf{r}_0 = \mathbf{0}$  and therefore we have the following chain of identities

$$\begin{aligned} u(\mathbf{r}_0) &= v(\mathbf{0}) = \frac{1}{4\pi\rho^2} \int_{\{\|\mathbf{r}\|=\rho\}} v(\mathbf{r}) dS \\ &= \frac{1}{4\pi\rho^2} \int_{\{\|\mathbf{r}\|=\rho\}} u(\mathbf{r}_0 - \mathbf{r}) dS = \frac{1}{4\pi\rho^2} \int_{\{\|\mathbf{r}' - \mathbf{r}_0\|=\rho\}} u(\mathbf{r}') dS \end{aligned}$$

where we used the fact that the translation  $\mathbf{r}' = \mathbf{r}_0 - \mathbf{r}$  does not change the surface element  $dS$ .

Hence, we shall prove the theorem for  $\mathbf{r}_0 = \mathbf{0}$ .

If  $u$  is a harmonic function in  $B$ , then  $\Delta u = 0$  in  $B$ . Moreover, for the sphere of radius  $r$  centered at  $\mathbf{0}$ , the unit outward normal is given by  $\mathbf{n} = r^{-1}(x, y, z)$  and therefore the outward normal derivative is given by

$$\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u = r^{-1}(xu_x + yu_y + zu_z) = \frac{\partial u}{\partial r} \quad (10.2.3)$$

where the last equality follows from the Chain Rule:  $u_r = u_x x_r + u_y y_r + u_z z_r$  and the fact that, since  $r^2 = x^2 + y^2 + z^2$ , we have  $2rr_x = 2x, 2rr_y = 2y, 2rr_z = 2z$ .

Using these two results with  $v \equiv 1$  we can write (10.2.1) in the form

$$\int_{\partial B} \frac{\partial u}{\partial r} dS = 0,$$

or in polar coordinates

$$\int_0^{2\pi} \int_0^\pi u_r(r, \theta, \phi)|_{r=\rho} \rho^2 \sin \theta d\theta d\phi = 0.$$

Now, this result is valid for any  $\rho$  and we can divide both sides by  $4\pi\rho^2$  (the area of the sphere) and pull out the derivative outside the integral to get

$$\left( \frac{\partial}{\partial r} \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi u(r, \theta, \phi) \sin \theta d\theta d\phi \right) \Big|_{r=\rho} = 0.$$

The notation is a little awkward but the above formula tells us that if we take the function of  $r$

$$F(r) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi u(r, \theta, \phi) \sin \theta d\theta d\phi$$

differentiate it with respect to  $r$  and substitute  $r = \rho$ , then for any  $\rho$  this derivative is equal to zero, which means that  $F(r)$  is independent of  $r$ . In particular

$$F(r) = F(0)$$

but since  $u(0, \theta, \phi) = u(0, 0, 0) = u(\mathbf{0})$ , we obtain

$$F(0) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi u(0, 0, 0) \sin \theta d\theta d\phi = u(\mathbf{0})$$

and we have proved that

$$\frac{1}{\text{area of } \partial B} \int_{\partial B} u \, dS = \frac{1}{4\pi\rho^2} \int_0^{2\pi} \int_0^\pi u(\rho, \theta, \phi) \rho^2 \sin \theta \, d\theta \, d\phi = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi u(\rho, \theta, \phi) \sin \theta \, d\theta \, d\phi u(\mathbf{0})$$

which ends the proof. ■

This theorem can be used to prove the *maximum principle* for harmonic functions.

**Theorem 2.2** *Let  $\Omega$  be an open, bounded and connected (that is consisting of one piece) region in  $\mathbb{R}^n$  and let  $u$  be a harmonic function in  $\Omega$  which is continuous in  $\bar{\Omega} = \Omega \cup \partial\Omega$ . Then the maximum and minimum values of  $u$  are attained on  $\partial\Omega$  and nowhere inside (unless  $u$  is a constant).*

**Proof.** Let  $u$  be a harmonic function in  $\Omega$  and continuous on the closure of  $\Omega$ . By continuity,  $u$  takes on the maximum value, say,  $M$ , somewhere at  $\mathbf{r}_M \in \bar{\Omega}$ . Assume that  $\mathbf{r}_M$  is an interior point of  $\Omega$ . Then by the definition of the maximum, there is a ball  $B(\mathbf{r}_M, \rho_0) \subset \Omega$  such that

$$u(\mathbf{r}) \leq u(\mathbf{r}_M) = M,$$

for all  $\mathbf{r} \in B(\mathbf{r}_M, \rho_0)$ . Let us take any  $0 < \rho < \rho_0$  and use the mean value theorem for  $u(\mathbf{r}_M)$ . Denoting  $S_\rho = \partial B(\mathbf{r}_M, \rho)$ , we have

$$M = u(\mathbf{r}_M) = \frac{1}{\text{area of } S_\rho} \int_{S_\rho} u(\mathbf{r}) \, dS_\rho \leq \max_{\mathbf{r} \in S_\rho} u(\mathbf{r}) \leq M,$$

since  $M$  is the absolute maximum. Thus the average over the sphere is equal to the maximum over the same sphere. However, this is possible only when the function is constant over the sphere. In fact, let for some  $\mathbf{r}_1 \in S_\rho$ ,  $u(\mathbf{r}_1) = M_1 < M$ , then by continuity there is a subset of the sphere, say  $S_1$ , with nonzero area, such that for  $\mathbf{r} \in S_1$  we have  $u(\mathbf{r}) \leq M - \epsilon < M$  for some  $\epsilon > 0$ . Thus,

$$\begin{aligned} \frac{1}{\text{area of } S_\rho} \int_{S_\rho} u(\mathbf{r}) \, dS_\rho &= \frac{1}{\text{area of } S_\rho} \left( \int_{S_1} u(\mathbf{r}) \, dS_\rho + \int_{S_\rho \setminus S_1} u(\mathbf{r}) \, dS_\rho \right) \\ &\leq \frac{1}{\text{area of } S_\rho} (\text{area of } S_\rho \cdot (M - \epsilon) + \text{area of } (S_\rho \setminus S_1) \cdot M) \\ &= M - \epsilon (\text{area of } S_1) < M \end{aligned}$$

where we used: area of  $S_\rho = \text{area of } S_1 + \text{area of } (S_\rho \setminus S_1)$ . This contradiction proves the statement, thus  $u(\mathbf{r}) = M$  for  $\mathbf{r} \in S_\rho$ , and since it is valid for any  $\rho < \rho_0$ , we obtain that  $u(\mathbf{r}) = M$  for all  $\mathbf{r} \in B(\mathbf{r}_M, \rho_0)$ .

Next we fix any point  $\mathbf{x}$  on the sphere  $S_\rho$ . We obtained that at  $\mathbf{x}$  the function  $u$  attains absolute maximum equal to  $M$  and we can repeat the consideration above for the ball  $B(\mathbf{x}, \rho_1)$  for some  $\rho_1 > 0$  and we obtain that also on this ball the function  $u$  has a constant value equal to  $M$ .

Since the domain  $\Omega$  is connected, it is possible to join any two points by a chain of overlapping balls. In particular, any point, say  $\mathbf{y}$ , can be joined by such a chain with the original ball  $B(\mathbf{r}_0, \rho_0)$ . By the argument above  $u(\mathbf{y}) = M$ , and since  $\mathbf{y}$  is arbitrary, we obtain that  $u \equiv M$  throughout  $\Omega$ . Hence, if  $u$  takes on a maximum somewhere in the interior, then it must be necessarily constant in  $\Omega$ .

The reasoning for the minimum is exactly the same. ■

As for the diffusion equation, also here the maximum principle leads directly to the uniqueness of the solution to the problem (7.0.1). We have the following result.

**Theorem 2.3** *There is at most one solution to the problem (7.0.1).*



**Proof.** Let  $u_1$  and  $u_2$  be two solutions to (7.0.1). Then  $u = u_1 - u_2$  is the solution to the homogeneous problem for the Laplace equation.

$$\Delta u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

By the maximum principle,  $u$  attains its maximum and minimum on  $\partial\Omega$  but since both maximum and minimum are 0, the function  $u$  is identically equal to zero and  $u_1 = u_2$ . ■

**Remark 2.1** Note that this theorem is valid only for bounded domains. Indeed, consider for example  $\Omega = \{(x, y); y > 0\}$  and the Dirichlet problem  $\Delta u = 0$  in  $\Omega$  and  $u(x, 0) = 0$ . Clearly, one solution is  $u \equiv 0$ . However, there are other solutions:  $u(x, y) = y$ ,  $u(x, y) = xy$ . The reason is that the maximum principle breaks down in infinite regions—we still don't have local maxima, but the function can increase with arguments tending to  $\infty$ .

Yet another application of Green's formula is the so-called Dirichlet principle which is related to the physical concept of energy. Recall that in Lecture 3 we introduced the concept of energy in the wave motion - the total energy of a vibrating system occupying at rest the region  $\Omega$  was given by

$$E(t) = E_K(t) + E_P(t) = \frac{\rho}{2} \int_{\Omega} u_t^2(\mathbf{r}, t) d\mathbf{r} + \frac{T}{2} \int_{\Omega} |\nabla u|^2 d\mathbf{r},$$

The Laplace equation can be looked at as a frozen wave equation - independent of time. In such a case no kinetic energy is present and the energy of, say, elastic string or membrane, with shape given by the function  $u$  is given by

$$E(u) = \frac{T}{2} \int_{\Omega} |\nabla u|^2 d\mathbf{r}. \quad (10.2.4)$$

If this string or membrane does not move, it means that the shape  $u$  is the one describing the equilibrium of this string or membrane. According to physics, at this shape the system should have the least energy. Mathematically, the corresponding statement is called the *Dirichlet principle*.

**Theorem 2.4** Let  $u$  be the unique harmonic function in  $\Omega$  that satisfies the boundary condition

$$u(\mathbf{r}) = h(\mathbf{r}), \text{ for } \mathbf{r} \in \partial\Omega. \quad (10.2.5)$$

If  $w$  is any other function satisfying (10.2.5), then

$$E(w) \geq E(u). \quad (10.2.6)$$

**Proof.** Define  $v = u - w$  and evaluate

$$\begin{aligned} E(w) &= E(u + v) = \frac{T}{2} \int_{\Omega} |\nabla u(\mathbf{r}) + \nabla v(\mathbf{r})|^2 d\mathbf{r} \\ &= \frac{T}{2} \int_{\Omega} ((u_x + v_x)^2 + (u_y + v_y)^2 + (u_z + v_z)^2) d\mathbf{r} \\ &= \frac{T}{2} \int_{\Omega} ((u_x^2 + v_x^2 + u_y^2 + v_y^2 + u_z^2 + v_z^2 + 2(u_x v_x + u_y v_y + u_z v_z))) d\mathbf{r} \\ &= \frac{T}{2} \left( \int_{\Omega} |\nabla u(\mathbf{r})|^2 d\mathbf{r} + \frac{T}{2} \int_{\Omega} |\nabla v(\mathbf{r})|^2 d\mathbf{r} + 2 \int_{\Omega} \nabla u \cdot \nabla v d\mathbf{r} \right) \\ &= E(u) + E(v) + 2 \int_{\Omega} \nabla u \cdot \nabla v d\mathbf{r} \end{aligned}$$

Since  $u$  is a harmonic function ( $\Delta u = 0$ ) and  $v = 0$  on the boundary  $\partial\Omega$ , from Green's identity (10.2.1) we obtain that  $\int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{r} = 0$  and therefore

$$E(w) = E(u) + E(v).$$

But the energy  $E$  is a nonnegative function, thus for any  $w$  (which satisfies the boundary condition (10.2.5)) we have

$$E(w) \geq E(u)$$

that is the minimum energy is attained at the harmonic function  $u$ . ■

### 3 Second Green's formula with applications

Second Green's formula is an immediate consequence of the fact that the first formula is symmetric with respect to  $u$  and  $v$ : we have both

$$\int_{\partial\Omega} v \frac{\partial u}{\partial n} dS = \int_{\Omega} \nabla v \cdot \nabla u \, d\mathbf{r} + \int_{\Omega} v \cdot \Delta u \, d\mathbf{r}$$

and

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS = \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{r} + \int_{\Omega} u \cdot \Delta v \, d\mathbf{r}.$$

Subtracting one from the other we get

$$\int_{\Omega} (u \Delta v - v \Delta u) \, d\mathbf{r} = \int_{\partial\Omega} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS. \quad (10.3.1)$$

This identity leads to an important definition. We call a boundary condition *symmetric* for the operator  $\Delta$  if the right-hand side of (10.3.1) vanishes for all pairs of functions that satisfy the boundary condition. For example, if both  $u$  and  $v$  satisfy the homogeneous Dirichlet boundary condition, then the right-hand side of (10.3.1) clearly vanishes. Similar situation is for Neumann boundary conditions. For the Robin conditions we have  $\frac{\partial u}{\partial n} + au = 0$  and  $\frac{\partial v}{\partial n} + av = 0$  on the boundary, and hence we have

$$u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} = -auv + avu = 0.$$

Thus Robin boundary conditions are also symmetric.

We use second Green's formula to prove the formula which gives the value of a harmonic function at a point in  $\Omega$  in terms of the values of the function (and its normal derivative) at the boundary. In two dimensions such a representation can be obtained from the Cauchy formula (recall that any harmonic function of two variables can be obtained as a real part of an analytic function).

We prove the following result.

**Theorem 3.1** *If  $u$  is a harmonic function in  $\Omega \subset \mathbb{R}^3$ , then for any  $\mathbf{r}_0 \in \Omega$*

$$u(\mathbf{r}_0) = \frac{1}{4\pi} \int_{\partial\Omega} \left( -u(\mathbf{r}) \frac{\partial}{\partial n} \left( \frac{1}{\|\mathbf{r} - \mathbf{r}_0\|} \right) + \frac{1}{\|\mathbf{r} - \mathbf{r}_0\|} \frac{\partial u}{\partial n}(\mathbf{r}) \right) dS \quad (10.3.2)$$

**Proof.** We shall use (10.3.1) with  $v(\mathbf{r}_0, \mathbf{r}) = -\frac{1}{4\pi}\Phi_3(\mathbf{r}_0 - \mathbf{r})$  (where  $\Phi_3$  is given by (10.1.2)). However, this is not immediate, because  $\Phi_3$  is not differentiable at zero, that is  $v$  is not differentiable at  $\mathbf{r} = \mathbf{r}_0$ . We shall get around this difficulty by cutting from  $\Omega$  a ball  $B_\epsilon$  centered at  $\mathbf{r} = \mathbf{r}_0$  with radius  $\epsilon$  (small enough for  $B_\epsilon \subset \Omega$ ). Outside this ball both  $v$  and  $u$  are harmonic ( $v$  is harmonic since  $\Phi_3$  is harmonic everywhere except at 0). Denoting  $\Omega_\epsilon = \Omega \setminus B_\epsilon$  we obtain

$$0 = \int_{\Omega_\epsilon} (u\Delta v - v\Delta u) d\mathbf{r} = \int_{\partial\Omega_\epsilon} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS. \quad (10.3.3)$$

The boundary  $\partial\Omega_\epsilon$  consists of two separate parts:  $\partial\Omega_\epsilon = \partial\Omega \cup S_\epsilon$  where  $S_\epsilon$  is the sphere centered at  $\mathbf{r} = \mathbf{r}_0$  and radius  $\epsilon$ .

For simplicity, let  $\mathbf{r}_0 = \mathbf{0}$  (see the proof of the mean value theorem).

By (10.2.3) the outward normal derivative on the sphere is equal to  $\frac{\partial u}{\partial r}$ . Here however, the outward normal vector for the sphere points inward  $\Omega_\epsilon$ , hence the outward normal derivative with respect to  $\Omega_\epsilon$  on  $S_\epsilon$  is given by  $-\frac{\partial u}{\partial r}$ . With this in mind we obtain from (10.3.3)

$$\int_{\partial\Omega} \left( u \frac{\partial}{\partial n} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS = \int_{S_\epsilon} \left( u \frac{\partial}{\partial r} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial r} \right) dS. \quad (10.3.4)$$

The left hand side of (10.3.4) corresponds now to the right-hand side of (10.3.2) (with minus sign), so the theorem will follow if we can prove that the right-hand side of (10.3.4) tends to  $-4\pi u(\mathbf{0})$  as  $\epsilon \rightarrow 0$ .

Since on  $S_\epsilon$  we have  $r = \epsilon$  we have that

$$\frac{\partial}{\partial r} \left( \frac{1}{r} \right) = -\frac{1}{r^2} = -\frac{1}{\epsilon^2}$$

so that

$$\int_{S_\epsilon} \left( u \frac{\partial}{\partial r} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial r} \right) dS = -\frac{1}{\epsilon^2} \int_{S_\epsilon} u dS + \frac{1}{\epsilon} \int_{S_\epsilon} \frac{\partial u}{\partial r} dS.$$

Since  $\frac{\partial u}{\partial r}$  is a bounded function, for the second integral we obtain the following estimate

$$\left| \frac{1}{\epsilon} \int_{S_\epsilon} \frac{\partial u}{\partial r} dS \right| \leq \max \left| \frac{\partial u}{\partial r} \right| \frac{1}{\epsilon} \int_{S_\epsilon} dS = \max \left| \frac{\partial u}{\partial r} \right| \frac{4\pi\epsilon^2}{\epsilon} \leq M\epsilon$$

for some constant  $M$ . Hence this term tends to zero if  $\epsilon \rightarrow 0$ . For the first term we note that we can write  $4\pi\epsilon^2 u(\mathbf{0}) = \int_{S_\epsilon} u(\mathbf{0}) dS_\epsilon$  and therefore

$$\begin{aligned} \left| \frac{1}{\epsilon^2} \int_{S_\epsilon} u dS - 4\pi u(\mathbf{0}) \right| &\leq \frac{1}{\epsilon^2} \int_{S_\epsilon} |u(\mathbf{r}) - u(\mathbf{0})| dS \\ &\leq 4\pi \max_{\|\mathbf{r}\| \leq \epsilon} |u(\mathbf{r}) - u(\mathbf{0})| \end{aligned}$$

and the last term tends to zero as  $\epsilon \rightarrow 0$  by the continuity of  $u$ . Therefore

$$\lim_{\epsilon \rightarrow 0} -\frac{1}{\epsilon^2} \int_{S_\epsilon} u dS + \frac{1}{\epsilon} \int_{S_\epsilon} \frac{\partial u}{\partial r} dS = -4\pi u(\mathbf{0})$$

and the theorem is proved. ■

In two dimensions we have to use  $\Phi_2(r) = \log(x^2 + y^2)$  as the function  $v$ . With this provision the proof is the same leading to the following representation formula

$$u(\mathbf{r}_0) = \frac{1}{2\pi} \int_{\partial\Omega} \left( -u(\mathbf{r}) \frac{\partial}{\partial n} \log \|\mathbf{r} - \mathbf{r}_0\| + \log \|\mathbf{r} - \mathbf{r}_0\| \frac{\partial u}{\partial n}(\mathbf{r}) \right) dS, \quad (10.3.5)$$

where  $\mathbf{r}_0 \in \Omega \subset \mathbb{R}^2$  and  $dS$  is the line element of the boundary curve  $\partial\Omega$ .

**Remark 3.1** If the function  $u$  is not harmonic but instead satisfies the Poisson equation in  $\Omega$  we have  $\Delta u = f$  and the term  $\int_{\Omega} v \Delta u \, d\mathbf{r} = \int_{\Omega} v f \, d\mathbf{r}$  is carried through the whole calculations of Theorem 3.1 leading to the following formula

$$u(\mathbf{r}_0) = \frac{1}{4\pi} \int_{\partial\Omega} \left( -u(\mathbf{r}) \frac{\partial}{\partial n} \left( \frac{1}{\|\mathbf{r} - \mathbf{r}_0\|} \right) + \frac{1}{\|\mathbf{r} - \mathbf{r}_0\|} \frac{\partial u}{\partial n}(\mathbf{r}) \right) dS + \frac{1}{4\pi} \int_{\Omega} \frac{f(\mathbf{r})}{\|\mathbf{r} - \mathbf{r}_0\|} d\mathbf{r}. \quad (10.3.6)$$

In two dimensions the formula (10.3.6) should be similarly modified using the logarithm instead of  $1/\|\mathbf{r}\|$ .

## 4 Green's function for the Dirichlet problem

The representation formula (10.3.2) would give us the solution to the following problem

$$\begin{aligned} \Delta u &= 0, & \text{in } \Omega, \\ u &= f, \quad \frac{\partial u}{\partial n} = g, & \text{on } \partial\Omega. \end{aligned}$$

However, this problem is overdetermined – we know that it is enough to know only the value of the harmonic function on the boundary to determine it uniquely. Thus we have to modify (10.3.2) so that the term involving the normal derivative vanishes. This leads to the concept of *Green's function*.

**Definition 4.1** *Green's function  $G(\mathbf{r})$  of the operator  $\Delta$  and the domain  $\Omega$  at the point  $\mathbf{r}_0$  is a function defined for  $\mathbf{r} \in \Omega$  such that:*

1.  $G(\mathbf{r})$  possesses continuous second derivatives and  $\Delta G = 0$  in  $\Omega$  except at the point  $\mathbf{r} = \mathbf{r}_0$ ,
2.  $G(\mathbf{r}) = 0$  for  $\mathbf{r} \in \partial\Omega$ ,
3. the function  $H(\mathbf{r}) = G(\mathbf{r}) + \frac{1}{4\pi\|\mathbf{r} - \mathbf{r}_0\|}$  is finite at  $\mathbf{r}_0$ , has continuous second derivatives and  $\Delta H = 0$  in  $\Omega$ .

Note that Green's function is the solution of the special Dirichlet problem

$$\begin{aligned} \Delta_r G &= -\frac{1}{4\pi} \Delta_r \left( \frac{1}{\|\mathbf{r} - \mathbf{r}_0\|} \right) & \mathbf{r} \in \Omega, \\ G(\mathbf{r}) &= 0, & \mathbf{r} \in \partial\Omega, \end{aligned}$$

where the subscript  $r$  indicates that  $\mathbf{r}$  is the independent variable of the equation (and  $\mathbf{r}_0$  is a parameter).

It can be shown that Green's function exists and is unique. Since it depends on the point  $\mathbf{r}_0$ , it is convenient to acknowledge this dependence by writing  $G(\mathbf{r}, \mathbf{r}_0)$ . We have the following result

**Theorem 4.1** *If  $G(\mathbf{r}, \mathbf{r}_0)$  is the Green function for  $\Omega$  at  $\mathbf{r}_0$ , then the unique solution of the Dirichlet problem is given by the formula*

$$u(\mathbf{r}_0) = \int_{\partial\Omega} u(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} dS. \quad (10.4.1)$$

**Proof.** Since  $u$  and  $H = G + \frac{1}{4\pi\|\mathbf{r}-\mathbf{r}_0\|}$  are harmonic functions, we have from (10.3.1) and (10.3.4)

$$\begin{aligned} 0 &= \int_{\partial\Omega} \left( u \frac{\partial H}{\partial n} - H \frac{\partial u}{\partial n} \right) dS \\ &= \int_{\partial\Omega} \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dS + \frac{1}{4\pi} \int_{\partial\Omega} \left( u \frac{\partial}{\partial n} \left( \frac{1}{\|\mathbf{r}-\mathbf{r}_0\|} \right) - \frac{1}{\|\mathbf{r}-\mathbf{r}_0\|} \frac{\partial u}{\partial n} \right) dS \\ &= \int_{\partial\Omega} \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dS - u(\mathbf{r}_0), \end{aligned}$$

thus

$$u(\mathbf{r}_0) = \int_{\partial\Omega} \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dS.$$

However, from the definition we have that  $G = 0$  on  $\partial\Omega$  so that the second term in the integral vanishes and we obtain

$$u(\mathbf{r}_0) = \int_{\partial\Omega} u \frac{\partial G}{\partial n} dS,$$

which is exactly (10.4.1). ■

Note the similarity of this concept with that of the source function for the diffusion equation—the source function was also obtained as the solution of a special problem and the solution of a given initial value problem was obtained by integration of this particular solution and the datum.

**Remark 4.1** We can use Green's function also to represent the solution of the Poisson equation. In fact, since  $\Delta u = f$  and  $H$  is harmonic, we get as above

$$\begin{aligned} \int_{\Omega} f(\mathbf{r}) H(\mathbf{r}) d\mathbf{r} &= \int_{\partial\Omega} \left( u \frac{\partial H}{\partial n} - H \frac{\partial u}{\partial n} \right) dS \\ &= \int_{\partial\Omega} \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dS + \frac{1}{4\pi} \int_{\partial\Omega} \left( u \frac{\partial}{\partial n} \left( \frac{1}{\|\mathbf{r}-\mathbf{r}_0\|} \right) - \frac{1}{\|\mathbf{r}-\mathbf{r}_0\|} \frac{\partial u}{\partial n} \right) dS \\ &= \int_{\partial\Omega} \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dS - u(\mathbf{r}_0) + \frac{1}{4\pi} \int_{\Omega} \frac{f(\mathbf{r})}{\|\mathbf{r}-\mathbf{r}_0\|} d\mathbf{r}, \end{aligned}$$

thus by  $H = G + 1/\|\mathbf{r}-\mathbf{r}_0\|$ , and using the boundary value of  $G$  we obtain

$$u(\mathbf{r}_0) = \int_{\partial\Omega} u \frac{\partial G}{\partial n} dS + \int_{\Omega} f(\mathbf{r}) G(\mathbf{r}_0, \mathbf{r}) d\mathbf{r}. \quad (10.4.2)$$

The only thing wrong with (10.4.1) is that usually it is not easy to find  $G$  explicitly. However, this can be done for some special geometries, as we shall see below.

## 5 Applications of Green's function

### 5.1 Green's function for the half-space

Let  $\Omega$  be the half-space

$$\Omega = \{\mathbf{r} = (x, y, z); z > 0\}$$

We use the fundamental solution  $\Phi_3(\mathbf{r}) = \frac{1}{\|\mathbf{r}\|}$  as the building block and modify it so that all the properties of Green's function are satisfied. First note that for any fixed  $\mathbf{r}_0 = (x_0, y_0, z_0)$  with  $z > 0$ , the function  $\tilde{G}(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{\|\mathbf{r} - \mathbf{r}_0\|}$  satisfies conditions 1. and 2. of the definition, so that we must modify it only to ensure that the boundary condition is satisfied. First note that adding to  $\tilde{G}$  any harmonic function  $v$  doesn't destroy these two properties, so we should find a harmonic function in  $\{z > 0\}$  such that

$$\tilde{G}(\mathbf{r}, \mathbf{r}_0)|_{z=0} = -\frac{1}{4\pi\sqrt{(x-x_0)^2 + (y-y_0)^2 + z_0^2}} = v(\mathbf{r})|_{z=0}.$$

This suggests an appropriate candidate for  $v$ : let us take the same function as before, but constructed for the point symmetric to  $\mathbf{r}_0$ , that is,  $v(\mathbf{r}) = \frac{1}{4\pi\|\mathbf{r} - \mathbf{r}_0^*\|}$  where  $\mathbf{r}_0^* = (x_0, y_0, -z_0)$ . Because  $v$  is unbounded only at  $\mathbf{r} = \mathbf{r}_0^*$  which doesn't belong to  $\Omega$ , it is clear that  $v$  is a harmonic function in  $\Omega$ . Furthermore, since at  $z = 0$  we have

$$\|\mathbf{r} - \mathbf{r}_0\|_{z=0} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z_0^2} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (-z_0)^2} = \|\mathbf{r} - \mathbf{r}_0^*\|_{z=0}$$

we see that the sought Green's function for the half space is given by

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}_0) &= \tilde{G}(\mathbf{r}, \mathbf{r}_0) + v = -\frac{1}{4\pi\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} + \frac{1}{4\pi\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}} \\ &= -\frac{1}{4\pi\|\mathbf{r} - \mathbf{r}_0\|} + \frac{1}{4\pi\|\mathbf{r} - \mathbf{r}_0^*\|}. \end{aligned} \quad (10.5.1)$$

To get the formula for the solution we have to calculate the derivative of  $G$  in the direction of the outward normal to  $\Omega$  at the boundary  $\{z = 0\}$ . But the direction of the normal here coincides with the negative direction of the  $z$  axis of the coordinate system, that is,

$$\left. \frac{\partial G}{\partial n} \right|_{\partial\Omega} = -\left. \frac{\partial G}{\partial z} \right|_{z=0}.$$

Using the formula

$$\frac{\partial}{\partial z} \left( \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \right) = -\frac{z-z_0}{(\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2})^3}$$

we obtain for  $z = 0$

$$\left. \frac{\partial G}{\partial n} \right|_{\partial\Omega} = \frac{1}{4\pi} \left( \frac{z+z_0}{\|\mathbf{r} - \mathbf{r}_0\|^3} - \frac{z-z_0}{\|\mathbf{r} - \mathbf{r}_0^*\|^3} \right) \Big|_{z=0} = \frac{1}{2\pi} \frac{z_0}{\|\mathbf{r} - \mathbf{r}_0\|^3}$$

and consequently the solution to the Dirichlet problem

$$\begin{aligned} \Delta u &= 0, & \{(x, y, z); z > 0\} \\ u(x, y, 0) &= h(x, y) \end{aligned}$$

is given by

$$u(\mathbf{r}_0) = \frac{z_0}{2\pi} \int_{\{z=0\}} \frac{h(\mathbf{r})}{\|\mathbf{r} - \mathbf{r}_0\|^3} dS_r,$$

where the subscript  $r$  denotes that the integration is carried with respect to the variable  $\mathbf{r}$ . Explicitly we get

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h(x, y)}{((x_0 - x)^2 + (y_0 - y)^2 + z_0^2)^{3/2}} dx dy.$$

## 5.2 Green's function for the sphere

In this subsection  $\Omega$  is the ball centered at  $\mathbf{0}$  with radius  $a > 0$ :

$$\Omega = \{\mathbf{r}; \|\mathbf{r}\| \leq a\}.$$

The method we use here is similar to that of the previous subsection, that is, we use the method of reflection. However, the reflection with respect to the sphere is a little more complicated than in the case of the plane.

Let  $\|\mathbf{r}_0\| < a$ . Following the idea above, we are looking for the point  $\mathbf{r}_0^*$  outside the ball so that Green's function will have the form

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi\|\mathbf{r} - \mathbf{r}_0\|} + \frac{\alpha}{4\pi\|\mathbf{r} - \mathbf{r}_0^*\|},$$

where  $\alpha$  is a constant (this allows a greater flexibility in our search without changing the main property of the second term, that is, its being a harmonic function). Thus, for any  $\alpha$  the properties 1. and 3. of the definition of the Green function are satisfied.

Now, we must pick up the constant  $\alpha$  and the point  $\mathbf{r}_0^*$  so that for any  $\mathbf{r}_0$  inside the sphere, the function  $G(\mathbf{r}, \mathbf{r}_0) = 0$  whenever  $\|\mathbf{r}\| = a$ .

It is natural to look first at points which are colinear with the origin  $\mathbf{0}$  and  $\mathbf{r}_0$ , that is, of the form  $\mathbf{r}_0 = \beta\mathbf{r}_0^*$  with the constant  $\beta$  to be determined. Thus, we are looking for the Green function among the functions of the form

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi\|\mathbf{r} - \mathbf{r}_0\|} + \frac{\alpha}{4\pi\|\mathbf{r} - \beta\mathbf{r}_0\|}.$$

To determine the constants we shall assume that  $\|\mathbf{r}_0\| \neq 0$  and use two particular points on the sphere for which the calculations are the easiest, that is, the points colinear with  $\mathbf{0}$  and  $\mathbf{r}_0$ . These points are determined to be  $\mathbf{r} = \pm \frac{a\mathbf{r}_0}{\|\mathbf{r}_0\|}$  and for  $G$  to be the Green function it is necessary that

$$\alpha \left\| \pm \frac{a\mathbf{r}_0}{\|\mathbf{r}_0\|} - \mathbf{r}_0 \right\| = \left\| \pm \frac{a\mathbf{r}_0}{\|\mathbf{r}_0\|} - \beta\mathbf{r}_0 \right\|. \quad (10.5.2)$$

These equations can be transformed into the following system

$$\begin{aligned} \alpha|a - \|\mathbf{r}_0\|| &= |a - \beta\|\mathbf{r}_0\|| \\ \alpha|a + \|\mathbf{r}_0\|| &= |a + \beta\|\mathbf{r}_0\|| \end{aligned}$$

Since we are interested in finding points outside the sphere, we must have  $\beta > a/\|\mathbf{r}_0\|$ , hence dropping the absolute value bars we obtain

$$\begin{aligned} \alpha(a - \|\mathbf{r}_0\|) &= (-a + \beta\|\mathbf{r}_0\|) \\ \alpha(a + \|\mathbf{r}_0\|) &= (a + \beta\|\mathbf{r}_0\|). \end{aligned}$$

Thus

$$\alpha = \frac{a}{\|\mathbf{r}_0\|}, \quad \beta = \frac{a^2}{\|\mathbf{r}_0\|^2}$$

and

$$\mathbf{r}_0^* = \beta\mathbf{r}_0 = \frac{a^2}{\|\mathbf{r}_0\|^2}\mathbf{r}_0.$$

To complete the construction, we must show that the property (10.5.2) holds for all points on the sphere and not only for these particular two.

Thus, let  $\mathbf{r}, \|\mathbf{r}\| = a$ , be any point on the sphere. The point  $\mathbf{r}' = \frac{\|\mathbf{r}_0\|}{a}\mathbf{r}$  is at the distance  $\|\mathbf{r}_0\|$  from the origin and therefore the triangles with vertices at  $\mathbf{0}, \frac{a}{\|\mathbf{r}_0\|}\mathbf{r}_0, \mathbf{r}'$  and  $\mathbf{0}, \|\mathbf{r}\|, \mathbf{r}_0$  are congruent. This means that

$$\|\mathbf{r} - \mathbf{r}_0\| = \left\| \frac{a}{\|\mathbf{r}_0\|}\mathbf{r}_0 - \frac{\|\mathbf{r}_0\|}{a}\mathbf{r} \right\| = \frac{\|\mathbf{r}_0\|}{a} \left\| \mathbf{r} - \frac{a^2}{\|\mathbf{r}_0\|^2}\mathbf{r}_0 \right\| = \alpha^{-1}\|\mathbf{r} - \mathbf{r}_0^*\|,$$

so that we have determined the Green function as

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi\|\mathbf{r} - \mathbf{r}_0\|} + \frac{a}{\mathbf{r}_0} \frac{1}{4\pi\|\mathbf{r} - \mathbf{r}_0^*\|}, \quad (10.5.3)$$

where

$$\mathbf{r}_0^* = \frac{a^2}{\|\mathbf{r}_0\|^2} \mathbf{r}_0.$$

Now, to express the solution we must find the normal derivative of  $G$  on the sphere (remember that  $\mathbf{r}_0$  is fixed so that the differentiation is carried with respect to  $\mathbf{r}$  only). Thus

$$\nabla\|\mathbf{r} - \mathbf{r}_0\| = \frac{\mathbf{r} - \mathbf{r}_0}{\|\mathbf{r} - \mathbf{r}_0\|}, \quad \nabla\|\mathbf{r} - \mathbf{r}_0^*\| = \frac{\mathbf{r} - \mathbf{r}_0^*}{\|\mathbf{r} - \mathbf{r}_0^*\|}$$

and differentiating  $G$  we obtain

$$\nabla_r G(\mathbf{r}, \mathbf{r}_0) = \frac{\mathbf{r} - \mathbf{r}_0}{4\pi\|\mathbf{r} - \mathbf{r}_0\|^3} - \frac{a}{\|\mathbf{r}_0\|} \frac{\mathbf{r} - \mathbf{r}_0^*}{\|\mathbf{r} - \mathbf{r}_0^*\|^3}. \quad (10.5.4)$$

On the sphere  $\|\mathbf{r}\| = a$  we have from the above

$$\|\mathbf{r} - \mathbf{r}_0^*\| = \frac{a\|\mathbf{r} - \mathbf{r}_0\|}{\|\mathbf{r}_0\|},$$

thus substituting into (10.5.4) we get

$$\nabla_r G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi\|\mathbf{r} - \mathbf{r}_0\|^3} \left( \mathbf{r} - \mathbf{r}_0 - \frac{\|\mathbf{r}_0\|^2}{a^2} \mathbf{r} - \mathbf{r}_0 \right) = \frac{a^2 - \|\mathbf{r}_0\|^2}{4\pi\|\mathbf{r} - \mathbf{r}_0\|^3} \mathbf{r}.$$

Since the normal is given by  $\mathbf{n} = \mathbf{r}/a$ , the normal derivative is given by

$$\left. \frac{\partial G}{\partial n} \right|_{\|\mathbf{r}\|=a} = \frac{\mathbf{r}}{a} \cdot \nabla G = \frac{a^2 - \|\mathbf{r}_0\|^2}{4\pi a \|\mathbf{r} - \mathbf{r}_0\|^3}$$

and the final formula for the solution of the Dirichlet problem can be written as

$$u(\mathbf{r}_0) = \frac{a^2 - \|\mathbf{r}_0\|^2}{4\pi a} \int_{\{\|\mathbf{r}\|=a\}} \frac{h(\mathbf{r})}{\|\mathbf{r} - \mathbf{r}_0\|^3} dS. \quad (10.5.5)$$

The above calculations have been carried out for  $\mathbf{r}_0 \neq \mathbf{0}$ . However,  $\mathbf{r}_0$  is the centre of the ball so that we can use the mean value theorem which asserts that

$$u(\mathbf{0}) = \frac{1}{4\pi} \int_{\{\|\mathbf{r}\|=a\}} h(\mathbf{r}) dS$$

which agrees exactly with the right-hand side of (10.5.5) evaluated at  $\mathbf{r}_0 = \mathbf{0}$ .

As before, we rewrite (10.5.5) in an explicit form, this time using spherical coordinates on the sphere. Then  $dS = a^2 \sin \theta d\theta d\phi$  and using the cosine rule

$$\|\mathbf{r} - \mathbf{r}_0\|^2 = \|\mathbf{r}\|^2 + \|\mathbf{r}_0\|^2 - 2\|\mathbf{r}\|\|\mathbf{r}_0\| \cos \psi = r_0^2 + a^2 - 2r_0 a \cos \psi,$$

where  $\psi$  is the angle between  $\mathbf{r}$  and  $\mathbf{r}_0$ , and  $r_0 = \|\mathbf{r}_0\|$ , we obtain

$$u(\mathbf{r}_0) = u(r_0, \theta_0, \phi_0) = \frac{a(a^2 - r_0^2)}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{h(\theta, \phi)}{(a^2 + r_0^2 - 2ar_0 \cos \psi)^{3/2}} \sin \theta d\theta d\phi. \quad (10.5.6)$$



## 6 Laplace equation in two dimensions

The Laplace equation in two dimensional space is exceptional due to a direct link with the complex functions theory. In fact, let  $f : \Omega \rightarrow \mathbb{C}$  be an analytic function in a certain domain  $\Omega \subset \mathbb{C}$ . Then, writing  $z = x + iy$  and  $f(z) = u(x, y) + iv(x, y)$ , where  $u$  and  $v$  are real-valued, we know that  $u$  and  $v$  satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x,$$

which, upon differentiation, produce

$$u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0$$

and

$$v_{xx} + v_{yy} = -u_{yx} - u_{xy} = 0,$$

hence both real and imaginary parts of any analytic function are harmonic functions. Conversely, it can be proved that if  $u$  is a harmonic function in a connected domain, then there exists an analytic function  $f$  such that  $u = \operatorname{Re} f$ .

This allows to use a number of complex analysis techniques to solve problems in PDE's. In this section we shall describe a few such cases.

### 6.1 Poisson's formula for a disk

Let  $\Omega = \{(x, y); x^2 + y^2 < a^2\} = \{(r, \phi) \mid 0 < r < a, 0 \leq \phi < 2\pi\}$ . In this subsection we consider the Dirichlet problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad \text{in } \Omega, \\ u(a, \phi) &= h(\phi) \end{aligned} \tag{10.6.1}$$

Our aim is to derive the two-dimensional version of (10.5.6). We could have mimicked the approach of the previous section and construct the Green's function for the disk in the same way as we have found the Green's function for the ball. However, to illustrate the applicability of the complex functions theory to PDE's we shall use an alternative approach based on the Cauchy formula. Let us recall that if  $f$  is an analytic in the disk  $\Omega = \{z \in \mathbb{C}; |z| < a\}$  and continuous on its circumference  $\gamma$ , then for any  $z \in \Omega$  we have the Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} dw \tag{10.6.2}$$

It is known that the real (and imaginary) parts of any analytic function  $f = u + iv$  is a harmonic function so that taking real part of the above formula we could obtain the values of the harmonic function  $u$  inside the circle  $\gamma$  but, due to the presence of a complicated algebraic expression under the sign of the integral, it would be given in terms of the values of both  $u$  and  $v$  on the boundary. Such an approach has a limited applicability but it is possible to modify Cauchy formula in such a way that it involves only real part of the function  $f$ .

To achieve this we modify the function  $f$ : for any given fixed  $z \in \Omega$  we consider the function

$$\Phi(w) = f(w) \frac{a^2 - |z|^2}{a^2 - w\bar{z}},$$

where  $\bar{z}$  denotes the complex conjugate of  $z$ . If  $f$  is analytic in  $\Omega$  and continuous on  $\partial\Omega$ , then  $\Phi$  also enjoys this property (the factor  $(a^2 - |z|^2)/(a^2 - w\bar{z})$  has the singular point if  $|z| = a$  and  $w = z$ , thus on the circle, but this is a removable singularity). Thus the Cauchy formula (10.6.2) is valid for  $\Phi$  and we have

$$\Phi(w) = \frac{a^2 - r^2}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - w)(a^2 - \xi\bar{z})} d\xi, \tag{10.6.3}$$

where we used  $|z| = r$ . In particular, putting  $w = z$  in the formula, we obtain  $\Phi(z) = f(z)$  and passing to polar coordinates with  $\xi = ae^{i\phi}$  and  $z = re^{i\theta}$  we obtain

$$\begin{aligned} f(re^{i\theta}) &= \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(ae^{i\phi})ae^{i\phi}}{(ae^{i\phi} - re^{i\theta})(a^2 - are^{i\phi}e^{-i\theta})} d\phi \\ &= \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(ae^{i\phi})}{(ae^{-i\phi} - re^{-i\theta})(ae^{i\phi} - re^{i\theta})} d\phi \\ &= \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(ae^{i\phi})}{a^2 + r^2 - 2ar \cos(\phi - \theta)} d\phi \end{aligned}$$

Taking harmonic function  $u$  such that  $u = \operatorname{Re} f$  we obtain the *Poisson formula*

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(a, \phi)}{a^2 + r^2 - 2ar \cos(\phi - \theta)} d\phi, \quad (10.6.4)$$

which gives the value of a harmonic function inside the circle in terms of its boundary values, thus solving the Dirichlet problem for the Laplace equation.

## 7 Some methods of solving the Laplace and Poisson equations

In this section we shall describe some methods of finding explicit solutions to the Laplace and Poisson equations.

### 7.1 Spherically symmetric data

The simplest case is when the all the data of the equation are spherically symmetric, that is, they depend only on  $r^2 = \|\mathbf{r}\|^2 = x^2 + y^2 + z^2$  for  $n = 3$ , or  $r^2 = x^2 + y^2$  for  $n = 2$ .

**Example 7.1 Spherically symmetric Poisson equation.** Let  $\Omega = \{\mathbf{r} = (x, y, z) \in \mathbf{R}^3; 0.5 \leq \|\mathbf{r}\| \leq 1\}$ . Find the solution to the problem

$$\begin{aligned} \Delta u &= 12\|\mathbf{r}\|, \quad \text{in } \Omega, \\ u(\mathbf{r})|_{\|\mathbf{r}\|=0.5} &= 1, \\ u(\mathbf{r})|_{\|\mathbf{r}\|=1} &= 3, \end{aligned}$$

Since all the data are spherically symmetric, we shall use the expression for the Laplacian in spherical coordinates without the angle dependent terms (7.1.1) and convert the above problem into the boundary value problem for the second order ordinary differential equation

$$\begin{aligned} \Delta u &= u_{rr} + \frac{2}{r}u_r = 12r, \\ u(0.5) &= 1, \quad u(1) = 3 \end{aligned}$$

Multiplying the differential equation by  $r^2$  and using  $r^2 u_{rr} + 2r u_r = (r^2 u_r)_r$  we rewrite the equation as

$$(r^2 u_r)_r = 12r^3,$$

which upon the first integration produces

$$r^2 u_r = 3r^4 + C_1$$

and next

$$u(r) = r^3 - \frac{C_1}{3r^3} + C_2.$$

Since  $C_1$  is arbitrary, we can write the above equation in a more compact form

$$u(r) = r^3 + \frac{C_1}{r^3} + C_2.$$

Using the boundary conditions we obtain the system of equations for  $C_1$  and  $C_2$

$$\begin{aligned} \frac{1}{8} + 8C_1 + C_2 &= 1, \\ 1 + C_1 + C_2 &= 3, \end{aligned}$$

which has the solution

$$C_1 = -\frac{9}{56}, \quad C_2 = \frac{121}{56}.$$

The solution to the problem is given thus by

$$u(r) = r^3 - \frac{9}{56r^3} + \frac{121}{56}$$

where  $r = x^2 + y^2 + z^2$ .

## 7.2 The Laplace equation in the half-plane

Let us consider the following boundary value problem: find a bounded function  $u$  satisfying

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad -\infty < x < \infty, y > 0, \\ u(x, 0) &= h(x), \end{aligned} \tag{10.7.1}$$

where  $h$  is a given function. To solve this problem we use the Fourier transform method. Let us recall that the Fourier transform is defined by

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx,$$

and that for functions  $f$  satisfying  $f(-\infty) = f'(-\infty) = f(\infty) = f'(\infty) = 0$  we have

$$\mathcal{F}[f_{xx}](\omega) = -\omega^2 \mathcal{F}[f](\omega).$$

Applying the Fourier transform to both equations of (7.1.1) we obtain the initial value problem for the second-order ordinary differential equation with a parameter:

$$\begin{aligned} -\omega^2 \hat{u} + \hat{u}_{yy} &= 0, \quad -\infty < \omega < \infty, y > 0, \\ \hat{u}(\omega, 0) &= \hat{h}(\omega). \end{aligned} \tag{10.7.2}$$

The characteristic equation of the above differential equation is

$$r^2 - \omega^2 = 0$$

which has two solutions:  $r_1 = |\omega|$  and  $r_2 = -|\omega|$  (the absolute values has been introduced here to keep the sign of the roots independent of the sign of  $\omega$ ). Consequently,

$$\hat{u}(\omega, y) = C_1(\omega) e^{-|\omega|y} + C_2(\omega) e^{|\omega|y}.$$

Since we are looking for solutions which are bounded, and  $e^{|\omega|y}$  tends to infinity as  $y \rightarrow \infty$ , we infer that  $C_2(\omega) = 0$ . Using the initial value we obtain finally

$$\hat{u}(\omega, y) = \hat{h}(\omega)e^{-|\omega|y}.$$

To get  $u$  we apply the inverse Fourier transform and the convolution theorem

$$\mathcal{F}^{-1}[\hat{f} \cdot \hat{g}] = \frac{1}{\sqrt{2\pi}} f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)g(y-x)dx.$$

Therefore

$$u(x, y) = \frac{1}{\sqrt{2\pi}} h * \mathcal{F}^{-1}[e^{-|\omega|y}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(\lambda) \mathcal{F}^{-1}[e^{-|\omega|y}](x-\lambda) d\lambda,$$

where the inverse Fourier transform is evaluated with respect to  $\omega$ . Since

$$\begin{aligned} \mathcal{F}^{-1}[e^{-|\omega|y}](x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\omega|y} e^{i\omega x} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \left( \int_0^{\infty} e^{-\omega y} e^{i\omega x} d\omega + \int_{-\infty}^0 e^{\omega y} e^{i\omega x} d\omega \right) \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\omega y} (e^{i\omega x} + e^{-i\omega x}) d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\omega y} \cos \omega x d\omega = \sqrt{\frac{2}{\pi}} \frac{y}{x^2 + y^2}, \end{aligned}$$

we obtain

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(\lambda)}{y^2 + (\lambda - x)^2} d\lambda. \quad (10.7.3)$$

We shall illustrate the application of this formula solving one easy and one more difficult problem.

**Example 7.2** Solve the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & -\infty < x < \infty, y > 0, \\ u(x, 0) &= h(x), \end{aligned}$$

where  $h(x) = 1$  for  $-1 < x < 1$  and  $h(x) = 0$  elsewhere.

Using the formula (7.1.3) we obtain

$$\begin{aligned} u(x, y) &= \frac{y}{\pi} \int_{-1}^1 \frac{d\lambda}{y^2 + (\lambda - x)^2} \\ &= \frac{1}{\pi} \int_{\frac{-1-x}{y}}^{\frac{1-x}{y}} \frac{dz}{1 + z^2} \\ &= \frac{1}{\pi} \left( \tan^{-1} \frac{1-x}{y} + \tan^{-1} \frac{1+x}{y} \right) \end{aligned}$$

where we used the change of variables  $z = (\lambda - x)/y$  and the fact that the function  $\tan^{-1}$  is odd.

Using some geometry we can write the above formula in a more compact form. In fact, denoting  $\alpha = \tan^{-1} \frac{1-x}{y}$  and  $\beta = \tan^{-1} \frac{1+x}{y}$  we observe that  $\alpha + \beta$  is always the interior angle at the vertex  $(x, y)$  of the triangle with vertices  $(-1, 0), (x, y), (0, 1)$ . Thus  $0 < \alpha + \beta < \pi$ . Also  $-\pi/2 < \alpha, \beta < \pi/2$  (from the definition of  $\tan^{-1}$ ). Hence one can use the trigonometric formula

$$\cot(\alpha + \beta) = \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}$$

getting

$$\cot \left( \tan^{-1} \frac{1-x}{y} + \tan^{-1} \frac{1+x}{y} \right) = \frac{1 - \frac{(1+x)(1-x)}{y^2}}{\frac{1+x}{y} + \frac{1-x}{y}} = \frac{x^2 + y^2 - 1}{2y}$$

and finally

$$u(x, y) = \frac{1}{\pi} \cot^{-1} \frac{x^2 + y^2 - 1}{2y}.$$

**Example 7.3** Solve the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & -\infty < x < \infty, y > 0, \\ u(x, 0) &= \frac{1}{1+x^2}, \end{aligned}$$

Using the formula (7.1.3) we obtain

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{(1+\lambda)^2(y^2 + (\lambda - x)^2)}. \quad (10.7.4)$$

To evaluate this integral we use the residue method from complex analysis. Factorizing the denominator we obtain

$$(\lambda + i)(\lambda - i)(\lambda - x + iy)(\lambda - x - iy),$$

so that in the upper half-plane we have two simple poles at  $\lambda = i$  and  $\lambda = x + iy$ . Evaluating residues at these points we obtain

$$Res_{\lambda=i} = \frac{1}{2i((i-x)^2 + y^2)}$$

and

$$Res_{\lambda=x+iy} = \frac{1}{2iy(1 + (x + iy)^2)}.$$

From the residue theorem we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\lambda}{(1+\lambda)^2(y^2 + (\lambda - x)^2)} &= 2\pi i (Res_{\lambda=x+iy} + Res_{\lambda=i}) \\ &= \frac{\pi}{y(1 + (x + iy)^2)} + \frac{\pi}{((i-x)^2 + y^2)} = \frac{\pi}{y(1 + x^2 + 2ixy - y^2)} + \frac{\pi}{y^2 + x^2 + 2ix - 1} \\ &= \frac{\pi(1 + x^2 - y^2 - 2ixy)}{y((1 + x^2 - y^2)^2 + 4x^2y^2)} + \frac{\pi(y^2 + x^2 - 1 + 2ix)}{(y^2 + x^2 - 1)^2 + 4x^2}. \end{aligned}$$

It follows that

$$(1 + x^2 - y^2)^2 + 4x^2y^2 = (y^2 + x^2 - 1) + 4x^2 = (y^2 + x^2 + 1)^2 - 4y^2,$$

thus the last sum can be written as

$$\begin{aligned} \frac{\pi(1 + x^2 - y^2 - 2ixy) + y(y^2 + x^2 - 1 + 2ix)}{y((y^2 + x^2 + 1)^2 - 4y^2)} &= \frac{\pi(1 + x^2 - y^2) + y(y^2 + x^2 - 1)}{y(y^2 + x^2 + 1 - 2y)(y^2 + x^2 + 1 + 2y)} \\ &= \frac{\pi(1 + y)(y^2 + x^2 + 1 - 2y)}{y(y^2 + x^2 + 1 - 2y)(y^2 + x^2 + 1 + 2y)} = \frac{\pi(1 + y)}{y(y^2 + x^2 + 1 + 2y)} \end{aligned}$$

Inserting the above into (7.1.4) we obtain finally the solution

$$u(x, y) = \frac{1 + y}{y^2 + x^2 + 1 + 2y}.$$

### 7.3 Laplace and Poisson equations in regions with polar symmetry

In this subsection we shall show a variant of the method of separation of variables.

Throughout this subsection we denote by  $\Omega$  one of the following sets:

- (i)  $\Omega = \{(x, y); x^2 + y^2 < a^2\} = \{(r, \phi); r < b, 0 \leq \phi < 2\pi\}$  (the interior of a disc),
- (ii)  $\Omega = \{(x, y); a^2 < x^2 + y^2 < b^2\} = \{(r, \phi); a < r < b, 0 \leq \phi < 2\pi\}$  (an annulus),
- (iii)  $\Omega = \{(x, y); b^2 < x^2 + y^2\} = \{(r, \phi); b < r, 0 \leq \phi < 2\pi\}$  (the exterior of the disc).

Let us consider the Laplace equation in  $\Omega$ :

$$\Delta u = 0, \quad \text{in } \Omega, \quad (10.7.5)$$

with one of the following boundary conditions:

(I)

In the case (i):

$$u(a, \phi) = g(\phi), \quad u(r, \phi) \text{ bounded as } r \rightarrow 0^+, \quad (10.7.6)$$

(II)

In the case (ii):

$$\begin{aligned} u(a, \phi) &= g_a(\phi), \\ u(b, \phi) &= g_b(\phi), \end{aligned} \quad (10.7.7)$$

(III)

In the case (iii):

$$u(b, \phi) = g(\phi), \quad u(r, \phi) \text{ bounded as } r \rightarrow +\infty. \quad (10.7.8)$$

Solutions to (I)-(III) can be obtained as particular cases of the general solution to (11.2.18).

Using the Chain Rule we can write the Laplace equation in polar coordinates as

$$r^{-1}(ru_r)_r + r^2 u_{\phi\phi} = 0. \quad (10.7.9)$$

To apply the separation of variables technique, we look for particular solutions in the form

$$u(r, \phi) = R(r)\Phi(\phi). \quad (10.7.10)$$

Inserting (11.2.23) into the equation (11.2.22) we obtain

$$\frac{r(rR_r)_r}{R} + \frac{\Phi_{\phi\phi}}{\Phi} = 0,$$

which yields that both terms are constant, and leads to two ordinary differential equations

$$\Phi'' + \lambda\Phi = 0, \quad \text{for } 0 \leq \phi < 2\pi, \quad (10.7.11)$$

and

$$r(rR')' - \lambda R = 0, \quad \text{for } r > 0. \quad (10.7.12)$$

Let us consider first Eq. (11.2.24). It is not immediate what kind of boundary conditions one should assign to this equation. To find the proper ones we note that the restriction  $0 \leq \phi < 2\pi$  is artificial as geometrically the point  $(r, \phi)$  coincides with the point  $(r, \phi + 2\pi)$ , thus  $u(r, \phi)$  must be equal to  $u(r, \phi + 2\pi)$ , for any  $\phi$ . This observation shows that  $\Phi$  must be periodic with period  $2\pi$ , that is, we require

$$\Phi(\phi) = \Phi(\phi + 2\pi), \quad \text{for } 0 \leq \phi < 2\pi. \quad (10.7.13)$$

Next let us note that (11.2.26) yields

$$\Phi'(\phi) = \Phi'(\phi + 2\pi), \quad \text{for } 0 \leq \phi < 2\pi,$$

and therefore we shall consider the following problem

$$\begin{aligned} \Phi'' + \lambda\Phi &= 0, \quad \text{for } 0 \leq \phi < 2\pi, \\ \Phi(0) &= \Phi(2\pi) \\ \Phi'(0) &= \Phi'(2\pi). \end{aligned} \quad (10.7.14)$$

The general solution of (11.2.24) is given by

$$u(\phi) = C_1 e^{\sqrt{-\lambda}\phi} + C_2 e^{-\sqrt{-\lambda}\phi}$$

and the boundary conditions produce the system of equations

$$\begin{aligned} C_1 + C_2 &= C_1 e^{\sqrt{-\lambda}2\pi} + C_2 e^{-\sqrt{-\lambda}2\pi} \\ C_1 \sqrt{-\lambda} - C_2 \sqrt{-\lambda} &= \sqrt{-\lambda} C_1 e^{\sqrt{-\lambda}2\pi} - \sqrt{-\lambda} C_2 e^{-\sqrt{-\lambda}2\pi}. \end{aligned}$$

Assuming for a moment that  $\lambda \neq 0$ , we can divide the second equation by  $\sqrt{-\lambda}$  so that we get the following system

$$\begin{aligned} C_1(1 - e^{\sqrt{-\lambda}2\pi}) + C_2(1 - e^{-\sqrt{-\lambda}2\pi}) &= 0 \\ C_1(1 - e^{\sqrt{-\lambda}2\pi}) - C_2(1 - e^{-\sqrt{-\lambda}2\pi}) &= 0. \end{aligned}$$

The determinant of this system is given by

$$-2(1 - e^{\sqrt{-\lambda}2\pi})(1 - e^{-\sqrt{-\lambda}2\pi})$$

which is zero only if either

$$1 - e^{\sqrt{-\lambda}2\pi} = 0$$

or

$$1 - e^{-\sqrt{-\lambda}2\pi} = 0.$$

From these equations we obtain

$$e^{\pm\sqrt{-\lambda}2\pi} = 1 = e^{i2n\pi}$$

which yields

$$\lambda = n^2$$

where  $n$  is a nonzero integer. Consequently, we obtain possible solutions in the form

$$\tilde{\Phi}_n(\phi) = C_{1n} e^{in\phi} + C_{2n} e^{-in\phi}. \quad (10.7.15)$$

These functions can be written in the real form yielding

$$\{\sin n\phi, \cos n\phi\}_{n=\pm 1, \pm 2, \dots}$$

as the set of eigenfunctions.

Consider now the exceptional case  $\lambda = 0$ . Then the equation (11.2.24) has the solution  $\Phi(\phi) = C_0\phi + C'_0$  which is periodic only when  $C_0 = 0$ . Thus  $\Phi_0$  is constant and is included in the general formula if we allow  $n = 0$ .

Having determined the eigenvalues and eigenfunctions for the angle dependent part of the equation, we can now tackle the equation for  $R$ :

$$r^2 R'' + rR' - n^2 R = 0. \quad (10.7.16)$$

This is so-called Euler equation which can be solved by using trial solutions of the form  $R_\alpha = r^\alpha$ . Inserting such a function into the equation (11.2.29), we obtain

$$r^\alpha(\alpha(\alpha - 1) + \alpha - n^2) = 0$$

which is satisfied if and only if  $\alpha$  is a root of the following quadratic equation

$$\alpha(\alpha - 1) + \alpha - n^2 = 0$$

or simply

$$\alpha^2 = n^2.$$

So, if  $n \neq 0$ , we obtain two distinct solutions

$$\alpha_1 = n, \quad \alpha_2 = -n,$$

where  $n \in \mathbb{N}$ , and consequently

$$R_n(r) = a_n r^n + b_n r^{-n}.$$

If  $n = 0$ , this procedure produces only one solution  $R_0(r) = a_0$ , but then the equation can be integrated directly to give

$$R_0(r) = a_0 + b_0 \ln r.$$

Thus the general solution of the Laplace equation in any region with polar symmetry is given by

$$u(r, \phi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\phi + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \sin n\phi. \quad (10.7.17)$$

Now, let us specify the form of the solution so that it can be the solution of the boundary value problems (I)-(III).

Case (I)

In the case (I) we require the solution to be bounded as  $r \rightarrow 0^+$ , which forces  $B_0 = B_n = D_n = 0$  for all  $n \geq 1$ . Thus we obtain

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi. \quad (10.7.18)$$

The constants  $A_n$  and  $C_n$  are to be determined from the boundary condition, that is, we must have

$$g(\phi) = u(a, \phi) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos n\phi + \sum_{n=1}^{\infty} C_n a^n \sin n\phi.$$

Therefore

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi, \\ A_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \cos n\phi d\phi, \\ C_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \sin n\phi d\phi, \end{aligned} \quad (10.7.19)$$



for  $n > 0$ , where we used the orthogonality of all trigonometric functions on  $[0, 2\pi]$  and the normalizing relation

$$\int_0^{2\pi} \cos^2 n\phi d\phi = \int_0^{2\pi} \sin^2 n\phi d\phi = \pi.$$

Case (II)

Case (II) requires the full expansion (11.2.30). The coefficients are to be determined from the infinite system of equations:

$$\begin{aligned} A_0 + B_0 \ln a &= \frac{1}{2\pi} \int_0^{2\pi} g_a(\phi) d\phi, \\ A_0 + B_0 \ln b &= \frac{1}{2\pi} \int_0^{2\pi} g_b(\phi) d\phi, \end{aligned} \quad (10.7.20)$$

and for  $n \geq 1$ ,

$$\begin{aligned} A_n a^n + B_n a^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_a(\phi) \cos n\phi d\phi, \\ A_n b^n + B_n b^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_b(\phi) \cos n\phi d\phi, \end{aligned} \quad (10.7.21)$$

and

$$\begin{aligned} C_n a^n + D_n a^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_a(\phi) \sin n\phi d\phi, \\ C_n b^n + D_n b^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_b(\phi) \sin n\phi d\phi. \end{aligned} \quad (10.7.22)$$

Note that these equations are solvable as the determinant of each system is equal to  $(a/b)^n - (b/a)^n \neq 0$  for  $n \geq 1$  and  $\ln(b/a) \neq 0$  for  $n = 0$ .

Case (III)

Case (III) is similar to the Case (I) with the difference being that the requirement of boundedness of  $u(r, \phi)$  as  $r \rightarrow \infty$  forces  $B_0 = A_n = C_n$  for all  $n \geq 0$ . The solution is then given by

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} B_n r^{-n} \cos n\phi + \sum_{n=1}^{\infty} D_n r^{-n} \sin n\phi. \quad (10.7.23)$$

The constants  $A_n$  and  $C_n$  are defined by

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi, \\ B_n &= \frac{b^n}{\pi} \int_0^{2\pi} g(\phi) \cos n\phi d\phi, \\ D_n &= \frac{b^n}{\pi} \int_0^{2\pi} g(\phi) \sin n\phi d\phi, \end{aligned} \quad (10.7.24)$$

for  $n > 1$ .

**Example 7.4** Find the solution to:

$$\begin{aligned}\Delta u &= 0, & 1 < r < 2, 0 \leq \phi < 2\pi, \\ u(1, \phi) &= \cos^2 \phi + 1, \\ u(2, \phi) &= 1 + \sin \phi.\end{aligned}$$

To solve the problem we first write  $\cos^2 \phi = \frac{1}{2} + \frac{\cos 2\phi}{2}$ . With this change we use (11.2.30)

$$u(r, \phi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\phi + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \sin n\phi.$$

There is, however, no need to use the integral formulas for coefficients of the expansion, as our data are already given in the form of (finite) Fourier expansion. By comparison we have

$$\frac{3}{2} + \frac{\cos 2\phi}{2} = u(1, \phi) = A_0 + \sum_{n=1}^{\infty} (A_n + B_n) \cos n\phi + \sum_{n=1}^{\infty} (C_n + D_n) \sin n\phi,$$

where we used  $\ln 1 = 0$ . Similarly

$$1 + \sin \phi = u(2, \phi) = A_0 + B_0 \ln 2 + \sum_{n=1}^{\infty} (A_n 2^n + B_n 2^{-n}) \cos n\phi + \sum_{n=1}^{\infty} (C_n 2^n + D_n 2^{-n}) \sin n\phi.$$

By the uniqueness of the Fourier expansion we get the following equations

$$\begin{aligned}A_0 &= \frac{3}{2}, \\ A_0 + B_0 \ln 2 &= 1, \\ C_1 + D_1 &= 0, \\ 2C_1 + 2^{-1}D_1 &= 1, \\ A_2 + B_2 &= \frac{1}{2} \\ 2^2 A_2 + 2^{-2}B_2 &= 0.\end{aligned}$$

All the other coefficients, being the solutions of homogeneous linear systems with non-zero determinants, are zero. Solving the above system, we obtain

$$A_0 = \frac{3}{2}, B_0 = -\frac{1}{2 \ln 2}, C_1 = \frac{2}{3}D_1 = -\frac{2}{3}, A_2 = -\frac{1}{30}, B_2 = \frac{8}{15},$$

and thus the solution is given by

$$u(r, \phi) = \frac{3}{2} - \frac{1}{2 \ln 2} \ln r + \frac{2}{3}(r + r^{-1}) \sin \phi + \left( -\frac{1}{30}r^2 + \frac{8}{15}r^{-2} \right) \cos 2\phi.$$

The next example is a little more sophisticated and requires some complex analysis.

**Example 7.5** Find the solution to:

$$\begin{aligned}\Delta u &= 0, & r < 1, 0 \leq \phi < 2\pi, \\ u(1, \phi) &= g(\phi),\end{aligned}$$

where

$$g(\phi) = \begin{cases} 1 & \text{for } 0 \leq \phi \leq \pi \\ 0 & \text{for } \pi < \phi < 2\pi. \end{cases}$$

According to the formula (11.2.31) the solution is given by

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi,$$

where in our case

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{\pi} d\phi = \frac{1}{2}, \\ A_n &= \frac{1}{\pi} \int_0^{\pi} \cos n\phi d\phi = \frac{1}{n\pi} (\sin n\pi - \sin 0) = 0, \\ C_n &= \frac{1}{\pi} \int_0^{\pi} \sin n\phi d\phi = -\frac{1}{n\pi} (\cos n\pi - \cos 0) = -\frac{1}{n\pi} ((-1)^n - 1). \end{aligned}$$

The coefficient  $C_n$  can be written in the following way

$$C_n = \begin{cases} 0 & \text{for } n = 2k \\ \frac{2}{(2k+1)\pi} & \text{for } n = 2k+1, \end{cases}$$

where  $k = 0, 1, \dots$  and the solution takes the form

$$u(r, \phi) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \sin(2k+1)\phi. \quad (10.7.25)$$

An interesting thing about the series above is that it can actually be summed. In fact

$$\sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \sin(2k+1)\phi = \operatorname{Im} \sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} (\cos(2k+1)\phi + i \sin(2k+1)\phi) = \operatorname{Im} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}.$$

The last series is quite well-known. In fact, since

$$\begin{aligned} \ln(1-z) &= -\sum_{k=1}^{\infty} \frac{z^k}{k}, \\ \ln(1+z) &= -\sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k}, \end{aligned}$$

we have

$$\ln \frac{1+z}{1-z} = \ln(1+z) - \ln(1-z) = 2 \sum_{k=1}^{\infty} \frac{z^{2k+1}}{2k+1}.$$

We have to calculate  $\operatorname{Im} \ln \frac{1+z}{1-z}$ . To do this, we recall that for complex logarithm we have the formula  $\ln z = \ln|z| + i \arg z$ , so that we have to find  $\arg \frac{1+z}{1-z}$ . For  $z = re^{i\phi} = r(\cos \phi + i \sin \phi)$  we have

$$\frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z})} = \frac{1+2ir \sin \phi - r^2}{1-2r \cos \phi + r^2}$$

and since the denominator is a real number, we have

$$\arg \frac{1+z}{1-z} = \arg(1+2ir \sin \phi - r^2).$$

Now,  $1-r^2 > 0$ , so that  $-\pi/2 < \arg(1+2ir \sin \phi - r^2) < \pi/2$  and we can use  $\tan^{-1}$  to express it analytically as

$$\arg \frac{1+z}{1-z} = \tan^{-1} \frac{r \sin \phi}{1-r^2}.$$

Returning to the solution of our problem we obtain

$$u(r, \phi) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{2r \sin \phi}{1-r^2}.$$

The above example suggests that it may be possible to sum the whole series (11.2.30). To simplify the matters we shall concentrate on the solution for the interior of the disc

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi, \quad (10.7.26)$$

where the constants  $A_n$  and  $C_n$ ,  $n \geq 0$  are given by

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi, \\ A_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \cos n\phi d\phi, \\ C_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \sin n\phi d\phi. \end{aligned} \quad (10.7.27)$$

Inserting the coefficient formulae into (7.1.26) we obtain

$$\begin{aligned} u(r, \phi) &= \frac{1}{2\pi} \int_0^{2\pi} g(\psi) d\psi \\ &\quad + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} g(\psi) \frac{r^n}{a^n} (\cos n\phi \cos n\psi + \sin n\phi \sin n\psi) d\psi \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\psi) \left( 1 + 2 \sum_{n=1}^{\infty} \frac{r^n}{a^n} \cos n(\phi - \psi) \right) d\psi. \end{aligned}$$

Using Euler's formula we obtain

$$\frac{r^n}{a^n} \cos n(\phi - \psi) = \operatorname{Re} \left( \frac{r}{a} e^{i(\phi - \psi)} \right)^n$$

so that denoting  $z = \frac{r}{a} e^{i(\phi - \psi)}$  we see that

$$\begin{aligned} 1 + 2 \sum_{n=1}^{\infty} \frac{r^n}{a^n} \cos n(\phi - \psi) &= 1 + 2 \operatorname{Re} \sum_{n=1}^{\infty} z^n = 1 + 2 \operatorname{Re} \frac{z}{1 - z} \\ &= 1 + 2 \operatorname{Re} \frac{z - |z|^2}{1 - (z + \bar{z}) + |z|^2} = 1 + 2 \frac{r \cos(\phi - \psi) + r^2}{a^2 + r^2 - 2r \cos(\phi - \psi)} \\ &= \frac{a^2 - r^2}{a^2 + r^2 - 2r \cos(\phi - \psi)}. \end{aligned}$$

Thus we obtained the Poisson formula for the solution of the Laplace equation in the disc

$$u(r, \phi) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{g(\psi)}{a^2 + r^2 - 2r \cos(\phi - \psi)} d\psi. \quad (10.7.28)$$

This formula is very important from the theoretical point of view. Below we shall see how it works in practice. To compare it with the Fourier expansion method we shall solve the same problem as in Example 1.5.

**Example 7.6** Find the solution to:

$$\begin{aligned}\Delta u &= 0, \quad r < 1, 0 \leq \phi < 2\pi, \\ u(1, \phi) &= g(\phi),\end{aligned}$$

where

$$g(\phi) = \begin{cases} 1 & \text{for } 0 \leq \phi \leq \pi \\ 0 & \text{for } \pi < \phi < 2\pi. \end{cases}$$

According to (7.1.28) we have

$$\begin{aligned}u(r, \phi) &= \frac{1-r^2}{2\pi} \int_0^\pi \frac{d\psi}{1+r^2-2r\cos(\psi-\phi)} = \frac{1-r^2}{2\pi(1+r^2)} \int_0^\pi \frac{d\psi}{1-\alpha\cos(\psi-\phi)} \\ &= \frac{1-r^2}{2\pi(1+r^2)} \int_{-\phi}^{\pi-\phi} \frac{d\xi}{1-\alpha\cos\xi},\end{aligned}$$

where we denoted  $\alpha = 2r/(1+r^2)$  and  $\xi = \psi - \phi$ . Since the possible intervals of integration stretch from  $-2\pi$  to  $\pi$ , the direct substitution  $t = \tan \xi/2$  is not feasible as it is applicable only on  $-\pi < \xi < \pi$ . Thus we shall split the integral as follows

$$\int_{-\phi}^{\pi-\phi} = \int_0^\pi + \int_{-\phi}^0 - \int_{\pi-\phi}^\pi$$

Next we use

$$\int_{-\phi}^0 \frac{d\xi}{1-\alpha\cos\xi} = \int_0^\phi \frac{d\xi}{1-\alpha\cos\xi},$$

since  $\cos$  is an even function, and

$$\int_{\pi-\phi}^\pi \frac{d\xi}{1-\alpha\cos\xi} d\psi = \int_0^\phi \frac{d\xi}{1+\alpha\cos\xi} d\psi.$$

Thus

$$\begin{aligned}\int_{-\phi}^{\pi-\phi} \frac{d\xi}{1-\alpha\cos\xi} &= \int_0^\pi \frac{d\xi}{1-\alpha\cos\xi} + \int_0^\phi \frac{d\xi}{1-\alpha\cos\xi} - \int_0^\pi \frac{d\xi}{1+\alpha\cos\xi} \\ &= \int_0^\pi \frac{d\xi}{1-\alpha\cos\xi} + 2\alpha \int_0^\phi \frac{\cos\xi d\xi}{1-\alpha^2\cos^2\xi}.\end{aligned}$$

To evaluate the first integral we can use the substitution  $t = \tan \xi/2$ ; then we have  $d\xi = 2dt/(1+t^2)$  and  $\cos \xi = (1-t^2)/(1+t^2)$ , which yields

$$\begin{aligned}\int_0^\pi \frac{d\xi}{1-\alpha\cos\xi} &= 2 \int_0^\infty \frac{dt}{1-\alpha+(1+\alpha)t^2} = 2 \frac{1}{\sqrt{1-\alpha^2}} \tan^{-1} \left( t \sqrt{\frac{1+\alpha}{1-\alpha}} \right) \Big|_0^\infty \\ &= \frac{\pi}{\sqrt{1-\alpha^2}}.\end{aligned}$$

The second integral is evaluated as follows

$$\begin{aligned} \int_0^\phi \frac{\cos \xi d\xi}{1 - \alpha^2 \cos^2 \xi} &= \int_0^\phi \frac{\cos \xi d\xi}{1 - \alpha^2 + \alpha^2 \sin^2 \xi} \\ &= \frac{1}{\alpha \sqrt{1 - \alpha^2}} \int_0^{\frac{\alpha}{\sqrt{1 - \alpha^2}} \sin \phi} \frac{dt}{1 + t^2} = \frac{1}{\alpha \sqrt{1 - \alpha^2}} \tan^{-1} \frac{\alpha}{\sqrt{1 - \alpha^2}} \sin \phi, \end{aligned}$$

where we used  $t = \frac{\alpha}{\sqrt{1 - \alpha^2}} \sin \xi$ .

Returning to the old notation we find that

$$\frac{1}{\sqrt{1 - \alpha^2}} = \frac{1 + r^2}{1 - r^2},$$

and

$$\frac{\alpha}{\sqrt{1 - \alpha^2}} = \frac{2r}{1 - r^2},$$

thus combining all the partial results we obtain, as before, the solution in the form

$$u(r, \phi) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{2r}{1 - r^2} \sin \phi.$$

**Example 7.7 Poisson equation in a disc.** Let us start with a general comment concerning the solvability of the Poisson equation which is a nonhomogeneous Laplace equation. Suppose that we are to solve the problem

$$\begin{aligned} \Delta u &= f, & \text{in } \Omega, \\ u &= g, & \text{on } \partial\Omega. \end{aligned} \tag{10.7.29}$$

In many cases it is possible to guess easily a particular solution to the Poisson equation in the whole space, that is, to find a function  $v$  which satisfies

$$\Delta v = f, \quad \text{in } \mathbb{R}^n.$$

Then, if  $u$  is the solution to (7.1.29) (as yet unknown), then the function  $w = u - v$  satisfies

$$\Delta w = \Delta u - \Delta v = f - f = 0, \quad \text{in } \Omega.$$

and

$$w = g - v|_{\partial\Omega}$$

on  $\partial\Omega$ . Thus the Dirichlet problem for the Poisson equation (7.1.29) has been reduced to the Dirichlet problem for the Laplace equation

$$\begin{aligned} \Delta w &= 0, & \text{in } \Omega, \\ w &= g - v|_{\partial\Omega}, & \text{on } \partial\Omega, \end{aligned} \tag{10.7.30}$$

which, for example in a disc, we can solve.

We shall illustrate this method by solving the problem

$$\begin{aligned} \Delta u &= -xy, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ . It is easy to check that  $v = -\frac{1}{12}(x^3y + y^3x)$  is a particular solution to the Poisson equation in the whole space. Since we are working in the unit disc, it is convenient to convert all the function to polar coordinates. We have

$$v = -\frac{1}{12}(x^3y + y^3x) = -\frac{1}{12}xy(x^2 + y^2) = -\frac{r^4}{12} \sin \phi \cos \phi = -\frac{r^4}{24} \sin 2\phi.$$

Following our general strategy we define

$$w = u + \frac{r^4}{24} \sin 2\phi,$$

so that  $w$  satisfies

$$\begin{aligned} \Delta w &= 0, & \text{in } \Omega, \\ w &= \frac{1}{24} \sin 2\phi, & \text{on } \partial\Omega, \end{aligned}$$

Using the general formula

$$w(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi,$$

we see that

$$\frac{1}{24} \sin 2\phi = w(1, \phi) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\phi + \sum_{n=1}^{\infty} C_n \sin n\phi,$$

and from the uniqueness of Fourier expansion we infer that  $A_n = 0$  for all  $n$  and  $C_n = 0$  for all  $n \neq 2$ . Thus

$$w(r, \phi) = \frac{r^2}{24} \sin 2\phi,$$

and

$$u(r, \phi) = w(r, \phi) + v(r, \phi) = \frac{r^2}{24} \sin 2\phi - \frac{r^4}{24} \sin 2\phi.$$

**Example 7.8 Boundary value problem for the Laplace equation in a rectangle.** Let us consider now the case when  $\Omega$  is a rectangle, say,  $\Omega = \{(x, y); 0 < x < a, 0 < y < b\}$ . In this case the explicit formulation of the problem reads

$$\begin{aligned} \Delta u &= 0, & \text{for } 0 < x < a, 0 < y < b \\ u(0, y) &= h_1(y), & \text{for } 0 < y < b, \\ u(a, y) &= h_2(y), & \text{for } 0 < y < b, \\ u(x, 0) &= h_3(y), & \text{for } 0 < x < a, \\ u(x, b) &= h_4(y), & \text{for } 0 < x < a, \end{aligned} \tag{10.7.31}$$

To solve this problem we split it into four simpler problems: by  $u_i$  we shall denote the solution of the Laplace equation in  $\Omega$  satisfying the boundary condition  $u_i = h_i$  on the appropriate part of the boundary and homogeneous boundary conditions on the three remaining parts. If we manage to solve all four problems, then the solution to the full problem (7.1.31) is given by  $u = u_1 + u_2 + u_3 + u_4$ .

Let us consider now one of these four partial problems and solve, say,

$$\begin{aligned} \Delta u &= 0, & \text{for } 0 < x < a, 0 < y < b \\ u(0, y) &= y, & \text{for } 0 < y < b, \\ u(a, y) &= 0, & \text{for } 0 < y < b, \\ u(x, 0) &= 0, & \text{for } 0 < x < a, \\ u(x, b) &= 0, & \text{for } 0 < x < a, \end{aligned}$$

We start with separation of the variables. Inserting  $u(x, y) = X(x)Y(y)$  into the Laplace equation we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

so that there is a constant  $\lambda$  such that

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

Next we observe that  $Y$  must satisfy homogeneous boundary conditions  $Y(0) = Y(b) = 0$ , thus we start with the equation for  $Y$ . For the sake of completeness we shall repeat the derivation of the solution to this problem.

The general solution to the equation is given by

$$Y(y) = \tilde{C}_1 e^{\sqrt{\lambda}y} + \tilde{C}_2 e^{-\sqrt{\lambda}y},$$

hence using the boundary conditions we obtain

$$\begin{aligned} \tilde{C}_1 + \tilde{C}_2 &= 0 \\ \tilde{C}_1 e^{b\sqrt{\lambda}} + \tilde{C}_2 e^{-b\sqrt{\lambda}} &= 0. \end{aligned}$$

For a homogeneous system of linear algebraic equations to have a non-zero solution it is necessary and sufficient that

$$e^{-b\sqrt{\lambda}} - e^{b\sqrt{\lambda}} = 0,$$

which gives

$$e^{2b\sqrt{\lambda}} = 1 = e^{i2n\pi}$$

for  $n \in \mathbb{Z}$ . Thus we obtain the sequence of eigenvalues

$$\lambda_n = -\frac{n^2\pi^2}{b^2}; \quad (10.7.32)$$

consequently we have

$$Y(y) = \tilde{C}_1 e^{in\pi y/b} - \tilde{C}_1 e^{-in\pi y/b} = 2i\tilde{C}_1 \sin \frac{n\pi y}{b},$$

and the sequence of real-valued eigenfunctions is given by

$$\tilde{Y}_n(y) = \sin \frac{n\pi y}{b}$$

We obtain

$$\int_0^b \sin \frac{n\pi y}{b} \sin \frac{k\pi y}{b} dy = 0$$

for  $n \neq k$ . On the other hand, for  $n = k$  we have

$$\left\| \sin \frac{n\pi y}{b} \right\|^2 = \int_0^b \sin^2 \frac{n\pi y}{b} dy = \frac{1}{2} \int_0^b \left( 1 - \cos \frac{2n\pi y}{b} \right) dy = \frac{b}{2}. \quad (10.7.33)$$

Next we shall deal with the equation  $X'' + \lambda X$ . Since  $\lambda_n = -n^2\pi^2/b^2$ , we obtain the equation

$$X'' - \frac{n^2\pi^2}{b^2} X = 0,$$

which has the general solution

$$X_n(x) = A'_n e^{xn\pi/b} + B'_n e^{-xn\pi/b} = A_n \cosh \frac{xn\pi}{b} + B_n \sinh \frac{xn\pi}{b}.$$



Using the homogeneous boundary condition at  $x = a$  we obtain

$$X_n(0) = A_n \cosh \frac{an\pi}{b} + B_n \sinh \frac{an\pi}{b} = 0,$$

which leads to

$$\frac{A_n}{B_n} = -\tanh \frac{an\pi}{b}.$$

Thus

$$X_n(x) = B_n \left( -\tanh \frac{an\pi}{b} \cosh \frac{xn\pi}{b} + \sinh \frac{xn\pi}{b} \right) = B_n \frac{\sinh n\pi(a+x)/b}{\cosh an\pi/b}.$$

The solution satisfying homogeneous boundary conditions on three sides of the rectangle is thus given by

$$u(x, y) = \sum_{n=1}^{\infty} B_n \frac{\sinh n\pi(a+x)/b}{\cosh an\pi/b} \sin \frac{n\pi y}{b}.$$

The last boundary condition will be satisfied if we find the constants  $B_n$  such that

$$u(0, y) = y = \sum_{n=1}^{\infty} B_n \tanh \frac{an\pi}{b} \sin \frac{n\pi y}{b}. \quad (10.7.34)$$

This is simply the Fourier expansion of  $y$  on  $[0, b]$ . Since

$$\int_0^b y \sin \frac{n\pi y}{b} dy = (-1)^n \frac{b^2}{n\pi},$$

multiplying both sides of (7.1.34) and integrating over  $[0, b]$  we obtain

$$\frac{(-1)^n b^2}{n\pi} = B_n \tanh \frac{an\pi}{b} \frac{b}{2},$$

where we used (7.1.33). Hence

$$B_n = \frac{(-1)^n 2b}{n\pi} \coth \frac{an\pi}{b}$$

and finally the solution is given by

$$u(x, y) = \frac{2b}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\sinh n\pi(a+x)/b}{\sinh an\pi/b} \sin \frac{n\pi y}{b}.$$



# Lecture 7

## The Laplace equation

In this lecture we shall discuss the some methods of solving the boundary value problem for the last of the basic equations of classical mathematical physics -the Poisson equation:

$$\begin{aligned}\Delta u &= f \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u &= \phi, \quad \text{on } \partial\Omega,\end{aligned}\tag{7.0.1}$$

where  $\Delta u = u_{xx} + u_{yy}$  if  $n = 2$  and  $\Delta u = u_{xx} + u_{yy} + u_{zz}$  if  $n = 3$ .

Of special interest are solutions to (7.0.1) with  $f = 0$ , that is, the solutions to the Laplace equation. Any solution to the Laplace equation is called a *harmonic function*. Let us recall that both real and imaginary parts of any analytic function  $f(z)$ , where  $z = x + iy \in \mathbb{C}$  are harmonic functions of two variables  $x, y$ .

### 1 Some methods of solving the Laplace and Poisson equations

In this section we shall describe some methods of finding explicit solutions to the Laplace and Poisson equations.

#### 1.1 Spherically symmetric data

The simplest case is when the all the data of the equation are spherically symmetric, that is, they depend only on  $r^2 = \|\mathbf{r}\|^2 = x^2 + y^2 + z^2$  for  $n = 3$ , or  $r^2 = x^2 + y^2$  for  $n = 2$ .

**Example 1.1 Spherically symmetric Poisson equation.** Let  $\Omega = \{\mathbf{r} = (x, y, z) \in \mathbb{R}^3; 0.5 \leq \|\mathbf{r}\| \leq 1\}$ . Find the solution to the problem

$$\begin{aligned}\Delta u &= 12\|\mathbf{r}\|, \quad \text{in } \Omega, \\ u(\mathbf{r})|_{\|\mathbf{r}\|=0.5} &= 1, \\ u(\mathbf{r})|_{\|\mathbf{r}\|=1} &= 3,\end{aligned}$$

Since all the data are spherically symmetric, we shall use the expression for the Laplacian in spherical coordinates without the angle dependent terms (7.1.1) and convert the above problem into the boundary value problem for the second order ordinary differential equation

$$\begin{aligned}\Delta u &= u_{rr} + \frac{2}{r}u_r = 12r, \\ u(0.5) &= 1, \quad u(1) = 3\end{aligned}$$

Multiplying the differential equation by  $r^2$  and using  $r^2 u_{rr} + 2r u_r = (r^2 u_r)_r$  we rewrite the equation as

$$(r^2 u_r)_r = 12r^3,$$

which upon the first integration produces

$$r^2 u_r = 3r^4 + C_1$$

and next

$$u(r) = r^3 - \frac{C_1}{3r^3} + C_2.$$

Since  $C_1$  is arbitrary, we can write the above equation in a more compact form

$$u(r) = r^3 + \frac{C_1}{r^3} + C_2.$$

Using the boundary conditions we obtain the system of equations for  $C_1$  and  $C_2$

$$\begin{aligned} \frac{1}{8} + 8C_1 + C_2 &= 1, \\ 1 + C_1 + C_2 &= 3, \end{aligned}$$

which has the solution

$$C_1 = -\frac{9}{56}, \quad C_2 = \frac{121}{56}.$$

The solution to the problem is given thus by

$$u(r) = r^3 - \frac{9}{56r^3} + \frac{121}{56}$$

where  $r = x^2 + y^2 + z^2$ .

## 1.2 The Laplace equation in the half-plane

Let us consider the following boundary value problem: find a bounded function  $u$  satisfying

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad -\infty < x < \infty, y > 0, \\ u(x, 0) &= h(x), \end{aligned} \tag{7.1.1}$$

where  $h$  is a given function. To solve this problem we use the Fourier transform method. Let us recall that the Fourier transform is defined by

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx,$$

and that for functions  $f$  satisfying  $f(-\infty) = f'(-\infty) = f(\infty) = f'(\infty) = 0$  we have

$$\mathcal{F}[f_{xx}](\omega) = -\omega^2 \mathcal{F}[f](\omega).$$

Applying the Fourier transform to both equations of (7.1.1) we obtain the initial value problem for the second-order ordinary differential equation with a parameter:

$$\begin{aligned} -\omega^2 \hat{u} + \hat{u}_{yy} &= 0, \quad -\infty < \omega < \infty, y > 0, \\ \hat{u}(\omega, 0) &= \hat{h}(\omega). \end{aligned} \tag{7.1.2}$$

The characteristic equation of the above differential equation is

$$r^2 - \omega^2 = 0$$

which has two solutions:  $r_1 = |\omega|$  and  $r_2 = -|\omega|$  (the absolute values has been introduced here to keep the sign of the roots independent of the sign of  $\omega$ ). Consequently,

$$\hat{u}(\omega, y) = C_1(\omega) e^{-|\omega|y} + C_2(\omega) e^{|\omega|y}.$$

Since we are looking for solutions which are bounded, and  $e^{|\omega|y}$  tends to infinity as  $y \rightarrow \infty$ , we infer that  $C_2(\omega) = 0$ . Using the initial value we obtain finally

$$\hat{u}(\omega, y) = \hat{h}(\omega)e^{-|\omega|y}.$$

To get  $u$  we apply the inverse Fourier transform and the convolution theorem

$$\mathcal{F}^{-1}[\hat{f} \cdot \hat{g}] = \frac{1}{\sqrt{2\pi}} f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)g(y-x)dx.$$

Therefore

$$u(x, y) = \frac{1}{\sqrt{2\pi}} h * \mathcal{F}^{-1}[e^{-|\omega|y}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(\lambda) \mathcal{F}^{-1}[e^{-|\omega|y}](x - \lambda) d\lambda,$$

where the inverse Fourier transform is evaluated with respect to  $\omega$ . Since

$$\begin{aligned} \mathcal{F}^{-1}[e^{-|\omega|y}](x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\omega|y} e^{i\omega x} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \left( \int_0^{\infty} e^{-\omega y} e^{i\omega x} d\omega + \int_{-\infty}^0 e^{\omega y} e^{i\omega x} d\omega \right) \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\omega y} (e^{i\omega x} + e^{-i\omega x}) d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\omega y} \cos \omega x d\omega = \sqrt{\frac{2}{\pi}} \frac{y}{x^2 + y^2}, \end{aligned}$$

we obtain

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(\lambda)}{y^2 + (\lambda - x)^2} d\lambda. \quad (7.1.3)$$

We shall illustrate the application of this formula solving one easy and one more difficult problem.

**Example 1.2** Solve the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & -\infty < x < \infty, y > 0, \\ u(x, 0) &= h(x), \end{aligned}$$

where  $h(x) = 1$  for  $-1 < x < 1$  and  $h(x) = 0$  elsewhere.

Using the formula (7.1.3) we obtain

$$\begin{aligned} u(x, y) &= \frac{y}{\pi} \int_{-1}^1 \frac{d\lambda}{y^2 + (\lambda - x)^2} \\ &= \frac{1}{\pi} \int_{\frac{-1-x}{y}}^{\frac{1-x}{y}} \frac{dz}{1 + z^2} \\ &= \frac{1}{\pi} \left( \tan^{-1} \frac{1-x}{y} + \tan^{-1} \frac{1+x}{y} \right) \end{aligned}$$

where we used the change of variables  $z = (\lambda - x)/y$  and the fact that the function  $\tan^{-1}$  is odd.

Using some geometry we can write the above formula in a more compact form. In fact, denoting  $\alpha = \tan^{-1} \frac{1-x}{y}$  and  $\beta = \tan^{-1} \frac{1+x}{y}$  we observe that  $\alpha + \beta$  is always the interior angle at the vertex  $(x, y)$  of the triangle with vertices  $(-1, 0), (x, y), (0, 1)$ . Thus  $0 < \alpha + \beta < \pi$ . Also  $-\pi/2 < \alpha, \beta < \pi/2$  (from the definition of  $\tan^{-1}$ ). Hence one can use the trigonometric formula

$$\cot(\alpha + \beta) = \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}$$

getting

$$\cot \left( \tan^{-1} \frac{1-x}{y} + \tan^{-1} \frac{1+x}{y} \right) = \frac{1 - \frac{(1+x)(1-x)}{y^2}}{\frac{1+x}{y} + \frac{1-x}{y}} = \frac{x^2 + y^2 - 1}{2y}$$

and finally

$$u(x, y) = \frac{1}{\pi} \cot^{-1} \frac{x^2 + y^2 - 1}{2y}.$$

**Example 1.3** Solve the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & -\infty < x < \infty, y > 0, \\ u(x, 0) &= \frac{1}{1+x^2}, \end{aligned}$$

Using the formula (7.1.3) we obtain

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{(1+\lambda)^2(y^2 + (\lambda-x)^2)}. \quad (7.1.4)$$

To evaluate this integral we use the residue method from complex analysis. Factorizing the denominator we obtain

$$(\lambda + i)(\lambda - i)(\lambda - x + iy)(\lambda - x - iy),$$

so that in the upper half-plane we have two simple poles at  $\lambda = i$  and  $\lambda = x + iy$ . Evaluating residues at these points we obtain

$$Res_{\lambda=i} = \frac{1}{2i((i-x)^2 + y^2)}$$

and

$$Res_{\lambda=x+iy} = \frac{1}{2iy(1 + (x+iy)^2)}.$$

From the residue theorem we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\lambda}{(1+\lambda)^2(y^2 + (\lambda-x)^2)} &= 2\pi i (Res_{\lambda=x+iy} + Res_{\lambda=i}) \\ &= \frac{\pi}{y(1 + (x+iy)^2)} + \frac{\pi}{((i-x)^2 + y^2)} = \frac{\pi}{y(1 + x^2 + 2ixy - y^2)} + \frac{\pi}{y^2 + x^2 + 2ix - 1} \\ &= \frac{\pi(1 + x^2 - y^2 - 2ixy)}{y((1 + x^2 - y^2)^2 + 4x^2y^2)} + \frac{\pi(y^2 + x^2 - 1 + 2ix)}{(y^2 + x^2 - 1)^2 + 4x^2}. \end{aligned}$$

It follows that

$$(1 + x^2 - y^2)^2 + 4x^2y^2 = (y^2 + x^2 - 1) + 4x^2 = (y^2 + x^2 + 1)^2 - 4y^2,$$

thus the last sum can be written as

$$\begin{aligned} \frac{\pi(1 + x^2 - y^2 - 2ixy) + y(y^2 + x^2 - 1 + 2ix)}{y((y^2 + x^2 + 1)^2 - 4y^2)} &= \frac{\pi(1 + x^2 - y^2) + y(y^2 + x^2 - 1)}{y(y^2 + x^2 + 1 - 2y)(y^2 + x^2 + 1 + 2y)} \\ &= \frac{\pi(1 + y)(y^2 + x^2 + 1 - 2y)}{y(y^2 + x^2 + 1 - 2y)(y^2 + x^2 + 1 + 2y)} = \frac{\pi(1 + y)}{y(y^2 + x^2 + 1 + 2y)} \end{aligned}$$

Inserting the above into (7.1.4) we obtain finally the solution

$$u(x, y) = \frac{1 + y}{y^2 + x^2 + 1 + 2y}.$$

### 1.3 Laplace and Poisson equations in regions with polar symmetry

In this subsection we shall show a variant of the method of separation of variables.

Throughout this subsection we denote by  $\Omega$  one of the following sets:

- (i)  $\Omega = \{(x, y); x^2 + y^2 < a^2\} = \{(r, \phi); r < a, 0 \leq \phi < 2\pi\}$  (the interior of a disc),
- (ii)  $\Omega = \{(x, y); a^2 < x^2 + y^2 < b^2\} = \{(r, \phi); a < r < b, 0 \leq \phi < 2\pi\}$  (an annulus),
- (iii)  $\Omega = \{(x, y); b^2 < x^2 + y^2\} = \{(r, \phi); b < r, 0 \leq \phi < 2\pi\}$  (the exterior of the disc).

Let us consider the Laplace equation in  $\Omega$ :

$$\Delta u = 0, \quad \text{in } \Omega, \quad (7.1.5)$$

with one of the following boundary conditions:

(I)

In the case (i):

$$u(a, \phi) = g(\phi), \quad u(r, \phi) \text{ bounded as } r \rightarrow 0^+, \quad (7.1.6)$$

(II)

In the case (ii):

$$\begin{aligned} u(a, \phi) &= g_a(\phi), \\ u(b, \phi) &= g_b(\phi), \end{aligned} \quad (7.1.7)$$

(III)

In the case (iii):

$$u(b, \phi) = g(\phi), \quad u(r, \phi) \text{ bounded as } r \rightarrow +\infty. \quad (7.1.8)$$

Solutions to (I)-(III) can be obtained as particular cases of the general solution to (11.2.18).

Using the Chain Rule we can write the Laplace equation in polar coordinates as

$$r^{-1}(ru_r)_r + r^2 u_{\phi\phi} = 0. \quad (7.1.9)$$

To apply the separation of variables technique, we look for particular solutions in the form

$$u(r, \phi) = R(r)\Phi(\phi). \quad (7.1.10)$$

Inserting (11.2.23) into the equation (11.2.22) we obtain

$$\frac{r(rR_r)_r}{R} + \frac{\Phi_{\phi\phi}}{\Phi} = 0,$$

which yields that both terms are constant, and leads to two ordinary differential equations

$$\Phi'' + \lambda\Phi = 0, \quad \text{for } 0 \leq \phi < 2\pi, \quad (7.1.11)$$

and

$$r(rR')' - \lambda R = 0, \quad \text{for } r > 0. \quad (7.1.12)$$

Let us consider first Eq. (11.2.24). It is not immediate what kind of boundary conditions one should assign to this equation. To find the proper ones we note that the restriction  $0 \leq \phi < 2\pi$  is artificial as geometrically the point  $(r, \phi)$  coincides with the point  $(r, \phi + 2\pi)$ , thus  $u(r, \phi)$  must be equal to  $u(r, \phi + 2\pi)$ , for any  $\phi$ . This observation shows that  $\Phi$  must be periodic with period  $2\pi$ , that is, we require

$$\Phi(\phi) = \Phi(\phi + 2\pi), \quad \text{for } 0 \leq \phi < 2\pi. \quad (7.1.13)$$

Next let us note that (11.2.26) yields

$$\Phi'(\phi) = \Phi'(\phi + 2\pi), \quad \text{for } 0 \leq \phi < 2\pi,$$

and therefore we shall consider the following problem

$$\begin{aligned} \Phi'' + \lambda\Phi &= 0, \quad \text{for } 0 \leq \phi < 2\pi, \\ \Phi(0) &= \Phi(2\pi) \\ \Phi'(0) &= \Phi'(2\pi). \end{aligned} \quad (7.1.14)$$

The general solution of (11.2.24) is given by

$$u(\phi) = C_1 e^{\sqrt{-\lambda}\phi} + C_2 e^{-\sqrt{-\lambda}\phi}$$

and the boundary conditions produce the system of equations

$$\begin{aligned} C_1 + C_2 &= C_1 e^{\sqrt{-\lambda}2\pi} + C_2 e^{-\sqrt{-\lambda}2\pi} \\ C_1 \sqrt{-\lambda} - C_2 \sqrt{-\lambda} &= \sqrt{-\lambda} C_1 e^{\sqrt{-\lambda}2\pi} - \sqrt{-\lambda} C_2 e^{-\sqrt{-\lambda}2\pi}. \end{aligned}$$

Assuming for a moment that  $\lambda \neq 0$ , we can divide the second equation by  $\sqrt{-\lambda}$  so that we get the following system

$$\begin{aligned} C_1(1 - e^{\sqrt{-\lambda}2\pi}) + C_2(1 - e^{-\sqrt{-\lambda}2\pi}) &= 0 \\ C_1(1 - e^{\sqrt{-\lambda}2\pi}) - C_2(1 - e^{-\sqrt{-\lambda}2\pi}) &= 0. \end{aligned}$$

The determinant of this system is given by

$$-2(1 - e^{\sqrt{-\lambda}2\pi})(1 - e^{-\sqrt{-\lambda}2\pi})$$

which is zero only if either

$$1 - e^{\sqrt{-\lambda}2\pi} = 0$$

or

$$1 - e^{-\sqrt{-\lambda}2\pi} = 0.$$

From these equations we obtain

$$e^{\pm\sqrt{-\lambda}2\pi} = 1 = e^{i2n\pi}$$

which yields

$$\lambda = n^2$$

where  $n$  is a nonzero integer. Consequently, we obtain possible solutions in the form

$$\tilde{\Phi}_n(\phi) = C_{1n} e^{in\phi} + C_{2n} e^{-in\phi}. \quad (7.1.15)$$

These functions can be written in the real form yielding

$$\{\sin n\phi, \cos n\phi\}_{n=\pm 1, \pm 2, \dots}$$

as the set of eigenfunctions.



Consider now the exceptional case  $\lambda = 0$ . Then the equation (11.2.24) has the solution  $\Phi(\phi) = C_0\phi + C'_0$  which is periodic only when  $C_0 = 0$ . Thus  $\Phi_0$  is constant and is included in the general formula if we allow  $n = 0$ .

Having determined the eigenvalues and eigenfunctions for the angle dependent part of the equation, we can now tackle the equation for  $R$ :

$$r^2 R'' + rR' - n^2 R = 0. \quad (7.1.16)$$

This is so-called Euler equation which can be solved by using trial solutions of the form  $R_\alpha = r^\alpha$ . Inserting such a function into the equation (11.2.29), we obtain

$$r^\alpha(\alpha(\alpha - 1) + \alpha - n^2) = 0$$

which is satisfied if and only if  $\alpha$  is a root of the following quadratic equation

$$\alpha(\alpha - 1) + \alpha - n^2 = 0$$

or simply

$$\alpha^2 = n^2.$$

So, if  $n \neq 0$ , we obtain two distinct solutions

$$\alpha_1 = n, \quad \alpha_2 = -n,$$

where  $n \in \mathbb{N}$ , and consequently

$$R_n(r) = a_n r^n + b_n r^{-n}.$$

If  $n = 0$ , this procedure produces only one solution  $R_0(r) = a_0$ , but then the equation can be integrated directly to give

$$R_0(r) = a_0 + b_0 \ln r.$$

Thus the general solution of the Laplace equation in any region with polar symmetry is given by

$$u(r, \phi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\phi + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \sin n\phi. \quad (7.1.17)$$

Now, let us specify the form of the solution so that it can be the solution of the boundary value problems (I)-(III).

Case (I)

In the case (I) we require the solution to be bounded as  $r \rightarrow 0^+$ , which forces  $B_0 = B_n = D_n = 0$  for all  $n \geq 1$ . Thus we obtain

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi. \quad (7.1.18)$$

The constants  $A_n$  and  $C_n$  are to be determined from the boundary condition, that is, we must have

$$g(\phi) = u(a, \phi) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos n\phi + \sum_{n=1}^{\infty} C_n a^n \sin n\phi.$$

Therefore

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi, \\ A_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \cos n\phi d\phi, \\ C_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \sin n\phi d\phi, \end{aligned} \quad (7.1.19)$$

for  $n > 0$ , where we used the orthogonality of all trigonometric functions on  $[0, 2\pi]$  and the normalizing relation

$$\int_0^{2\pi} \cos^2 n\phi d\phi = \int_0^{2\pi} \sin^2 n\phi d\phi = \pi.$$

Case (II)

Case (II) requires the full expansion (11.2.30). The coefficients are to be determined from the infinite system of equations:

$$\begin{aligned} A_0 + B_0 \ln a &= \frac{1}{2\pi} \int_0^{2\pi} g_a(\phi) d\phi, \\ A_0 + B_0 \ln b &= \frac{1}{2\pi} \int_0^{2\pi} g_b(\phi) d\phi, \end{aligned} \quad (7.1.20)$$

and for  $n \geq 1$ ,

$$\begin{aligned} A_n a^n + B_n a^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_a(\phi) \cos n\phi d\phi, \\ A_n b^n + B_n b^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_b(\phi) \cos n\phi d\phi, \end{aligned} \quad (7.1.21)$$

and

$$\begin{aligned} C_n a^n + D_n a^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_a(\phi) \sin n\phi d\phi, \\ C_n b^n + D_n b^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_b(\phi) \sin n\phi d\phi. \end{aligned} \quad (7.1.22)$$

Note that these equations are solvable as the determinant of each system is equal to  $(a/b)^n - (b/a)^n \neq 0$  for  $n \geq 1$  and  $\ln(b/a) \neq 0$  for  $n = 0$ .

Case (III)

Case (III) is similar to the Case (I) with the difference being that the requirement of boundedness of  $u(r, \phi)$  as  $r \rightarrow \infty$  forces  $B_0 = A_n = C_n$  for all  $n \geq 0$ . The solution is then given by

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} B_n r^{-n} \cos n\phi + \sum_{n=1}^{\infty} D_n r^{-n} \sin n\phi. \quad (7.1.23)$$

The constants  $A_n$  and  $C_n$  are defined by

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi, \\ B_n &= \frac{b^n}{\pi} \int_0^{2\pi} g(\phi) \cos n\phi d\phi, \\ D_n &= \frac{b^n}{\pi} \int_0^{2\pi} g(\phi) \sin n\phi d\phi, \end{aligned} \quad (7.1.24)$$

for  $n > 1$ .

**Example 1.4** Find the solution to:

$$\begin{aligned}\Delta u &= 0, & 1 < r < 2, 0 \leq \phi < 2\pi, \\ u(1, \phi) &= \cos^2 \phi + 1, \\ u(2, \phi) &= 1 + \sin \phi.\end{aligned}$$

To solve the problem we first write  $\cos^2 \phi = \frac{1}{2} + \frac{\cos 2\phi}{2}$ . With this change we use (11.2.30)

$$u(r, \phi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\phi + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \sin n\phi.$$

There is, however, no need to use the integral formulas for coefficients of the expansion, as our data are already given in the form of (finite) Fourier expansion. By comparison we have

$$\frac{3}{2} + \frac{\cos 2\phi}{2} = u(1, \phi) = A_0 + \sum_{n=1}^{\infty} (A_n + B_n) \cos n\phi + \sum_{n=1}^{\infty} (C_n + D_n) \sin n\phi,$$

where we used  $\ln 1 = 0$ . Similarly

$$1 + \sin \phi = u(2, \phi) = A_0 + B_0 \ln 2 + \sum_{n=1}^{\infty} (A_n 2^n + B_n 2^{-n}) \cos n\phi + \sum_{n=1}^{\infty} (C_n 2^n + D_n 2^{-n}) \sin n\phi.$$

By the uniqueness of the Fourier expansion we get the following equations

$$\begin{aligned}A_0 &= \frac{3}{2}, \\ A_0 + B_0 \ln 2 &= 1, \\ C_1 + D_1 &= 0, \\ 2C_1 + 2^{-1}D_1 &= 1, \\ A_2 + B_2 &= \frac{1}{2} \\ 2^2 A_2 + 2^{-2}B_2 &= 0.\end{aligned}$$

All the other coefficients, being the solutions of homogeneous linear systems with non-zero determinants, are zero. Solving the above system, we obtain

$$A_0 = \frac{3}{2}, B_0 = -\frac{1}{2 \ln 2}, C_1 = \frac{2}{3}D_1 = -\frac{2}{3}, A_2 = -\frac{1}{30}, B_2 = \frac{8}{15},$$

and thus the solution is given by

$$u(r, \phi) = \frac{3}{2} - \frac{1}{2 \ln 2} \ln r + \frac{2}{3}(r + r^{-1}) \sin \phi + \left( -\frac{1}{30}r^2 + \frac{8}{15}r^{-2} \right) \cos 2\phi.$$

The next example is a little more sophisticated and requires some complex analysis.

**Example 1.5** Find the solution to:

$$\begin{aligned}\Delta u &= 0, & r < 1, 0 \leq \phi < 2\pi, \\ u(1, \phi) &= g(\phi),\end{aligned}$$

where

$$g(\phi) = \begin{cases} 1 & \text{for } 0 \leq \phi \leq \pi \\ 0 & \text{for } \pi < \phi < 2\pi. \end{cases}$$

According to the formula (11.2.31) the solution is given by

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi,$$

where in our case

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{\pi} d\phi = \frac{1}{2}, \\ A_n &= \frac{1}{\pi} \int_0^{\pi} \cos n\phi d\phi = \frac{1}{n\pi} (\sin n\pi - \sin 0) = 0, \\ C_n &= \frac{1}{\pi} \int_0^{\pi} \sin n\phi d\phi = -\frac{1}{n\pi} (\cos n\pi - \cos 0) = -\frac{1}{n\pi} ((-1)^n - 1). \end{aligned}$$

The coefficient  $C_n$  can be written in the following way

$$C_n = \begin{cases} 0 & \text{for } n = 2k \\ \frac{2}{(2k+1)\pi} & \text{for } n = 2k+1, \end{cases}$$

where  $k = 0, 1, \dots$  and the solution takes the form

$$u(r, \phi) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \sin(2k+1)\phi. \quad (7.1.25)$$

An interesting thing about the series above is that it can actually be summed. In fact

$$\sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \sin(2k+1)\phi = \operatorname{Im} \sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} (\cos(2k+1)\phi + i \sin(2k+1)\phi) = \operatorname{Im} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}.$$

The last series is quite well-known. In fact, since

$$\begin{aligned} \ln(1-z) &= -\sum_{k=1}^{\infty} \frac{z^k}{k}, \\ \ln(1+z) &= -\sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k}, \end{aligned}$$

we have

$$\ln \frac{1+z}{1-z} = \ln(1+z) - \ln(1-z) = 2 \sum_{k=1}^{\infty} \frac{z^{2k+1}}{2k+1}.$$

We have to calculate  $\operatorname{Im} \ln \frac{1+z}{1-z}$ . To do this, we recall that for complex logarithm we have the formula  $\ln z = \ln|z| + i \arg z$ , so that we have to find  $\arg \frac{1+z}{1-z}$ . For  $z = re^{i\phi} = r(\cos \phi + i \sin \phi)$  we have

$$\frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z})} = \frac{1+2ir \sin \phi - r^2}{1-2r \cos \phi + r^2}$$

and since the denominator is a real number, we have

$$\arg \frac{1+z}{1-z} = \arg(1+2ir \sin \phi - r^2).$$

Now,  $1-r^2 > 0$ , so that  $-\pi/2 < \arg(1+2ir \sin \phi - r^2) < \pi/2$  and we can use  $\tan^{-1}$  to express it analytically as

$$\arg \frac{1+z}{1-z} = \tan^{-1} \frac{r \sin \phi}{1-r^2}.$$

Returning to the solution of our problem we obtain

$$u(r, \phi) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{2r \sin \phi}{1-r^2}.$$

The above example suggests that it may be possible to sum the whole series (11.2.30). To simplify the matters we shall concentrate on the solution for the interior of the disc

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi, \quad (7.1.26)$$

where the constants  $A_n$  and  $C_n$ ,  $n \geq 0$  are given by

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi, \\ A_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \cos n\phi d\phi, \\ C_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \sin n\phi d\phi. \end{aligned} \quad (7.1.27)$$

Inserting the coefficient formulae into (7.1.26) we obtain

$$\begin{aligned} u(r, \phi) &= \frac{1}{2\pi} \int_0^{2\pi} g(\psi) d\psi \\ &\quad + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} g(\psi) \frac{r^n}{a^n} (\cos n\phi \cos n\psi + \sin n\phi \sin n\psi) d\psi \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\psi) \left( 1 + 2 \sum_{n=1}^{\infty} \frac{r^n}{a^n} \cos n(\phi - \psi) \right) d\psi. \end{aligned}$$

Using Euler's formula we obtain

$$\frac{r^n}{a^n} \cos n(\phi - \psi) = \operatorname{Re} \left( \frac{r}{a} e^{i(\phi - \psi)} \right)^n$$

so that denoting  $z = \frac{r}{a} e^{i(\phi - \psi)}$  we see that

$$\begin{aligned} 1 + 2 \sum_{n=1}^{\infty} \frac{r^n}{a^n} \cos n(\phi - \psi) &= 1 + 2 \operatorname{Re} \sum_{n=1}^{\infty} z^n = 1 + 2 \operatorname{Re} \frac{z}{1 - z} \\ &= 1 + 2 \operatorname{Re} \frac{z - |z|^2}{1 - (z + \bar{z}) + |z|^2} = 1 + 2 \frac{r \cos(\phi - \psi) + r^2}{a^2 + r^2 - 2r \cos(\phi - \psi)} \\ &= \frac{a^2 - r^2}{a^2 + r^2 - 2r \cos(\phi - \psi)}. \end{aligned}$$

Thus we obtained the Poisson formula for the solution of the Laplace equation in the disc

$$u(r, \phi) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{g(\psi)}{a^2 + r^2 - 2r \cos(\phi - \psi)} d\psi. \quad (7.1.28)$$

This formula is very important from the theoretical point of view. Below we shall see how it works in practice. To compare it with the Fourier expansion method we shall solve the same problem as in Example 1.5.

**Example 1.6** Find the solution to:

$$\begin{aligned}\Delta u &= 0, \quad r < 1, 0 \leq \phi < 2\pi, \\ u(1, \phi) &= g(\phi),\end{aligned}$$

where

$$g(\phi) = \begin{cases} 1 & \text{for } 0 \leq \phi \leq \pi \\ 0 & \text{for } \pi < \phi < 2\pi. \end{cases}$$

According to (7.1.28) we have

$$\begin{aligned}u(r, \phi) &= \frac{1-r^2}{2\pi} \int_0^\pi \frac{d\psi}{1+r^2-2r\cos(\psi-\phi)} = \frac{1-r^2}{2\pi(1+r^2)} \int_0^\pi \frac{d\psi}{1-\alpha\cos(\psi-\phi)} \\ &= \frac{1-r^2}{2\pi(1+r^2)} \int_{-\phi}^{\pi-\phi} \frac{d\xi}{1-\alpha\cos\xi},\end{aligned}$$

where we denoted  $\alpha = 2r/(1+r^2)$  and  $\xi = \psi - \phi$ . Since the possible intervals of integration stretch from  $-2\pi$  to  $\pi$ , the direct substitution  $t = \tan \xi/2$  is not feasible as it is applicable only on  $-\pi < \xi < \pi$ . Thus we shall split the integral as follows

$$\int_{-\phi}^{\pi-\phi} = \int_0^\pi + \int_{-\phi}^0 - \int_{\pi-\phi}^\pi$$

Next we use

$$\int_{-\phi}^0 \frac{d\xi}{1-\alpha\cos\xi} = \int_0^\phi \frac{d\xi}{1-\alpha\cos\xi},$$

since  $\cos$  is an even function, and

$$\int_{\pi-\phi}^\pi \frac{d\xi}{1-\alpha\cos\xi} d\psi = \int_0^\phi \frac{d\xi}{1+\alpha\cos\xi} d\psi.$$

Thus

$$\begin{aligned}\int_{-\phi}^{\pi-\phi} \frac{d\xi}{1-\alpha\cos\xi} &= \int_0^\pi \frac{d\xi}{1-\alpha\cos\xi} + \int_0^\phi \frac{d\xi}{1-\alpha\cos\xi} - \int_0^\pi \frac{d\xi}{1+\alpha\cos\xi} \\ &= \int_0^\pi \frac{d\xi}{1-\alpha\cos\xi} + 2\alpha \int_0^\phi \frac{\cos\xi d\xi}{1-\alpha^2\cos^2\xi}.\end{aligned}$$

To evaluate the first integral we can use the substitution  $t = \tan \xi/2$ ; then we have  $d\xi = 2dt/(1+t^2)$  and  $\cos \xi = (1-t^2)/(1+t^2)$ , which yields

$$\begin{aligned}\int_0^\pi \frac{d\xi}{1-\alpha\cos\xi} &= 2 \int_0^\infty \frac{dt}{1-\alpha+(1+\alpha)t^2} = 2 \frac{1}{\sqrt{1-\alpha^2}} \tan^{-1} \left( t \sqrt{\frac{1+\alpha}{1-\alpha}} \right) \Big|_0^\infty \\ &= \frac{\pi}{\sqrt{1-\alpha^2}}.\end{aligned}$$

The second integral is evaluated as follows

$$\begin{aligned} \int_0^\phi \frac{\cos \xi d\xi}{1 - \alpha^2 \cos^2 \xi} &= \int_0^\phi \frac{\cos \xi d\xi}{1 - \alpha^2 + \alpha^2 \sin^2 \xi} \\ &= \frac{1}{\alpha \sqrt{1 - \alpha^2}} \int_0^{\frac{\alpha}{\sqrt{1 - \alpha^2}} \sin \phi} \frac{dt}{1 + t^2} = \frac{1}{\alpha \sqrt{1 - \alpha^2}} \tan^{-1} \frac{\alpha}{\sqrt{1 - \alpha^2}} \sin \phi, \end{aligned}$$

where we used  $t = \frac{\alpha}{\sqrt{1 - \alpha^2}} \sin \xi$ .

Returning to the old notation we find that

$$\frac{1}{\sqrt{1 - \alpha^2}} = \frac{1 + r^2}{1 - r^2},$$

and

$$\frac{\alpha}{\sqrt{1 - \alpha^2}} = \frac{2r}{1 - r^2},$$

thus combining all the partial results we obtain, as before, the solution in the form

$$u(r, \phi) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{2r}{1 - r^2} \sin \phi.$$

**Example 1.7 Poisson equation in a disc.** Let us start with a general comment concerning the solvability of the Poisson equation which is a nonhomogeneous Laplace equation. Suppose that we are to solve the problem

$$\begin{aligned} \Delta u &= f, & \text{in } \Omega, \\ u &= g, & \text{on } \partial\Omega. \end{aligned} \tag{7.1.29}$$

In many cases it is possible to guess easily a particular solution to the Poisson equation in the whole space, that is, to find a function  $v$  which satisfies

$$\Delta v = f, \quad \text{in } \mathbb{R}^n.$$

Then, if  $u$  is the solution to (7.1.29) (as yet unknown), then the function  $w = u - v$  satisfies

$$\Delta w = \Delta u - \Delta v = f - f = 0, \quad \text{in } \Omega.$$

and

$$w = g - v|_{\partial\Omega}$$

on  $\partial\Omega$ . Thus the Dirichlet problem for the Poisson equation (7.1.29) has been reduced to the Dirichlet problem for the Laplace equation

$$\begin{aligned} \Delta w &= 0, & \text{in } \Omega, \\ w &= g - v|_{\partial\Omega}, & \text{on } \partial\Omega, \end{aligned} \tag{7.1.30}$$

which, for example in a disc, we can solve.

We shall illustrate this method by solving the problem

$$\begin{aligned} \Delta u &= -xy, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ . It is easy to check that  $v = -\frac{1}{12}(x^3y + y^3x)$  is a particular solution to the Poisson equation in the whole space. Since we are working in the unit disc, it is convenient to convert all the function to polar coordinates. We have

$$v = -\frac{1}{12}(x^3y + y^3x) = -\frac{1}{12}xy(x^2 + y^2) = -\frac{r^4}{12} \sin \phi \cos \phi = -\frac{r^4}{24} \sin 2\phi.$$

Following our general strategy we define

$$w = u + \frac{r^4}{24} \sin 2\phi,$$

so that  $w$  satisfies

$$\begin{aligned} \Delta w &= 0, & \text{in } \Omega, \\ w &= \frac{1}{24} \sin 2\phi, & \text{on } \partial\Omega, \end{aligned}$$

Using the general formula

$$w(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi,$$

we see that

$$\frac{1}{24} \sin 2\phi = w(1, \phi) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\phi + \sum_{n=1}^{\infty} C_n \sin n\phi,$$

and from the uniqueness of Fourier expansion we infer that  $A_n = 0$  for all  $n$  and  $C_n = 0$  for all  $n \neq 2$ . Thus

$$w(r, \phi) = \frac{r^2}{24} \sin 2\phi,$$

and

$$u(r, \phi) = w(r, \phi) + v(r, \phi) = \frac{r^2}{24} \sin 2\phi - \frac{r^4}{24} \sin 2\phi.$$

**Example 1.8 Boundary value problem for the Laplace equation in a rectangle.** Let us consider now the case when  $\Omega$  is a rectangle, say,  $\Omega = \{(x, y); 0 < x < a, 0 < y < b\}$ . In this case the explicit formulation of the problem reads

$$\begin{aligned} \Delta u &= 0, & \text{for } 0 < x < a, 0 < y < b \\ u(0, y) &= h_1(y), & \text{for } 0 < y < b, \\ u(a, y) &= h_2(y), & \text{for } 0 < y < b, \\ u(x, 0) &= h_3(y), & \text{for } 0 < x < a, \\ u(x, b) &= h_4(y), & \text{for } 0 < x < a, \end{aligned} \tag{7.1.31}$$

To solve this problem we split it into four simpler problems: by  $u_i$  we shall denote the solution of the Laplace equation in  $\Omega$  satisfying the boundary condition  $u_i = h_i$  on the appropriate part of the boundary and homogeneous boundary conditions on the three remaining parts. If we manage to solve all four problems, then the solution to the full problem (7.1.31) is given by  $u = u_1 + u_2 + u_3 + u_4$ .

Let us consider now one of these four partial problems and solve, say,

$$\begin{aligned} \Delta u &= 0, & \text{for } 0 < x < a, 0 < y < b \\ u(0, y) &= y, & \text{for } 0 < y < b, \\ u(a, y) &= 0, & \text{for } 0 < y < b, \\ u(x, 0) &= 0, & \text{for } 0 < x < a, \\ u(x, b) &= 0, & \text{for } 0 < x < a, \end{aligned}$$



We start with separation of the variables. Inserting  $u(x, y) = X(x)Y(y)$  into the Laplace equation we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

so that there is a constant  $\lambda$  such that

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

Next we observe that  $Y$  must satisfy homogeneous boundary conditions  $Y(0) = Y(b) = 0$ , thus we start with the equation for  $Y$ . For the sake of completeness we shall repeat the derivation of the solution to this problem.

The general solution to the equation is given by

$$Y(y) = \tilde{C}_1 e^{\sqrt{\lambda}y} + \tilde{C}_2 e^{-\sqrt{\lambda}y},$$

hence using the boundary conditions we obtain

$$\begin{aligned} \tilde{C}_1 + \tilde{C}_2 &= 0 \\ \tilde{C}_1 e^{b\sqrt{\lambda}} + \tilde{C}_2 e^{-b\sqrt{\lambda}} &= 0. \end{aligned}$$

For a homogeneous system of linear algebraic equations to have a non-zero solution it is necessary and sufficient that

$$e^{-b\sqrt{\lambda}} - e^{b\sqrt{\lambda}} = 0,$$

which gives

$$e^{2b\sqrt{\lambda}} = 1 = e^{i2n\pi}$$

for  $n \in \mathbb{Z}$ . Thus we obtain the sequence of eigenvalues

$$\lambda_n = -\frac{n^2\pi^2}{b^2}; \quad (7.1.32)$$

consequently we have

$$Y(y) = \tilde{C}_1 e^{in\pi y/b} - \tilde{C}_1 e^{-in\pi y/b} = 2i\tilde{C}_1 \sin \frac{n\pi y}{b},$$

and the sequence of real-valued eigenfunctions is given by

$$\tilde{Y}_n(y) = \sin \frac{n\pi y}{b}$$

We obtain

$$\int_0^b \sin \frac{n\pi y}{b} \sin \frac{k\pi y}{b} dy = 0$$

for  $n \neq k$ . On the other hand, for  $n = k$  we have

$$\left\| \sin \frac{n\pi y}{b} \right\|^2 = \int_0^b \sin^2 \frac{n\pi y}{b} dy = \frac{1}{2} \int_0^b \left( 1 - \cos \frac{2n\pi y}{b} \right) dy = \frac{b}{2}. \quad (7.1.33)$$

Next we shall deal with the equation  $X'' + \lambda X$ . Since  $\lambda_n = -n^2\pi^2/b^2$ , we obtain the equation

$$X'' - \frac{n^2\pi^2}{b^2} X = 0,$$

which has the general solution

$$X_n(x) = A'_n e^{xn\pi/b} + B'_n e^{-xn\pi/b} = A_n \cosh \frac{xn\pi}{b} + B_n \sinh \frac{xn\pi}{b}.$$

Using the homogeneous boundary condition at  $x = a$  we obtain

$$X_n(0) = A_n \cosh \frac{an\pi}{b} + B_n \sinh \frac{an\pi}{b} = 0,$$

which leads to

$$\frac{A_n}{B_n} = -\tanh \frac{an\pi}{b}.$$

Thus

$$X_n(x) = B_n \left( -\tanh \frac{an\pi}{b} \cosh \frac{xn\pi}{b} + \sinh \frac{xn\pi}{b} \right) = B_n \frac{\sinh n\pi(a+x)/b}{\cosh an\pi/b}.$$

The solution satisfying homogeneous boundary conditions on three sides of the rectangle is thus given by

$$u(x, y) = \sum_{n=1}^{\infty} B_n \frac{\sinh n\pi(a+x)/b}{\cosh an\pi/b} \sin \frac{n\pi y}{b}.$$

The last boundary condition will be satisfied if we find the constants  $B_n$  such that

$$u(0, y) = y = \sum_{n=1}^{\infty} B_n \tanh \frac{an\pi}{b} \sin \frac{n\pi y}{b}. \quad (7.1.34)$$

This is simply the Fourier expansion of  $y$  on  $[0, b]$ . Since

$$\int_0^b y \sin \frac{n\pi y}{b} dy = (-1)^n \frac{b^2}{n\pi},$$

multiplying both sides of (7.1.34) and integrating over  $[0, b]$  we obtain

$$\frac{(-1)^n b^2}{n\pi} = B_n \tanh \frac{an\pi}{b} \frac{b}{2},$$

where we used (7.1.33). Hence

$$B_n = \frac{(-1)^n 2b}{n\pi} \coth \frac{an\pi}{b}$$

and finally the solution is given by

$$u(x, y) = \frac{2b}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\sinh n\pi(a+x)/b}{\sinh an\pi/b} \sin \frac{n\pi y}{b}.$$

# Lecture 11

## Method of separation of variables

So far, when we have discussed time dependent problems, the main emphasis was on the initial value problems. By avoiding additional complications resulting from the boundary values, this restriction allowed us to provide general formulas for solutions for arbitrary initial data. The price to pay is that these formulas are of limited use - in real life we don't encounter infinite space and the constructed free space solutions can be used only as approximations far away from real boundaries.

In this section we shall discuss a general method for finding solutions to initial-boundary value problems in bounded spatial domains, which is called the separation of variables method. It provides a general framework for representing the solution to an initial-boundary (or only boundary) value problem in terms of an infinite sum of functions which are solutions of simpler (lower dimensional) problems, and called the *eigenfunctions* of the problem. One drawback of this method is that the eigenfunctions can be found explicitly only for a small class of domains (roughly speaking, the domain should have "rectangular" form in some "nice" coordinates (examples are rectangles and boxes in cartesian coordinates, spheres, spherical angles in spherical coordinates, cylinders, wedges in cylindrical coordinates etc.) Another drawback of this method is that the representation of the solution as an infinite series is often difficult to handle. Yet, despite all this, the separation of variables is the most powerful analytical tool for solving boundary value problems.

### 1 General eigenvalue problem

Let us consider two classical time-dependent equations

$$u_t = D\Delta u \quad \text{and} \quad u_{tt} = c^2\Delta u \quad (11.1.1)$$

in a bounded domain  $\Omega \subset \mathbb{R}^n$ , where  $n = 1, 2, 3$  with one of the classical homogeneous boundary conditions and standard initial conditions. For brevity, we will use again the notation  $\mathbf{r} = (x, y, z)$  if  $n = 3$ ,  $\mathbf{r} = (x, y)$  if  $n = 2$ , and somewhat inconsequently,  $\mathbf{r} = x$  for  $n = 1$ .

The first step is to eliminate time from the equation. To achieve this, we shall look for particular solutions of either equation in the form

$$u(\mathbf{r}, t) = T(t)v(\mathbf{r}). \quad (11.1.2)$$

Inserting (11.1.2) into (11.1.1) we obtain

$$\frac{T'}{T} = \frac{\Delta v}{v}, \quad \text{or} \quad \frac{T''}{T} = \frac{\Delta v}{v}, \quad (11.1.3)$$

depending on whether we have diffusion or wave equation. Since the left-hand side of each equation depends only on  $t$  and the right-hand side only on  $\mathbf{r}$ , each side must be constant. In other words, to have the solution in the form (11.1.2), it is necessary for  $v$  to be a non-zero solution of the problem

$$\begin{aligned} -\Delta v &= \lambda v, \quad \text{in } \Omega \\ v &\text{ satisfies the stipulated boundary condition on } \partial\Omega \end{aligned} \quad (11.1.4)$$

for some constant  $\lambda$  (minus sign was introduced for later convenience). If this is the case, we say that this  $\lambda$  is an *eigenvalue*, and the corresponding solution  $v_\lambda$  is an *eigenfunction* of the problem. Note that the eigenfunction is never unique - any scalar multiple of  $v_\lambda$  is again an eigenfunction. Moreover, if there are more than one linearly independent eigenfunctions, then from the linearity of the problem it follows that any function belonging to the linear span of these eigenfunctions is again an eigenfunction, that is, all the eigenfunctions corresponding to the same eigenvalue form a linear subspace called the *eigenspace* corresponding to this eigenvalue. The dimension of this eigenspace is called the *multiplicity* of the corresponding eigenvalue.

Before we proceed any further we prove the following property of eigenvalues.

**Theorem 1.1** *All the eigenvalues of the problem (11.1.4) are real and*

- (i) *positive in the Dirichlet case and in the Robin case with nonvanishing and nonnegative function  $a$ ,*
- (ii) *non-negative in the Neumann case.*

*In the last case the eigenfunctions corresponding to the eigenvalue  $\lambda = 0$  are constant.*

**Proof.** To prove that the eigenvalues are all real numbers we note that since  $\Delta u = u_{xx} + u_{yy} + u_{zz} = \bar{u}_{xx} + \bar{u}_{yy} + \bar{u}_{zz} = \Delta \bar{u}$ , we have  $-\Delta u = \lambda u$  if and only if  $-\Delta \bar{u} = \bar{\lambda} \bar{u}$ . Moreover, it follows that if  $u$  satisfies either of the boundary conditions under considerations, then so does  $\bar{u}$  and therefore if  $u$  satisfies either Dirichlet or Neumann boundary conditions, then directly

$$\int_{\partial\Omega} \left( u \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial u}{\partial n} \right) dS = 0$$

and if  $u$  satisfies the Robin boundary conditions, then

$$\int_{\partial\Omega} \left( u \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial u}{\partial n} \right) dS = \int_{\partial\Omega} (-au\bar{u} + a\bar{u}u) dS = 0.$$

Thus, using Green's second identity with  $v = \bar{u}$  we obtain

$$\begin{aligned} 0 &= \int_{\partial\Omega} \left( u \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial u}{\partial n} \right) dS = \int_{\Omega} (u \Delta \bar{u} - \bar{u} \Delta u) d\mathbf{r} \\ &= (\lambda - \bar{\lambda}) \int_{\Omega} |u|^2 d\mathbf{r}. \end{aligned}$$

Since by definition the eigenfunction is not identically equal to zero, we must have  $\lambda - \bar{\lambda} = 0$  which yields that  $\text{Im} \lambda = 0$ .

To prove that the eigenvalues are nonnegative, we shall use Green's first identity (10.2.1) with  $u = v$

$$-\int_{\Omega} \bar{v} \cdot \Delta v d\mathbf{r} = -\int_{\partial\Omega} v \frac{\partial \bar{v}}{\partial n} dS + \int_{\Omega} |\nabla v|^2 d\mathbf{r}.$$

Using  $-\Delta v = \lambda v$  and Dirichlet boundary condition  $v = 0$  on  $\partial\Omega$ , we obtain

$$\lambda \int_{\Omega} |v|^2 d\mathbf{r} = \int_{\Omega} |\nabla v|^2 d\mathbf{r}. \quad (11.1.5)$$

However, we must have  $\int_{\Omega} |\nabla v|^2 d\mathbf{r} > 0$ . Otherwise, we would have  $|\nabla v| = 0$ , thus  $v = \text{constant}$ , but because  $v = 0$  on the boundary,  $v = 0$  throughout  $\Omega$ . This is impossible by the definition of an eigenfunction. Therefore, dividing (11.1.5) by  $\int_{\Omega} |v|^2 d\mathbf{r} > 0$ , we get  $\lambda > 0$ .

For the Neumann boundary condition we use  $\partial v / \partial n = 0$  on the boundary to obtain then same equation (11.1.5) for  $\lambda$ . However, in this case it is possible that  $v = \text{const}$  in  $\Omega$  and we obtain that it is possible to have zero eigenvalue; the corresponding eigenfunctions are then constant functions.

For Robin boundary conditions we use  $\partial v / \partial n = -av$  on the boundary to obtain

$$\lambda \int_{\Omega} |v|^2 d\mathbf{r} = \int_{\Omega} |\nabla v|^2 d\mathbf{r} + \int_{\partial\Omega} a|v|^2 dS. \quad (11.1.6)$$

From (11.1.6) we see that eigenvalues are non-negative. However, for  $\lambda$  to be zero we must have both terms on the right-hand side equal to zero. Vanishing of the first term implies as above that  $v = \text{const}$  in  $\Omega$  but then the second term equals  $|v|^2 \int_{\partial\Omega} a dS$  and as  $a$  is a nonnegative and nonvanishing function, this term is zero if and only if  $v \equiv 0$  which contradicts the definition of an eigenfunction. ■

Let us now suppose that, for some  $\lambda \geq 0$ , there exists an eigenfunction  $v_{\lambda}$ . Then the time-dependent part of (11.1.2) can be determined from (11.1.3). We have the following equations

$$T' = -\lambda DT, \quad \text{or} \quad T'' = \lambda c^2 T \quad (11.1.7)$$

depending on the case. Because we know that  $\lambda \geq 0$ , we can provide explicit solutions for the time-dependent part of the separated solution (11.1.2) in the form

$$T_{\lambda}(t) = A_{\lambda} e^{-D\lambda t} \quad (11.1.8)$$

for the diffusion equation, and

$$T_{\lambda}(t) = A_{\lambda} \sin(c\sqrt{\lambda}t) + B_{\lambda} \cos(c\sqrt{\lambda}t). \quad (11.1.9)$$

Thus, we obtain a particular solution

$$u_{\lambda}(\mathbf{r}, t) = A_{\lambda} e^{-D\lambda t} v_{\lambda}(\mathbf{r}) \quad (11.1.10)$$

for the diffusion equation, and

$$u_{\lambda}(\mathbf{r}, t) = (A_{\lambda} \sin(c\sqrt{\lambda}t) + B_{\lambda} \cos(c\sqrt{\lambda}t)) v_{\lambda}(\mathbf{r}). \quad (11.1.11)$$

Note that the knowledge about the properties of the eigenvalue  $\lambda$  allowed to recover the well-known properties of the solutions for large times: exponential decay in the case of the diffusion (for  $\lambda > 0$ ) and periodic motion for the wave equation.

Functions defined by (11.1.10) and (11.1.11) satisfy respective equations and appropriate boundary conditions. Clearly, we cannot expect them to satisfy an arbitrarily prescribed initial conditions - the only initial conditions available are the scalar multiples of the eigenfunction  $v_{\lambda}$ .

This latter observation shows a way forward. If we have more than one eigenvalue, say,  $\lambda_1, \dots, \lambda_k$  with corresponding eigenfunctions  $v_1, \dots, v_k$ , then, due to the linearity of the problem, any linear combination of the functions

$$u_k(\mathbf{r}, t) = \sum_{n=1}^k A_n e^{-D\lambda_n t} v_n(\mathbf{r}), \quad (11.1.12)$$

and

$$u_k(\mathbf{r}, t) = \sum_{n=1}^k (A_n \cos(c\sqrt{\lambda_n}t) + B_n \sin(c\sqrt{\lambda_n}t)) v_n(\mathbf{r}), \quad (11.1.13)$$

satisfies the respective equation with the prescribed boundary conditions. As far as the initial conditions are concerned, by properly choosing the constants  $A_n$  and  $B_n$  we can make these solutions satisfy all the initial conditions belonging to the linear span of the available eigenfunctions.

Clearly this is not enough but the next step is more subtle. We know that there are infinite sets of functions, like  $\{\sin n\pi x\}_{n \in \mathbb{N}}$  or  $\{\cos n\pi x\}_{n \in \mathbb{N}}$ , such that any "reasonable" function  $f$  can be represented as the infinite

linear combination of functions from such a set. Without going into details, we will say that the set of functions  $\{v_n\}_{n \in \mathbb{N}}$  is *complete* if any "reasonable" (e.g. continuous) function defined on  $\Omega$  can be written uniquely as the infinite linear combination

$$f(\mathbf{r}) = \sum_{n=1}^{\infty} a_n v_n(\mathbf{r}). \quad (11.1.14)$$

Here we assume at least pointwise convergence of the series in (11.1.14).

From now on we will be assuming that the eigenvalue problem (11.1.4) has a countable number of eigenvalues  $\{\lambda_n\}_{n \in \mathbb{N}}$  such that the corresponding set of eigenfunctions  $\{v_n(\mathbf{r})\}_{n \in \mathbb{N}}$  is complete. Thus, carrying out the plan outlined above we consider infinite series

$$u(\mathbf{r}, t) = \sum_{n=1}^{\infty} A_n e^{-D\lambda_n t} v_n(\mathbf{r}), \quad (11.1.15)$$

and

$$u(\mathbf{r}, t) = \sum_{n=1}^{\infty} (A_n \cos(c\sqrt{\lambda_n}t) + B_n \sin(c\sqrt{\lambda_n}t)) v_n(\mathbf{r}). \quad (11.1.16)$$

The presence of infinity in the above formulas makes things quite complicated. The infinite sum of continuous functions is not necessarily continuous and therefore even if each term of the series satisfies the boundary conditions, the sum not necessarily does so (for example it is possible to expand function  $f(x) = 1$  on  $[0, 1]$  into a series of sine functions which have the property that each is equal to zero at  $x = 0$ .) Similarly, infinite sum of differentiable functions need not be differentiable, thus even if each term of the series satisfies the diffusion or wave equation, the series may be not differentiable and therefore will fail to be the solution of the equation.

These problems as well as the question of completeness of the system of eigenfunctions are beyond the scope of this course. Our aim is merely to provide tools needed to construct the formal solution - that is to find an infinite series representation of the function which is the genuine solution if all the theoretical requirements mentioned above are met.

Having accepted this convention we are faced with the task of determining the coefficients  $A_n$  and  $B_n$  so that the initial values are satisfied. Therefore putting  $t = 0$  in (11.1.15) and (11.1.16) we obtain

$$\phi(\mathbf{r}) = u(\mathbf{r}, 0) = \sum_{n=1}^{\infty} A_n v_n(\mathbf{r}), \quad (11.1.17)$$

in the case of the diffusion equation, and

$$\begin{aligned} \phi(\mathbf{r}) &= u(\mathbf{r}, 0) = \sum_{n=1}^{\infty} A_n v_n(\mathbf{r}) \\ \psi(\mathbf{r}) &= u_t(\mathbf{r}, 0) = \sum_{n=1}^{\infty} c\sqrt{\lambda_n} B_n v_n(\mathbf{r}). \end{aligned} \quad (11.1.18)$$

This problem is similar to that of linear algebra - to find the coefficients of the expansion of a given vector in a prescribed base. In general, even in a finite dimensional case the problem is not easy. It becomes, however, much simpler if the base is orthonormal, that is, vectors are mutually orthogonal and of unit length. It turns out that in the space of functions one can also define an operation which is analogous to the dot product of  $\mathbb{R}^n$  and which can be used to introduce working definitions of the orthogonality and the norm of a function.

**Definition 1.1** Let  $f$  and  $g$  be continuous functions on  $\Omega \subset \mathbb{R}^n$ . We define the dot product of  $f$  and  $g$  by

$$(f, g) = \int_{\Omega} f(\mathbf{r}) \overline{g(\mathbf{r})} d\mathbf{r} \quad (11.1.19)$$

We say that  $f$  and  $g$  are orthogonal, if

$$(f, g) = \int_{\Omega} f(\mathbf{r}) \overline{g(\mathbf{r})} d\mathbf{r} = 0. \quad (11.1.20)$$

The number

$$\|f\| = \sqrt{\int_{\Omega} |f(\mathbf{r})|^2 d\mathbf{r}} \quad (11.1.21)$$

is called the norm of the function  $f$ . We say that the set of functions  $\{v_n(\mathbf{r})\}_{n \in \mathbb{N}}$  is orthogonal if  $v_k$  is orthogonal to  $v_j$  for any  $k, j \in \mathbb{N}$  such that  $k \neq j$ . The set  $\{v_n(\mathbf{r})\}_{n \in \mathbb{N}}$  is said to be orthonormal if it is orthogonal and  $\|v_k\| = 1$  for each  $k \in \mathbb{N}$ .

**Example 1.1** Possibly the best known systems of orthogonal functions are systems of trigonometric functions  $\{\sin n\pi x\}_{1 \leq n < \infty}$  or  $\{\cos n\pi x\}_{0 \leq n < \infty}$ . Taking into account that

$$\|\sin n\pi x\|^2 = \int_0^1 \sin^2 n\pi x dx = \frac{1}{2} \int_0^1 (1 - \cos 2n\pi x) dx = \frac{1}{2},$$

and similarly

$$\|\cos n\pi x\|^2 = \frac{1}{2},$$

for  $n > 1$  and

$$\|\cos 0\pi x\|^2 = \|1\|^2 = 1,$$

we obtain that

$$\left\{ \frac{1}{\sqrt{2}} \sin \pi x, \frac{1}{\sqrt{2}} \sin 2\pi x, \dots, \frac{1}{\sqrt{2}} \sin n\pi x, \dots \right\}$$

and

$$\{1, \cos \pi x, \cos 2\pi x, \dots, \cos n\pi x, \dots\},$$

are orthonormal systems of functions.

Fortunately, it turns out that any system of eigenfunctions of the Laplace equation with Dirichlet, Neumann or Robin boundary conditions is (or, more precisely can be made) orthogonal. We prove the following theorem.

**Theorem 1.2** *The eigenfunctions of the problem (11.1.4) corresponding to different eigenvalues are orthogonal. Moreover, all the eigenfunctions may be chosen to be real functions and orthogonal to each other.*

**Proof.** First, we shall show that all the eigenfunctions can be chosen to be real functions. In fact, if a complex valued function  $v$  is an eigenfunction corresponding to the eigenvalue  $\lambda$ , that is,  $-\Delta v = \lambda v$ , then due to Theorem 1.1,  $\lambda$  is a real number and therefore also the complex conjugate  $\bar{v}$  is an eigenfunction corresponding to the same eigenvalue:  $-\Delta \bar{v} = \lambda \bar{v}$ . Thus, the eigenspace corresponding to such a  $\lambda$  is at least two-dimensional. However, taking as new eigenfunctions  $v_1 = (v + \bar{v})/2 = \operatorname{Re} v$  and  $v_2 = (v - \bar{v})/2i = \operatorname{Im} v$  we obtain two new eigenfunctions (since  $v_1$  and  $v_2$  are linear combinations of the old ones) which are also linearly independent. In fact, let for some  $\alpha, \beta$

$$0 = \alpha v_1 + \beta v_2 = \frac{i\alpha + \beta}{2i} v + \frac{i\alpha - \beta}{2i} \bar{v}.$$

Since  $v$  and  $\bar{v}$  are linearly independent (provided  $\operatorname{Im} v \neq 0$ ), we must have

$$\begin{aligned} i\alpha + \beta &= 0 \\ i\alpha - \beta &= 0. \end{aligned}$$

The discriminant of this system is equal to  $-2i \neq 0$ , so the only solution is  $\alpha = \beta = 0$ , and  $v_1$  and  $v_2$  are linearly independent. Thus this two real eigenfunctions  $v_1$  and  $v_2$  span the same eigenspace as the eigenfunctions  $v$  and  $\bar{v}$  and therefore can be used instead of them. Continuing, we can replace all the complex

valued eigenfunctions (which necessarily appear in pairs - each one together with its complex conjugate) by real valued eigenfunctions corresponding to the same real eigenvalues without altering their span.

From now on we shall assume that all the eigenfunctions are real-valued. Let us recall Green's second identity (10.3.1)

$$\int_{\Omega} (u\Delta v - v\Delta u) d\mathbf{r} = \int_{\partial\Omega} \left( u \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v \right) dS$$

which is valid for any (twice differentiable) functions  $u, v$ . Let  $u = v_k$  and  $v = v_i$  be eigenfunctions corresponding to different eigenvalues  $\lambda_k \neq \lambda_i$ , that is,  $-\Delta v_k = \lambda_k v_k$  and  $-\Delta v_i = \lambda_i v_i$ . Then, if we consider Dirichlet or Neumann boundary conditions, then the right-hand side of Green's formula vanishes automatically. If we deal with the Robin conditions:  $\partial v_l / \partial n = -av_l$  for  $l = k, i$ , then

$$\int_{\partial\Omega} \left( u \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v \right) dS = \int_{\partial\Omega} (-av_k v_i + av_k v_i) dS = 0.$$

Thus always we have

$$0 = \int_{\Omega} (v_k \Delta v_i - v_i \Delta v_k) d\mathbf{r} = (\lambda_k - \lambda_i) \int_{\Omega} v_i v_k d\mathbf{r}$$

which shows that if  $\lambda_i \neq \lambda_k$ , then  $v_i, v_k$  are orthogonal.

In the last statement we claim that all the eigenfunctions can be made orthogonal. Note that this is not covered by the first statement of the theorem. The reason is that we may have several eigenfunctions corresponding to the same eigenvalue and in such a case the proof of the first statement fails (recall that the last step required dividing by  $\lambda_k - \lambda_i \neq 0$ .) Let us then suppose that  $v^{(1)}, v^{(2)}, \dots, v^{(r)}$  is the basis for the eigenspace corresponding to the same eigenvalue (it can be proved that to each eigenvalue of the problem (11.1.4) can correspond at most finite number of eigenfunctions, that is, the multiplicity of each eigenvalue is finite). Using the well-known Gramm-Schmidt procedure from linear algebra (with respect to the dot product defined by (11.1.19)) we can transform any finite basis of a subspace into an orthonormal basis for the same subspace. The procedure is algorithmic: we define the first element of the orthonormal basis by  $u^{(1)} = v^{(1)} / \|v^{(1)}\|$ , and next we subtract from  $v^{(2)}$  the orthogonal projection of  $v^{(2)}$  onto span of  $u^{(1)}$ , that is, we define

$$\tilde{u}^{(2)} = v^{(2)} - (v^{(2)}, u^{(1)}) u^{(1)}$$

and to obtain the second element of the basis we normalize:  $u^{(2)} = \tilde{u}^{(2)} / \|\tilde{u}^{(2)}\|$ . In the next step we subtract from  $v^{(3)}$  its orthogonal projection onto the linear span of  $u^{(1)}$  and  $u^{(2)}$  and again normalize the result to obtain  $u^{(3)}$ . The procedure continues in this way until all the elements of the original basis are exhausted; then that we are left with an orthonormal basis spanning exactly the same space as  $v^{(1)}, v^{(2)}, \dots, v^{(r)}$ . ■

According to the theorem we may assume that the system  $\{v_k\}_{k \in \mathbb{N}}$  of eigenfunction is orthonormal. Note that it is possible that in the corresponding sequence of eigenvalues  $\{\lambda_k\}_{k \in \mathbb{N}}$  the same numerical value of  $\lambda_k$  may correspond to several different  $k$  - this is due to the fact that in the set of eigenfunctions we may have several eigenfunction corresponding to the same eigenvalue.

The theorem provides us with a constructive way of finding the coefficients  $A_n$  and  $B_n$  in (11.1.17) and (11.1.18). In fact, multiplying both sides of (11.1.17) with  $v_k$  and integrating with over  $\Omega$  we obtain

$$\begin{aligned} \int_{\Omega} \phi(\mathbf{r}) v_k(\mathbf{r}) d\mathbf{r} &= \sum_{n=1}^{\infty} A_n \int_{\Omega} v_n(\mathbf{r}) v_k(\mathbf{r}) d\mathbf{r} \\ &= A_k \cdot 1, \end{aligned}$$

thanks to the fact that the eigenfunctions are orthonormal:

$$\int_{\Omega} v_k(\mathbf{r}) v_n(\mathbf{r}) d\mathbf{r} = \begin{cases} 0 & \text{for } n \neq k \\ 1 & \text{for } n = k. \end{cases}$$



Thus, in the case of the diffusion equation we obtain

$$A_n = \int_{\Omega} \phi(\mathbf{r}) v_n(\mathbf{r}) d\mathbf{r}. \quad (11.1.22)$$

Analogous procedure applied to (11.1.18) gives

$$\begin{aligned} A_n &= \int_{\Omega} \phi(\mathbf{r}) v_n(\mathbf{r}) d\mathbf{r} \\ B_n &= \frac{1}{c\sqrt{\lambda_k}} \int_{\Omega} \phi(\mathbf{r}) v_k(\mathbf{r}) d\mathbf{r} \end{aligned} \quad (11.1.23)$$

Summarizing, the (formal) solution to the diffusion equation is given by

$$u(\mathbf{r}, t) = \sum_{n=1}^{\infty} A_n e^{-D\lambda_n t} v_n(\mathbf{r}), \quad (11.1.24)$$

where the coefficients  $A_n$  are given by (11.1.22), and the (formal) solution to the wave equation is given by

$$u(\mathbf{r}, t) = \sum_{n=1}^{\infty} (A_n \cos(c\sqrt{\lambda_n} t) + B_n \sin(c\sqrt{\lambda_n} t)) v_n(\mathbf{r}). \quad (11.1.25)$$

with coefficients  $A_n$  and  $B_n$  given by (11.1.23).

Of course, any practical value of these formulas depend on the possibility of finding the eigenfunctions in an explicit form. The remaining part of this Lecture is devoted to this topic and also to the discussion of some variations of (11.1.24) and (11.1.25).

**Example 1.2** Let us solve the following initial-boundary value problem in the unit square  $\Omega = \{(x, y); 0 < x < 1, 0 < y < 1\}$

$$\begin{aligned} u_t &= \Delta u, \quad \text{in } \Omega, \\ u(x, y, 0) &= xy, \\ u(x, y, t) &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Separating time and spatial variables as in (11.1.2) we obtain the eigenvalue problem

$$\begin{aligned} -\Delta v &= \lambda v, \quad \text{in } \Omega, \\ v(x, y) &= 0, \quad \text{on } \partial\Omega. \end{aligned} \quad (11.1.26)$$

To solve this problem we once again apply the separation of variables technique and write  $v$  as the product

$$v(x, y) = X(x)Y(y).$$

Inserting such a function to the equation in (11.1.26) and reasoning exactly as when getting (11.1.3), we obtain

$$-\frac{X''}{X} = \frac{Y''}{Y} + \lambda = \gamma$$

for some constant  $\gamma$ . Taking into account the boundary condition, we arrive at two eigenvalue problems for ordinary differential equations

$$\begin{aligned} X'' + \gamma X &= 0, \\ X(0) = X(1) &= 0, \end{aligned}$$

and

$$\begin{aligned} Y'' + \mu Y &= 0, \\ Y(0) = Y(1) &= 0, \end{aligned} \quad (11.1.27)$$

where  $\mu = \lambda - \gamma$ . Although it is well-known, for the sake of completeness we shall repeat the argument leading to the solution of these problems.

Using e.g. formula (9.1.7) we obtain the general solution to the second order equation

$$X(x) = \tilde{C}_1 e^{\sqrt{\gamma}x} + \tilde{C}_2 e^{-\sqrt{\gamma}x}.$$

From the boundary conditions we obtain

$$\begin{aligned} \tilde{C}_1 + \tilde{C}_2 &= 0 \\ \tilde{C}_1 e^{\sqrt{\gamma}} + \tilde{C}_2 e^{-\sqrt{\gamma}} &= 0. \end{aligned}$$

For a homogeneous system of equations to have a non-zero solution it is necessary and sufficient that

$$e^{-\sqrt{\gamma}} - e^{\sqrt{\gamma}} = 0$$

which gives

$$e^{2\sqrt{\gamma}} = 1 = e^{i2n\pi}$$

for  $n \in \mathbb{Z}$ . Thus we obtain the sequence of eigenvalues

$$\gamma_n = -n^2\pi^2, \quad (11.1.28)$$

consequently we have

$$X(x) = \tilde{C}_1 e^{in\pi x} - \tilde{C}_1 e^{-in\pi x} = 2i\tilde{C}_1 \sin n\pi x,$$

and the sequence of real-valued eigenfunctions is given by

$$\tilde{X}_n(x) = \sin n\pi x$$

Since each function  $\sin n\pi x$  corresponds to a different eigenvalue  $-n^2\pi^2$ , we know that

$$\int_0^1 \sin n\pi x \sin k\pi x dx = 0$$

for  $n \neq k$ . On the other hand, for  $n = k$  we have

$$\|\sin n\pi x\|^2 = \int_0^1 \sin^2 n\pi x dx = \frac{1}{2} \int_0^1 (1 - \cos 2n\pi x) dx = \frac{1}{2},$$

thus the orthonormal system of eigenfunctions is given by  $\{\sqrt{2}\sin n\pi x\}_{n \geq 1}$ .

Having done that, we see that as the eigenvalues for (11.1.27) are given similarly by

$$\mu_m = -m^2\pi^2,$$

with eigenfunctions defined as

$$Y_m(y) = \sin m\pi y,$$

for  $m \geq 1$ .

We check that  $v(x, y) = \sin n\pi x \sin m\pi y$  is an eigenfunction of (11.1.26) corresponding to the eigenvalue  $\lambda = \pi^2(m^2 + n^2)$ . In fact

$$\Delta v = \sin m\pi y \frac{\partial^2}{\partial x^2} \sin n\pi x + \sin n\pi x \frac{\partial^2}{\partial y^2} \sin m\pi y = -(n^2\pi^2 + m^2\pi^2) \sin n\pi x \sin m\pi y = -\pi^2(n^2 + m^2)v.$$

It is important to note that in this case we have multiple eigenvalues – to each eigenvalue  $\lambda$  there correspond all the functions  $\sin n\pi x \sin m\pi y$  for which  $m^2 + n^2 = \lambda/\pi^2$  (with  $n, m \in \mathbb{N}$ ). For example,  $v_{21} = \sin 2\pi x \sin \pi y$  and  $v_{12} = \sin \pi x \sin 2\pi y$  are two (and only two) independent functions corresponding to the eigenvalue  $5\pi^2$  and this eigenvalue has multiplicity two. The next eigenvalue  $(2^2 + 2^2)\pi^2$  is single, and yet the next  $(3^2 + 1^2)\pi^2$  is again double.

Fortunately, in this case each two distinct eigenfunctions, as found above, are orthogonal, even when they belong to the same eigenspace so that there is no need to carry out the Gram-Schmidt orthogonalization.

Yet another remark pertains to the notation. Typically, in concrete cases, like this one, the numbering of eigenvalues and eigenfunctions with a single number (as we did in our general considerations) is not very convenient. In fact, to be in the agreement with the convention explained after the proof of Theorem 1.2 we should align the eigenfunctions as follows

$$\{\sin \pi x \sin \pi y, \sin 2\pi x \sin \pi y, \sin \pi x \sin 2\pi y, \dots\}$$

and define  $v_1(x, y) = \sin \pi x \sin \pi y$ ,  $v_2(x, y) = \sin 2\pi x \sin \pi y$ ,  $v_3(x, y) = \sin \pi x \sin 2\pi y$ , etc. and then the sequence of eigenvalues would look like

$$\{2\pi^2, 5\pi^2, 5\pi^2, \dots\}$$

that is each eigenvalue would be repeated in the sequence as many times as is its multiplicity. Clearly, such an arrangement is not the easiest to handle and it turns out that it is much more convenient to use multiple (in this case double) indexes. That is, we consider the set of eigenfunctions

$$\{v_{kl}(x, y)\}_{k,l \geq 1} = \{\sin k\pi x \sin l\pi y\}_{k,l \geq 1}$$

with the corresponding double sequence of eigenvalues

$$\{\lambda_{kl}\}_{k,l \geq 1} = \{(k^2 + l^2)\pi^2\}_{k,l \geq 1}.$$

Since

$$\begin{aligned} \|v_{kl}\|^2 &= \int_{\Omega} v_{kl}(x, y) dx dy = \int_0^1 \int_0^1 \sin k\pi x \sin l\pi y dx dy = \left( \int_0^1 \sin k\pi x dx \right) \left( \int_0^1 \sin l\pi y dy \right) \\ &= \frac{1}{4} \end{aligned}$$

the orthonormal set of eigenfunctions is given by

$$\{2 \sin k\pi x \sin l\pi y\}_{k,l \geq 1}. \quad (11.1.29)$$

The solution of the problem is then given by

$$u(\mathbf{r}, t) = 2 \sum_{k,l=1}^{\infty} A_{kl} e^{-t(k^2+l^2)\pi^2} \sin k\pi x \sin l\pi y$$

where

$$\begin{aligned} A_{kl} &= 2 \int_0^1 \int_0^1 xy \sin k\pi x \sin l\pi y dx dy = 2 \left( \int_0^1 x \sin k\pi x dx \right) \left( \int_0^1 y \sin l\pi y dy \right) \\ &= \frac{2(-1)^{k+l}}{\pi^2 kl}. \end{aligned}$$

Thus

$$u(\mathbf{r}, t) = \frac{4}{\pi^2} \sum_{k,l=1}^{\infty} \frac{(-1)^{k+l}}{kl} e^{-t(k^2+l^2)\pi^2} \sin k\pi x \sin l\pi y.$$

## 2 Nonhomogeneous problems

In this section we shall discuss how to use the technique of separation of variables to solve boundary value problem for nonhomogeneous equations with possibly nonhomogeneous boundary data.

We shall start with nonhomogeneous the diffusion equation with zero boundary data. To shorten the exposition, we restrict ourselves to the case of Dirichlet boundary conditions, the others being analogous.

### 2.1 Nonhomogeneous diffusion equation

Let us consider the initial-boundary value problem

$$\begin{aligned} u_t &= D\Delta u + f(\mathbf{r}, t), \quad \text{in } \Omega \\ u(\mathbf{r}, 0) &= \phi(\mathbf{r}), \\ u(\mathbf{r}, t) &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (11.2.1)$$

We cannot apply the separation of variable technique as, in general,  $f$  contains both variables. To solve the problem we shall use the same approach as for nonhomogeneous ordinary differential equations. We shall first find a general solution of the homogeneous equation and, using the postulated completeness of eigenfunctions of the problem, we shall expand the right-hand side and the initial value converting the time dependent part of the problem (11.2.1) into an infinite system of nonhomogeneous ordinary differential equations corresponding to (11.1.7) for the homogeneous case.

So, let  $\{v_k(\mathbf{r})\}_{k \in \mathbb{N}}$  be the complete set of orthonormal eigenfunctions of the problem

$$\begin{aligned} -\Delta u &= \lambda u, \quad \text{in } \Omega \\ u(\mathbf{r}) &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Then we can write

$$\begin{aligned} f(\mathbf{r}, t) &= \sum_{k=1}^{\infty} f_k(t) v_k(\mathbf{r}), \\ \phi(\mathbf{r}) &= \sum_{k=1}^{\infty} \phi_k v_k(\mathbf{r}), \end{aligned} \quad (11.2.2)$$

where  $f_k(t) = \int_{\Omega} f(\mathbf{r}, t) v_k(\mathbf{r}) d\mathbf{r}$  and  $\phi_k = \int_{\Omega} \phi(\mathbf{r}) v_k(\mathbf{r}) d\mathbf{r}$ . We also envisage the solution to be in the form by

$$u(\mathbf{r}, t) = \sum_{k=1}^{\infty} u_k(t) v_k(\mathbf{r}),$$

where  $u_k$  are to be determined. Inserting this formula into the diffusion equation, we obtain

$$\begin{aligned} u_t(\mathbf{r}, t) &= \sum_{k=1}^{\infty} u_{k,t}(t) v_k(\mathbf{r}) = \sum_{k=1}^{\infty} u_k(t) \Delta v_k(\mathbf{r}) + \sum_{k=1}^{\infty} f_k(t) v_k(\mathbf{r}) \\ &= \sum_{k=1}^{\infty} (-\lambda_k u_k(t) + f_k(t)) v_k(\mathbf{r}), \end{aligned}$$

and

$$u(\mathbf{r}, 0) = \sum_{k=1}^{\infty} u_k(0) v_k(\mathbf{r}) = \sum_{k=1}^{\infty} \phi_k v_k(\mathbf{r}) \quad (11.2.3)$$

which translates into the infinite system of nonhomogeneous ordinary linear differential equations

$$\begin{aligned} u_{k,t} &= -\lambda_k u_k + f_k, \\ u_k(0) &= \phi_k. \end{aligned} \quad (11.2.4)$$

These equations can be explicitly solved and we get

$$u_k(t) = \phi_k e^{-\lambda_k D t} + e^{-\lambda_k D t} \int_0^t e^{\lambda_k D s} f_k(s) ds. \quad (11.2.5)$$

Thus the final solution is given by the formula

$$u(\mathbf{r}, t) = \sum_{k=1}^{\infty} e^{-\lambda_k D t} \left( \phi_k + \int_0^t e^{\lambda_k D s} f_k(s) ds \right) v_k(\mathbf{r}). \quad (11.2.6)$$

Note that this solution can be written as a sum of two parts:

$$u(\mathbf{r}, t) = u^{(0)} + u^{(1)}$$

where

$$u^{(0)}(\mathbf{r}, t) = \sum_{k=1}^{\infty} e^{-\lambda_k D t} \phi_k v_k(\mathbf{r})$$

which, according to (11.1.24) is the solution to the homogeneous equation satisfying the initial condition  $u^{(0)}(\mathbf{r}, 0) = \phi(\mathbf{r})$ , and

$$u^{(1)}(\mathbf{r}, t) = \sum_{k=1}^{\infty} \left( e^{-\lambda_k D t} \int_0^t e^{\lambda_k D s} f_k(s) ds \right) v_k(\mathbf{r})$$

which gives the solution of the nonhomogeneous equation with zero initial condition.

## 2.2 Nonhomogeneous wave equation

Let us now consider Dirichlet problem for the wave equation

$$\begin{aligned} u_{tt} &= c^2 \Delta u + f(\mathbf{r}, t), \quad \text{in } \Omega \\ u(\mathbf{r}, 0) &= \phi(\mathbf{r}), \\ u_t(\mathbf{r}, 0) &= \psi(\mathbf{r}), \\ u(\mathbf{r}, t) &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (11.2.7)$$

As in the previous example we postulate the existence of a complete set of orthonormal eigenfunctions  $\{v_k(\mathbf{r})\}_{k \in \mathbb{N}}$  so that the data can be expanded as

$$\begin{aligned} f(\mathbf{r}, t) &= \sum_{k=1}^{\infty} f_k(t) v_k(\mathbf{r}), \\ \phi(\mathbf{r}) &= \sum_{k=1}^{\infty} \phi_k v_k(\mathbf{r}), \\ \psi(\mathbf{r}) &= \sum_{k=1}^{\infty} \psi_k v_k(\mathbf{r}), \end{aligned} \quad (11.2.8)$$

and we look for the solution in the form

$$u(\mathbf{r}, t) = \sum_{k=1}^{\infty} u_k(t) v_k(\mathbf{r}).$$

Following the same approach from the previous example, we obtain the infinite set of second order ordinary differential equations

$$\begin{aligned} u_{k,tt} &= -c^2 \lambda_k u_k + f_k, \\ u_k(0) &= \phi_k, \\ u_{k,t}(0) &= \psi_k. \end{aligned} \quad (11.2.9)$$

Using the fact that  $\lambda_k \geq 0$ , from formula (9.1.12) we obtain

$$u_k(t) = \phi_k \cos c\sqrt{\lambda_k}t + \frac{1}{c\sqrt{\lambda_k}}\psi_k \sin c\sqrt{\lambda_k}t + \frac{1}{c\sqrt{\lambda_k}} \int_0^t (\sin c\sqrt{\lambda_k}(t-s))f_k(s)ds. \quad (11.2.10)$$

Therefore the solution to the problem is given by

$$u(\mathbf{r}, t) = \sum_{k=1}^{\infty} \left( \phi_k \cos c\sqrt{\lambda_k}t + \frac{1}{c\sqrt{\lambda_k}}\psi_k \sin c\sqrt{\lambda_k}t + \frac{1}{c\sqrt{\lambda_k}} \int_0^t (\sin c\sqrt{\lambda_k}(t-s))f_k(s)ds \right) v_k(\mathbf{r}). \quad (11.2.11)$$

Similarly to the previous example, the solution can be written as the sum of three functions

$$u(\mathbf{r}, t) = u^{(0)} + u^{(1)} + u^{(2)}$$

where

$$u^{(0)}(\mathbf{r}, t) = \sum_{k=1}^{\infty} \phi_k \cos c\sqrt{\lambda_k}t v_k(\mathbf{r})$$

which, according to (11.1.25) is the solution to the homogeneous equation and satisfies the initial conditions  $u^{(0)}(\mathbf{r}, 0) = \phi(\mathbf{r})$  and  $u_t^{(0)}(\mathbf{r}, 0) = 0$ ,

$$u^{(1)}(\mathbf{r}, t) = \sum_{k=1}^{\infty} \frac{\psi_k}{c\sqrt{\lambda_k}} \sin c\sqrt{\lambda_k}t v_k(\mathbf{r})$$

which is the solution to the homogeneous equation and satisfies the initial conditions  $u^{(1)}(\mathbf{r}, 0) = 0$  and  $u_t^{(1)}(\mathbf{r}, 0) = \psi(\mathbf{r})$ , and

$$u^{(2)}(\mathbf{r}, t) = \sum_{k=1}^{\infty} \frac{1}{c\sqrt{\lambda_k}} \left( \int_0^t (\sin c\sqrt{\lambda_k}(t-s))f_k(s)ds \right) v_k(\mathbf{r})$$

which gives the solution of the nonhomogeneous equation with zero initial condition.

## 2.3 Nonhomogeneous boundary problems

Let us consider the following variations of initial-boundary value problems:

$$\begin{aligned} u_t &= D\Delta u, & \text{in } \Omega \\ u(\mathbf{r}, 0) &= \phi(\mathbf{r}), \\ u(\mathbf{r}, t) &= g(\mathbf{r}, t), & \text{on } \partial\Omega, \end{aligned} \quad (11.2.12)$$

for the diffusion equation, and

$$\begin{aligned} u_{tt} &= c^2\Delta u, & \text{in } \Omega \\ u(\mathbf{r}, 0) &= \phi(\mathbf{r}), \\ u_t(\mathbf{r}, 0) &= \psi(\mathbf{r}), \\ u(\mathbf{r}, t) &= g(\mathbf{r}, t), & \text{on } \partial\Omega, \end{aligned} \quad (11.2.13)$$

for the wave equation.

The difference now is that the solution is supposed to be equal to a prescribed function  $g$  on the boundary for all the time. Again, we cannot use the separation of variables technique directly, since the boundary condition is not linear: if  $v_1$  and  $v_2$  are equal to  $g$  at the boundary, then  $v_1 + v_2 = 2g$  on the boundary, instead of the required  $g$ . Both these problems in principle can be transformed to problems with nonhomogeneous

equations and homogeneous boundary conditions using the method outlined in Example 2.4.1. The idea is to find a function  $G(\mathbf{r}, t)$ , defined and sufficiently regular throughout  $\Omega$ , which satisfies  $G(\mathbf{r}, t)|_{\partial\Omega} = g(\mathbf{r}, t)$ . Then we introduce the new function  $w(\mathbf{r}, t) = u(\mathbf{r}, t) - G(\mathbf{r}, t)$  and it is easy to check that  $w$  is the solution to

$$\begin{aligned} w_t &= D\Delta w - G_t + D\Delta G, \quad \text{in } \Omega \\ w(\mathbf{r}, 0) &= \phi(\mathbf{r}) - G(\mathbf{r}, 0), \\ w(\mathbf{r}, t) &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (11.2.14)$$

in the case of the diffusion equation, and

$$\begin{aligned} w_{tt} &= c^2\Delta w - G_{tt} + c^2\Delta G, \quad \text{in } \Omega \\ w(\mathbf{r}, 0) &= \phi(\mathbf{r}) - G(\mathbf{r}, 0), \\ w_t(\mathbf{r}, 0) &= \psi(\mathbf{r}) - G_t(\mathbf{r}, 0), \\ w(\mathbf{r}, t) &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (11.2.15)$$

for the wave equation. Since  $G$  is supposed to be a known function, the problems are reduced to that solved in two previous subsections.

For non-homogeneous Neumann and Robin problems the idea is basically the same - we have to find (by guessing, trial and error, or otherwise) a function such that its behaviour on the boundary coincides with that stipulated for the solution.

As usual, the problem is that finding a suitable  $G$  is usually not easy, especially in higher dimensions. In the following example we shall construct such extensions for the interval  $[0, 1]$ .

**Example 2.1** Let us find a function  $G(x, t)$  defined on  $[0, 1]$  such that

- (i)  $G(0, t) = h_0(t)$  and  $G(1, t) = h_1(t)$ ,
- (ii)  $G_x(0, t) = h_0(t)$  and  $G_x(1, t) = h_1(t)$ ,
- (iii)  $-G_x(0, t) + aG(0, t) = h_0(t)$  and  $G_x(1, t) + bG(1, t) = h_1(t)$ ,  $a, b > 0$ .

(i) For the first case we can treat  $t$  as a parameter, so we need the (simplest) function of  $x$  which takes on prescribed values at  $x = 0$  and  $x = 1$ . There exists a linear in  $x$  functions with such properties and it follows that we can take

$$G(x, t) = h_0(t) + x(h_1(t) - h_0(t)). \quad (11.2.16)$$

(ii) In the second case we note that what we need is a function which derivative behaves according to (11.2.16). Thus we may take

$$G(x, t) = xh_0(t) + \frac{x^2}{2}(h_1(t) - h_0(t)). \quad (11.2.17)$$

(iii) For the Robin case we assume that  $G(x, t) = A(t)x + B(t)$ . Then  $-G_x(0, t) + aG(0, t) = -A(t) + aB(t) = h_0(t)$  and  $G_x(1, t) + bG(1, t) = A(t) + b(A(t) + B(t)) = h_1(t)$ . The determinant is equal to  $-(a + b) - ab \neq 0$  for  $a, b > 0$ , thus the system has a unique solution given by  $A(t) = (ah_1(t) - bh_0(t))/((a + b) + ab)$  and  $B(t) = (h_1(t) + (1 + b)h_0(t))/((a + b) + ab)$ . Hence

$$G(x, t) = \frac{1}{(a + b) + ab} (x(ah_1(t) - bh_0(t)) + (h_1(t) + (1 + b)h_0(t))).$$

Note that in neither case  $G$  is determined in a unique way but having in mind equations (11.2.14) and (11.2.15) we are trying to find possibly simplest functions satisfying required conditions.

In the next example we shall provide a full solution to a problem with non-homogeneous boundary condition.

**Example 2.2** Solve the problem

$$\begin{aligned} u_{tt} &= u_{xx}, & \text{for } 0 < x < 1, t > 0 \\ u(x, 0) &= 0, \\ u_t(x, 0) &= 0, \\ u(0, x) &= t^2, & u(1, t) = 0, & t > 0, \end{aligned}$$

According to the general theory we must find a function  $G(x, t)$  which coincides with the solution at the endpoints. Using the previous example with  $h_0(t) = t^2$  and  $h_1(t) = 0$  we find  $G(x, t) = t^2(1 - x)$ . Thus we define the new function

$$w(x, t) = u(x, t) - t^2(1 - x).$$

This function satisfies the following problem

$$\begin{aligned} w_{tt} &= w_{xx} + 2(x - 1), & \text{for } 0 < x < 1, t > 0 \\ w(x, 0) &= 0, \\ w_t(x, 0) &= 0, \\ w(0, x) &= 0, & w(1, t) = 0, & t > 0, \end{aligned}$$

hence we can use the formula (11.2.11). We know that the normalized eigenfunctions of this problem are given by

$$v_k(x) = \sqrt{2} \sin k\pi x$$

and the eigenvalues are

$$\lambda_k = -k^2\pi^2$$

for  $k \geq 1$ . Moreover,  $\phi_k = \psi_k = 0$  for all  $k \geq 1$  and for  $f_k(t)$  we obtain

$$f_k(t) = 2\sqrt{2} \int_0^1 (x - 1) \sin k\pi x dx = -\frac{2\sqrt{2}}{k\pi}$$

Thus, from (11.2.11) we get

$$\begin{aligned} w(x, t) &= -\frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^t (\sin k\pi(t - s)) ds \sin k\pi x \\ &= \frac{4}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^3} (\cos k\pi t - 1) \sin k\pi x, \end{aligned}$$

and the final solution is given by

$$u(x, t) = t^2(1 - x) + w(x, t) = t^2(1 - x) + \frac{4}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^3} (\cos k\pi t - 1) \sin k\pi x.$$

## 2.4 Laplace equation in the disc

In this subsection we shall show that the method of separation of variables is not restricted to the time-dependent problems. We have already seen such an application in Example 1.2, where this method was used to solve the eigenvalue problem for the Laplace equation in a square. Here we shall apply the method for boundary value problems for the Laplace equation in plane domains which have polar symmetry: interior and exterior of a circle, annulus and sectors.

Throughout this subsection we denote by  $\Omega$  one of the following sets:

- (i)  $\Omega = \{(x, y); x^2 + y^2 < b^2\} = \{(r, \phi); r < b, 0 \leq \phi < 2\pi\}$  (the interior of a disc),



- (ii)  $\Omega = \{(x, y); a^2 < x^2 + y^2 < b^2\} = \{(r, \phi); a < r < b, 0 \leq \phi < 2\pi\}$  (an annulus),  
 (iii)  $\Omega = \{(x, y); b^2 < x^2 + y^2\} = \{(r, \phi); b < r, 0 \leq \phi < 2\pi\}$  (the exterior of the disc).

Let us consider the Laplace equation in  $\Omega$ :

$$\Delta u = 0, \quad \text{in } \Omega, \quad (11.2.18)$$

with one of the following boundary conditions:

(I)

In the case (i):

$$u(b, \phi) = g(\phi), \quad u(r, \phi) \text{ bounded as } r \rightarrow 0^+, \quad (11.2.19)$$

(II)

In the case (ii):

$$\begin{aligned} u(a, \phi) &= g_a(\phi), \\ u(b, \phi) &= g_b(\phi), \end{aligned} \quad (11.2.20)$$

(III)

In the case (iii):

$$u(b, \phi) = g(\phi), \quad u(r, \phi) \text{ bounded as } r \rightarrow +\infty. \quad (11.2.21)$$

Solutions to (I)-(III) can be obtained as particular cases of the general solution to (11.2.18).

Using the Chain Rule we can write the Laplace equation in polar coordinates as

$$r^{-1}(ru_r)_r + r^2 u_{\phi\phi} = 0. \quad (11.2.22)$$

To apply the separation of variables technique, we look for particular solutions in the form

$$u(r, \phi) = R(r)\Phi(\phi). \quad (11.2.23)$$

Inserting into the equation (11.2.22) we obtain

$$\frac{r(rR_r)_r}{R} + \frac{\Phi_{\phi\phi}}{\Phi} = 0$$

which results in both terms being constant and leads to two ordinary differential equations

$$\Phi'' + \lambda\Phi = 0, \quad \text{for } 0 \leq \phi < 2\pi, \quad (11.2.24)$$

and

$$r(rR_r)_r - \lambda R = 0. \quad (11.2.25)$$

Let us consider first Eq. (11.2.24). It is not immediate what kind of boundary conditions one should assign to this equation. To find the proper ones we note that the restriction  $0 \leq \phi < 2\pi$  is artificial as geometrically the point  $(r, \phi)$  coincides with the point  $(r, \phi + 2\pi)$ , thus  $u(r, \phi)$  must be equal to  $u(r, \phi + 2\pi)$ , for any  $\phi$ . This observation shows that  $\Phi$  must be periodic with period  $2\pi$ , that is, we require

$$\Phi(\phi) = \Phi(\phi + 2\pi), \quad \text{for } 0 \leq \phi < 2\pi. \quad (11.2.26)$$

We haven't discussed boundary conditions of this form so far but as we shall see they define a proper eigenvalue problem. To solve it, let us first note that (11.2.26) yields

$$\Phi'(\phi) = \Phi'(\phi + 2\pi), \quad \text{for } 0 \leq \phi < 2\pi,$$

and therefore we shall consider the following eigenvalue problem

$$\begin{aligned}\Phi'' + \lambda\Phi &= 0, \quad \text{for } 0 \leq \phi < 2\pi, \\ \Phi(0) &= \Phi(2\pi) \\ \Phi'(0) &= \Phi'(2\pi).\end{aligned}\tag{11.2.27}$$

Unfortunately, we cannot use the general theory to determine the sign of eigenvalues and we have to start from a scratch. The general solution of (11.2.24) is given by

$$u(\phi) = C_1 e^{\sqrt{-\lambda}\phi} + C_2 e^{-\sqrt{-\lambda}\phi}$$

and the boundary conditions give the system of equations

$$\begin{aligned}C_1 + C_2 &= C_1 e^{\sqrt{-\lambda}2\pi} + C_2 e^{-\sqrt{-\lambda}2\pi} \\ C_1 \sqrt{-\lambda} - C_2 \sqrt{-\lambda} &= \sqrt{-\lambda} C_1 e^{\sqrt{-\lambda}2\pi} - \sqrt{-\lambda} C_2 e^{-\sqrt{-\lambda}2\pi}.\end{aligned}$$

Assuming for a moment that  $\lambda = 0$  is not an eigenvalue, we can divide the second equation by  $\sqrt{-\lambda}$  so that we get the following system

$$\begin{aligned}C_1(1 - e^{\sqrt{-\lambda}2\pi}) + C_2(1 - e^{-\sqrt{-\lambda}2\pi}) &= 0 \\ C_1(1 - e^{\sqrt{-\lambda}2\pi}) - C_2(1 - e^{-\sqrt{-\lambda}2\pi}) &= 0.\end{aligned}$$

The determinant of this system is given by

$$-2(1 - e^{\sqrt{-\lambda}2\pi})(1 - e^{-\sqrt{-\lambda}2\pi})$$

which is zero only if either

$$1 - e^{\sqrt{-\lambda}2\pi} = 0$$

or

$$1 - e^{-\sqrt{-\lambda}2\pi} = 0.$$

From these equations we obtain

$$e^{\pm\sqrt{-\lambda}2\pi} = 1 = e^{i2n\pi}$$

which yields

$$\lambda = n^2$$

where  $n$  is a nonzero integer. Consequently, we obtain the eigenfunctions

$$\tilde{\Phi}_n(\phi) = C_{1n} e^{in\phi} + C_{2n} e^{-in\phi}\tag{11.2.28}$$

and, as in the first part of the proof of Theorem 1.2, the eigenfunction can be written in the real form giving the

$$\{\sin n\phi, \cos n\phi\}_{n=\pm 1, \pm 2, \dots}$$

as the set of eigenfunctions.

Consider now the exceptional case  $\lambda = 0$ . Then the equation (11.2.24) has the solution  $\Phi(\phi) = C_0\phi + C'_0$  which is periodic only when  $C_0 = 0$ . Thus  $\Phi_0$  is constant and is included in the general formula if we allow  $n = 0$ .

Having determined the eigenvalues, we can now solve the equation for  $R$ :

$$r^2 R_{rr} + r R_r - n^2 R = 0.\tag{11.2.29}$$

This is so called Euler equation which can be solved by using trial solutions of the form  $R_\alpha = r^\alpha$ . Inserting such a function into the equation (11.2.29), we obtain

$$r^\alpha(\alpha(\alpha - 1) + \alpha - n^2) = 0$$

which is satisfied if and only if  $\alpha$  is a root of the following quadratic equation

$$\alpha(\alpha - 1) + \alpha - n^2 = 0$$

or simply

$$\alpha^2 = n^2.$$

So, if  $n \neq 0$ , we obtain two distinct solutions

$$\alpha_1 = n, \quad \alpha_2 = -n,$$

where  $n \in \mathbb{N}$ , and consequently

$$R_n(r) = a_n r^n + b_n r^{-n}.$$

If  $n = 0$ , this procedure produces only one solution  $R_0(r) = a_0$ , but then the equation can be integrated directly to give

$$R_0(r) = a_0 + b_0 \ln r.$$

Thus we can write the general solution of the Laplace equation

$$u(r, \phi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\phi + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \sin n\phi. \quad (11.2.30)$$

Now, let us specify the form of the solution so that it can be the solution of the boundary value problems (I)-(III).

Case (I)

In the case (I) we require the solution to be bounded as  $r \rightarrow 0^+$ , which forces  $B_0 = B_n = D_n = 0$  for  $n \geq 1$ . Thus we obtain

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi. \quad (11.2.31)$$

The constants  $A_n$  and  $C_n$  are to be determined by

$$g(\phi) = u(a, \phi) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos n\phi + \sum_{n=1}^{\infty} C_n a^n \sin n\phi$$

where

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi, \\ A_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \cos n\phi d\phi, \\ C_n &= \frac{1}{a^n \pi} \int_0^{2\pi} g(\phi) \sin n\phi d\phi, \end{aligned} \quad (11.2.32)$$

for  $n > 0$ , where we used the orthogonality of all trigonometric functions on  $[0, 2\pi]$  and the normalizing relation

$$\int_0^{2\pi} \cos^2 n\phi d\phi = \int_0^{2\pi} \sin^2 n\phi d\phi = \pi.$$

Case (II)

Case (II) requires the full expansion (11.2.30). The coefficients are to be determined from the infinite system of equations:

$$\begin{aligned} A_0 + B_0 \ln a &= \frac{1}{2\pi} \int_0^{2\pi} g_a(\phi) d\phi, \\ A_0 + B_0 \ln b &= \frac{1}{2\pi} \int_0^{2\pi} g_b(\phi) d\phi, \end{aligned} \quad (11.2.33)$$

and for  $n \geq 1$ ,

$$\begin{aligned} A_n a^n + B_n a^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_a(\phi) \cos n\phi d\phi, \\ A_n b^n + B_n b^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_b(\phi) \cos n\phi d\phi, \end{aligned} \quad (11.2.34)$$

and

$$\begin{aligned} C_n a^n + D_n a^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_a(\phi) \sin n\phi d\phi, \\ C_n b^n + D_n b^{-n} &= \frac{1}{\pi} \int_0^{2\pi} g_b(\phi) \sin n\phi d\phi. \end{aligned} \quad (11.2.35)$$

Note that these equations are solvable as the determinant of each system is equal to  $(a/b)^n - (b/a)^n \neq 0$  for  $n \geq 1$  and  $\ln(b/a) \neq 0$  for  $n = 0$ .

Case (III)

Case (III) is similar to the Case (I) with the difference that the requirement of boundedness of  $u(r, \phi)$  as  $r \rightarrow \infty$  forces  $B_0 = A_n = C_n$  for all  $n \geq 0$ . The solution is then given by

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} B_n r^{-n} \cos n\phi + \sum_{n=1}^{\infty} D_n r^{-n} \sin n\phi. \quad (11.2.36)$$

The constants  $A_n$  and  $C_n$  are defined by

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi, \\ B_n &= \frac{b^n}{\pi} \int_0^{2\pi} g(\phi) \cos n\phi d\phi, \end{aligned} \quad (11.2.37)$$

$$D_n = \frac{b^n}{\pi} \int_0^{2\pi} g(\phi) \sin n\phi d\phi, \quad (11.2.38)$$

for  $n > 1$ .

**Example 2.3** Find the solution to:

$$\begin{aligned} \Delta u &= 0, \quad 1 < r < 2, 0 \leq \phi < 2\pi, \\ u(1, \phi) &= \cos^2 \phi + 1, \\ u(2, \phi) &= 1 + \sin \phi. \end{aligned}$$

To solve the problem we first write  $\cos^2 \phi = \frac{1}{2} + \frac{\cos 2\phi}{2}$ . With this change we use (11.2.30)

$$u(r, \phi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\phi + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \sin n\phi.$$

There is, however, no need to use the integral formulas for coefficients of the expansion, as our data are already given in the form of (finite) Fourier expansion. By comparison we have

$$\frac{3}{2} + \frac{\cos 2\phi}{2} = u(1, \phi) = A_0 + \sum_{n=1}^{\infty} (A_n + B_n) \cos n\phi + \sum_{n=1}^{\infty} (C_n + D_n) \sin n\phi,$$

where we used  $\ln 1 = 0$ . Similarly

$$1 + \sin \phi = u(2, \phi) = A_0 + B_0 \ln 2 + \sum_{n=1}^{\infty} (A_n 2^n + B_n 2^{-n}) \cos n\phi + \sum_{n=1}^{\infty} (C_n 2^n + D_n 2^{-n}) \sin n\phi.$$

By the uniqueness of the Fourier expansion we get the following equations

$$\begin{aligned} A_0 &= \frac{3}{2}, \\ A_0 + B_0 \ln 2 &= 1, \\ C_1 + D_1 &= 0, \\ 2C_1 + 2^{-1}D_1 &= 1, \\ A_2 + B_2 &= \frac{1}{2} \\ 2^2 A_2 + 2^{-2} B_2 &= 0. \end{aligned}$$

All the other coefficients are zero being the solutions of homogeneous linear systems with non-zero determinants. Solving, we obtain

$$A_0 = \frac{3}{2}, B_0 = -\frac{1}{2 \ln 2}, C_1 = \frac{2}{3} D_1 = -\frac{2}{3}, A_2 = -\frac{1}{30}, B_2 = \frac{8}{15}.$$

Thus the solution is given by

$$u(r, \phi) = \frac{3}{2} - \frac{1}{2 \ln 2} \ln r + \frac{2}{3} (r + r^{-1}) \sin \phi + \left( -\frac{1}{30} r^2 + \frac{8}{15} r^{-2} \right) \cos 2\phi.$$

The next example is a little more sophisticated and requires some complex analysis.

**Example 2.4** Find the solution to:

$$\begin{aligned} \Delta u &= 0, \quad r < 1, 0 \leq \phi < 2\pi, \\ u(1, \phi) &= g(\phi), \end{aligned}$$

where

$$g(\phi) = \begin{cases} 1 & \text{for } 0 \leq \phi \leq \pi \\ 0 & \text{for } \pi < \phi < 2\pi. \end{cases}$$

According to the formula (11.2.31) the solution is given by

$$u(r, \phi) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\phi + \sum_{n=1}^{\infty} C_n r^n \sin n\phi,$$

where in our case

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^\pi d\phi = \frac{1}{2}, \\ A_n &= \frac{1}{\pi} \int_0^\pi \cos n\phi d\phi = \frac{1}{n\pi} (\sin n\pi - \sin 0) = 0, \\ C_n &= \frac{1}{\pi} \int_0^\pi \sin n\phi d\phi = -\frac{1}{n\pi} (\cos n\pi - \cos 0) = -\frac{1}{n\pi} ((-1)^n - 1). \end{aligned}$$

The coefficient  $C_n$  can be written in the following way

$$C_n = \begin{cases} 0 & \text{for } n = 2k \\ \frac{2}{(2k+1)\pi} & \text{for } n = 2k+1, \end{cases}$$

where  $k = 0, 1, \dots$  and the solution takes the form

$$u(r, \phi) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \sin(2k+1)\phi. \quad (11.2.39)$$

An interesting thing about the series above is that it can actually be summed. In fact

$$\sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \sin(2k+1)\phi = \operatorname{Im} \sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} (\cos(2k+1)\phi + i \sin(2k+1)\phi) = \operatorname{Im} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}.$$

The last series is quite well-known. In fact, since

$$\begin{aligned} \ln(1-z) &= -\sum_{k=1}^{\infty} \frac{z^k}{k}, \\ \ln(1+z) &= -\sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k}, \end{aligned}$$

we have

$$\ln \frac{1+z}{1-z} = \ln(1+z) - \ln(1-z) = 2 \sum_{k=1}^{\infty} \frac{z^{2k+1}}{2k+1}.$$

We have to calculate  $\operatorname{Im} \ln \frac{1+z}{1-z}$ . To do this, we recall that for complex logarithm we have the formula  $\ln z = \ln|z| + i \arg z$ , so that we have to find  $\arg \frac{1+z}{1-z}$ . In this regard we observe that this is equal to  $\arg(1+z) - \arg(1-z)$ . Next, note that for  $z$  in the circle  $\alpha = \arg(1+z)$  changes from  $-\pi/2$  to  $\pi/2$  and is positive for  $\operatorname{Im} z > 0$  and negative otherwise. Also  $\beta = \arg(1-z)$  changes between  $-\pi/2$  and  $\pi/2$  but  $\beta$  is negative for  $\operatorname{Im} z > 0$  and positive otherwise. Consider first  $\operatorname{Im} z \geq 0$ , then the triangle with vertices  $-1, -1, z$  has angles equal to  $\alpha, -\beta, \pi - (\alpha - \beta)$  at the respective vertices. On the other hand, this triangle has two vertices at the diameter of the unit circle and the third vertex,  $z$ , is inside the circle, hence we must have  $\pi/2 < \pi - (\alpha - \beta) \leq \pi$ , which yields  $0 \leq \alpha - \beta < \pi/2$ . Similarly, for  $\operatorname{Im} z \leq 0$  we obtain  $-\pi/2 < \alpha - \beta \leq 0$ , so in any case for  $z$  belonging to the interior of the circle we have

$$\arg \frac{1+z}{1-z} = \arg(1+z) - \arg(1-z) = \alpha - \beta \in (-\pi/2, \pi/2)$$

and in this range we can use the formula

$$\arg \frac{1+z}{1-z} = \tan^{-1} \frac{\operatorname{Im} \frac{1+z}{1-z}}{\operatorname{Re} \frac{1+z}{1-z}}.$$

Now, reverting to real variables so that  $r = |z|$  and  $Im\,z = r \sin \phi$ ,  $Re\,z = r \cos \phi$ , we have

$$\frac{1+z}{1-z} = \frac{1-r^2 + i2r \sin \phi}{1-2r \cos \phi + r^2}.$$

Thus

$$\arg \frac{1+z}{1-z} = \tan^{-1} \frac{2r \sin \phi}{1-r^2}$$

Returning to the solution of our problem we obtain

$$u(r, \phi) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{2r \sin \phi}{1-r^2}.$$





# Lecture 11

## The hydrogen atom

The motion of an electron in the hydrogen atom is described by the following Schrödinger equation:

$$-i\hbar\psi_t = \frac{\hbar^2}{2m}\Delta\psi + \frac{e^2}{||^2}\psi(\cdot, t), \quad (11.0.1)$$

where  $e$  is the charge of the electron and  $r$  the distance to the proton, which is supposed to be situated at the origin.

We rewrite this equation in the dimensionless form:

$$i\hbar\psi_t = -\frac{1}{2}\Delta\psi - \frac{1}{||^2}\psi(\cdot, t), \quad (11.0.2)$$

and, as in the previous lecture, we will be looking for solution to this equation satisfying the condition

$$\int_{\mathbb{R}^3} |\psi(\cdot, t)|^2 d\mathbf{x} < \infty. \quad (11.0.3)$$

Although this is the whole space problem and we observed that the separation of variables had been unsuccessful in the case of the free Schrödinger equation, we shall try this method here. Writing, as usual,  $\psi(\cdot, t) = T(t)v(\cdot)$ , we obtain

$$2i\frac{T'}{T} = \frac{-\Delta v - 2v/||^2}{v} = \lambda,$$

where  $\lambda$  is a constant. The time factor is easily found to be

$$T(t) = e^{-\frac{i\lambda t}{2}}$$

and  $v$  solves the following problem

$$-\Delta v - \frac{2}{||}v = \lambda v. \quad (11.0.4)$$

Any acceptable solution  $v$  must satisfy (11.0.3).

We recall that the solutions of (11.0.4), if they exist, describe the stable states of the hydrogen atom. In fact, in 1913, Niels Bohr observed that the hydrogen atom is stable only for special values of the energy of the electron and the possible values of energy are, in certain system of units, equal to the reciprocals of squares of consecutive integers. We shall obtain this result from the mathematical analysis of Eq. (11.0.4).

The general solution of Eq. (11.0.4) is very difficult. We simplify the problem by considering only spherically symmetric states. Physically it means that we don't discuss the spin of the electron.

Denoting  $r = ||$  we write Eq. (11.0.4) in spherical coordinates. Taking into account that  $v(\cdot) = R(r)$  is independent on angular coordinates, we obtain the following equation for  $R$ :

$$-R_{rr} - \frac{2}{r}R_r - \frac{2}{r}R = \lambda R. \quad (11.0.5)$$

The condition (11.0.3) in spherical coordinates will have the form:

$$\int_0^\infty R^2(r)r^2 dr < \infty \quad (11.0.6)$$

and it is understood that

$$R(0) \text{ is finite.} \quad (11.0.7)$$

This ordinary differential equation, called the Laguerre differential equation is still not easy to solve.

To find out the most convenient change of variable simplifying the equation (11.0.5) we observe that for large  $r$  the equation is close to the equation  $-R_{rr} = \lambda R$ , which has the solutions  $e^{\pm\beta r}$  where  $\beta = \sqrt{-\lambda}$ . We are interested in solutions decaying to 0 as  $r \rightarrow \infty$ , thus we take  $\lambda < 0$  (otherwise  $\beta$  would be purely imaginary and we would have oscillations) and select the solution  $e^{-\beta r}$  as the approximate solution to (11.0.5). To get the exact solution we introduce a new unknown function  $w$  according to the formula

$$R(r) = e^{-\beta r} w(r).$$

This gives  $R_r = (w_r - \beta w)e^{-\beta r}$ ,  $R_{rr} = (w_{rr} - 2\beta w_r)e^{-\beta r}$  and the equation (11.0.5) will take the form

$$\frac{1}{2}rw_{rr} - \beta rw_r + w_r + (1 - \beta)w = 0. \quad (11.0.8)$$

We solve this equation by the power series method, that is, we look for the solution to (11.0.8) in the form

$$w(r) = \sum_{k=0}^{\infty} a_k r^k.$$

Substituting into (11.0.8) and after some rearrangement of terms we get

$$\sum_{k=0}^{\infty} \left( \frac{1}{2}k(k-1) + k \right) a_k r^k + \sum_{k=1}^{\infty} (-\beta(k-1) + (1-\beta)) a_{k-1} r^{k-1} = 0$$

and, comparing coefficients at respective powers of  $r$ , we obtain the series of equations:

$$\frac{k(k+1)}{2}a_k = (\beta k - 1)a_{k-1}, \quad k = 1, 2, \dots \quad (11.0.9)$$

This means

$$\begin{aligned} a_1 &= (\beta - 1)a_0 \\ 3a_2 &= (2\beta - 1)a_1 \\ 6a_3 &= (3\beta - 1)a_2 \\ 10a_4 &= (4\beta - 1)a_3 \quad \text{etc.} \end{aligned}$$

Thus, if  $\beta$  happens to be a positive integer, the sequence of coefficients  $a_0, a_1, a_2, \dots$  terminates and we have polynomial solution to (11.0.8). Since  $v(\vec{r}) = R(r) = w(r)e^{-\beta r}$ , we see that  $v(\vec{r})$  satisfies the condition normalizing condition at infinity.

Before we give the interpretation of this result, let us discuss briefly other possible solutions of (11.0.8). For  $\beta = 1/n$  there is, of course, another linearly independent solution of this equation but it is unbounded at  $r = 0$  and is of no physical interest. On the other hand, if  $\beta \neq 1$  then the sequence of coefficients doesn't terminate and it turns out that  $R(r)$  behaves approximately as  $e^{\beta r}$  as  $r \rightarrow \infty$  and therefore it cannot satisfy the normalizing condition (11.0.6).

Thus we have found all possible spherically symmetric stationary states of the hydrogen atom. For the first few we have

$n$	$\lambda(\text{energy})$	$v(\vec{r})(\text{stationary state})$
1	-1	$e^{-r}$
2	$-\frac{1}{4}$	$e^{-r/2}(1 - r/2)$
3	$-\frac{1}{9}$	$e^{-r/3}(1 - 2r/3 + 2r^2/27)$

We see that the lowest energy levels are

$$-1, \quad -\frac{1}{4}, \quad -\frac{1}{9}, \dots$$

which corresponds exactly to the results of Bohr's experiment.

We must remember, however, that there are plenty of eigenfunctions that possess the angular dependence (spin). Also, we have a continuous spectrum so that our analysis of the hydrogen atom is by no means complete. However, it can be proved all other elements of the spectrum are larger than these obtained above and therefore we found indeed the states of the lowest energy.



# Lecture 12

## Telegraph and diffusion equation

In Lecture 6 we discussed differences between the wave and diffusion equations and observed that the properties of the solutions are totally different. Nevertheless, it follows that there is a connection between the two processes. We describe it taking as an example the system of telegraph equations:

$$\begin{aligned}u_t &= -\frac{1}{C}i_x - \frac{G}{C}u, \\i_t &= -\frac{1}{L}u_x - \frac{R}{L}i,\end{aligned}\tag{12.0.1}$$

where  $C$  is the capacitance,  $L$  is the self-inductance,  $R$  is the resistance and  $G$  is the leakage coefficient (per unit length) which can be reduced to the damped wave equation (1.6.16).

Let us assume that either the coefficient of self-induction  $L$  or the capacitance  $C$  are relatively small. Then, putting in the first case  $v = u$ ,  $w = i$ ,  $1/C = b$ ,  $G/C = a$ ,  $R = c$  and  $L = \epsilon$  and in the second case  $v = i$ ,  $u = w$ ,  $1/L = b$ ,  $R/L = a$ ,  $G = c$  and  $C = \epsilon$ , we can write the telegraph system in the following form:

$$\begin{aligned}v_t + av + bw_x &= 0, \\ \epsilon w_t + v_x + cw &= 0,\end{aligned}\tag{12.0.2}$$

where  $\epsilon$  is a small positive parameter. We use the approach called the perturbation method. Since  $\epsilon$  is assumed to be small, we look for the solution to (12.0.2) in the form of power series in  $\epsilon$ :

$$v = \bar{v}_0 + \epsilon v_1 + \dots, \quad \bar{w} = \bar{w}_0 + \epsilon w_1.\tag{12.0.3}$$

We insert this expansion to (12.0.2) and obtain

$$\begin{aligned}(v_0)_t + \epsilon(v_1)_t + \dots + av_0 + \epsilon av_1 + \dots + b(w_0)_x + \epsilon b(w_1)_x + \dots &= 0, \\ \epsilon(w_0)_t + \epsilon^2(w_1)_t + \dots + (v_0)_x + \epsilon(v_1)_x + \dots + cw_0 + \epsilon cw_1 &= 0.\end{aligned}$$

Next we compare coefficients at the same powers of  $\epsilon$ ; this yields, on the zero level of approximation, that is for  $\epsilon^0$ , the following system of equations:

$$\begin{aligned}(v_0)_t + av_0 + b(w_0)_x &= 0, \\ (v_0)_x + cw_0 &= 0.\end{aligned}$$

Eliminating from both equations  $w$  we arrive at the diffusion equation for  $v_0$ :

$$(v_0)_t = -av_0 + \frac{b}{c}(v_0)_x,\tag{12.0.4}$$

with  $w_0$  given in terms of  $v_0$  by

$$w_0 = -\frac{1}{c}(v_0)_{xx}. \quad (12.0.5)$$

It can be proved that if we solve (12.0.4) with the same initial condition as for the function  $v$  of the telegraph system, then

$$v(x, t) = v_0(x, t) + O(\epsilon)$$

so that the solution  $v$  of the telegraph system (12.0.2) can be, with a good accuracy, approximated by the solution to the diffusion equation (12.0.4). We note that, due to the properties which were discussed in Lecture 6, the diffusion equation is much easier to solve numerically.

To conclude the discussion we write down explicitly the diffusion approximation of the telegraph system when the self-induction  $L$  is small. Then the voltage  $u$  satisfies approximately the following diffusion equation:

$$u_t = -\frac{G}{C}u + \frac{1}{CR}u_{xx}.$$

The physical interpretation of this fact is as follows. The capacitance and self-inductivity of the transmission lines create delays in spreading of the signal and contribute to the wave character of the process. With vanishing self-inductance the potential  $u$  is likely to propagate with relatively large speed and thus the propagation can be adequately described by a diffusion equation. If the capacitance vanishes, then the same is true for the current.