J. Banasiak
Department of Mathematics and Applied Mathematics
University of Pretoria, Pretoria, South Africa

BANACH LATTICES IN APPLICATIONS

Contents

1 Introduction 5

2 Basics 7

1 Defining Order 7

2 Order and Norm 13

3 Sublattices, Ideals and Bands 17

3.1 Complexification 19

4 Some other stuff 21

3 Operators on Banach Lattices 23

1 Types of operators 23

1.1 Positive functionals 27

1.2 Positive operators on complex Banach lattices 27

2 Perron–Frobenius theorems 29

2.1 Basics of spectral theory 29

References 33

Index 35

1

Introduction

The common feature of many models coming from natural and engineering sciences is that the solutions represent the distribution of the particles of various sizes and thus they should be coordinate-wise nonnegative (provided the initial distribution is physically realistic; that is, also non-negative). Though mostly we work with spaces of continuous or integrable functions, or sequences, where the nonnegativity of an element f is understood as $f(x) \geq 0$ pointwise or μ-almost everywhere in the former case and $f_n \geq 0$ for any n in the latter case, ertain results are easier to formulate in a more general setting. Thus we shall briefly present basic concepts of the theory of Banach lattices; that is, Banach spaces with order compatible with the norm. The presentation is based on classical textbooks on Banach lattices, see e.g. [1, 2, 3, 4, 5, 6].

2

Basics

1 Defining Order

In a given vector space X an order can be introduced either geometrically, by defining the so-called *positive cone* (in other words, by saying what it means to be a *positive element* of X), or through the axiomatic definition:

Definition 2.1. *Let X be an arbitrary set. A partial order (or simply, an order) on X is a binary relation, denoted here by '$\geq$', which is reflexive, transitive, and antisymmetric, that is,*

(1) $x \geq x$ for each $x \in X$;

(2) $x \geq y$ and $y \geq x$ imply $x = y$ for any $x, y \in X$;

(3) $x \geq y$ and $y \geq z$ imply $x \geq z$ for any $x, y, z \in X$.

If $x, y \in X$, we write $x > y$ if $x \geq y$ and $x \neq y$ and $x \leq y$ if $y \geq x$.

Remark 2.2. In some books, e.g. [5], partial order is introduced through the relation '$>$' defined as above. More precisely, the relation '$>$' is defined by: for any $x, y, z \in X$

(1') $x > y$ excludes $x = y$;

(3') $x > y$ and $y > z$ imply $x > z$.

It is easy to see that both definitions are equivalent.

If $Y \subset X$, then $x \in X$ is called *an upper bound* (respectively, *lower bound*) of Y if $x \geq y$ (respectively, $y \geq x$) for any $y \in Y$. An element $x \in Y$ is said to be *maximal* if there is no $Y \ni y \neq x$ for which $y \geq x$. A *minimal element* is defined analogously. A *greatest element* (respectively, a *least element*) of Y is an $x \in Y$ satisfying $x \geq y$ (respectively, $x \leq y$) for all $y \in Y$.

We note here that in an ordered space there are generally elements that cannot be compared and hence the distinction between maximal and greatest elements is important. A maximal element is the 'largest' amongst all comparable elements in Y, whereas a greatest element is the 'largest' amongst all elements in Y. If a greatest (or least) element exists, it must be unique by axiom (2).

By the order interval $[x, y]$ we understand the set

$$[x, y] := \{z \in X;\ x \leq z \leq y\}.$$

The *supremum* of a set is its least upper bound and the *infimum* is the greatest lower bound. For a two-point set $\{x, y\}$ we write $x \wedge y$ or $\inf\{x, y\}$ to denote its infimum and $x \vee y$ or $\sup\{x, y\}$ to denote supremum. In the same way we define $\sup A$ and $\inf A$ for an arbitrary set A. The supremum and infimum of a set need not exist. We observe the following associative law.

Proposition 2.3. *Let a subset E of X be represented as $E = \bigcup_{\xi\in\Xi} E_\xi$. Then*

1. *If $y_\xi = \sup E_\xi$ exists for any $\xi \in \Xi$ and $y = \sup_{\xi\in\Xi} y_\xi$ exists, then $y = \sup E$;*
2. *If $z_\xi = \inf E_\xi$ exists for any $\xi \in \Xi$ and $z = \inf_{\xi\in\Xi} y_\xi$ exists, then $z = \inf E$.*

Proof. 1. If $x \in E$, then $x \in E_\xi$ for some ξ and thus $x \le y_\xi \le y$. Thus, y is an upper bound for E. If v is an arbitrary upper bound for E, then v is an upper bound for E_ξ for each ξ. Hence $y_\xi \le v$ for any $\xi \in \Xi$ and consequently $y \le v$. Thus v is the least upper bound and hence $y = \sup E$.

2. The proof is analogous. □

Definition 2.4. *We say that an ordered space X is a lattice if for any $x, y \in X$ both $x \wedge y$ and $x \vee y$ exist.*

Example 2.5. The set of all subsets of a given set X with partial order defined as inclusion is a lattice with X being its greatest element.

From now on, unless stated otherwise, any vector space X is real.

Definition 2.6. *An ordered vector space is a vector space X equipped with partial order which is compatible with its vector structure in the sense that*

(4) $x \ge y$ implies $x + z \ge y + z$ for all $x, y, z \in X$;

(5) $x \ge y$ implies $\alpha x \ge \alpha y$ for any $x, y \in X$ and $\alpha \ge 0$.

If the ordered vector space X is also a lattice, it is called a vector lattice, or a Riesz space. The set $X_+ = \{x \in X;\ x \ge 0\}$ is referred to as the positive cone of X.

Example 2.7. A convex cone in a vector space X is a set C characterised by the properties:

(i) $C + C \subset C$;

(ii) $\alpha C \subset C$ for any $\alpha \ge 0$;

(iii) $C \cap (-C) = \{0\}$.

We show that X_+ is a convex cone in X. In fact, from axiom (4) we see that if $x, y \ge 0$, then $x+y \ge 0+y = y \ge 0$, so (i) is satisfied. From (5) we immediately have (ii) and, again using (2), we see that if $x \ge 0$ and $-x \ge 0$, then $0 = x$. Convexity then follows as if $x, y \in C$, then $\alpha x, \beta y \in C$ for any $\alpha, \beta \ge 0$ (in particular, for $\alpha + \beta = 1$) and thus $\alpha x + \beta y \in C$.

On the other hand, let C be a convex cone in a vector space X. If we define the relation '$\ge$' in X by the formula $y \ge x$ if and only if $y - x \in C$, then X becomes an ordered vector space such that $X_+ = C$. In fact, because $x - x = 0 \in C$, we have $x \ge x$ for any $x \in X$ which gives (1). Next, let $x - y \in C$ and $y - x \in C$. Then by (iii) we obtain axiom (2). Furthermore, if $x - y \in C$ and $y - z \in C$, then we have $x - z = (x - y) + (y - z) \in C$ by (i). Hence $\ge$ is a partial order on X. To prove that X is an ordered vector space, we consider $x - y \in C$ and $z \in X$; then $(x + z) - (y + z) = x - y \in C$ which establishes (4). Finally, if $x - y \in C$ and $\alpha \ge 0$, then $\alpha x - \alpha y = \alpha(x - y) \in C$ by (ii) so that (5) is satisfied. Moreover, $X_+ = \{x \in X;\ x \ge 0\} = \{x \in X;\ x - 0 \in C\} = C$.

The cone C of X is called *generating* if $X = C - C$; that is, if every vector can be written as a difference of two positive vectors or, equivalently, if for any $x \in X$ there is $y \in X_+$ satisfying $y \ge x$.

The Archimedean property of real numbers is that there are no infinitely large or small numbers. In other words, for any $r \in \mathbb{R}_+$, $\lim_{n\to\infty} nr = \infty$ or, equivalently, $\lim_{n\to\infty} n^{-1}r = 0$. Following this, we say that a Riesz space X is *Archimedean* if $\inf_{n\in\mathbb{N}}\{n^{-1}x\} = 0$ holds for any $x \in X_+$. In this book we only deal with Archimedean Riesz spaces.

The operations of taking supremum or infimum have several useful properties which make them similar to the numerical case.

Proposition 2.8. *For arbitrary elements x, y, z of a Riesz space, the following identities hold.*

1. $x + y = \sup\{x, y\} + \inf\{x, y\}$;
2. $x + \sup\{y, z\} = \sup\{x + y, x + z\}$ *and* $x + \inf\{y, z\} = \inf\{x + y, x + z\}$;
3. $\sup\{x, y\} = -\inf\{-x, -y\}$ *and* $\inf\{x, y\} = -\sup\{-x, -y\}$;
4. $\alpha \sup\{x, y\} = \sup\{\alpha x, \alpha y\}$ *and* $\alpha \inf\{x, y\} = \inf\{\alpha x, \alpha y\}$ *for* $\alpha \geq 0$.
5. *For any $x, y, z \in X_+$ we have*
$$\inf\{x + y, z\} \leq \inf\{x, z\} + \inf\{y, z\}.$$

Proof. 1. From $\inf\{x, y\} \leq y$ we obtain $x + \inf\{x, y\} \leq x + y$ so that $x \leq x + y - \inf\{x, y\}$ and similarly $y \leq x + y - \inf\{x, y\}$. Hence, $\sup\{x, y\} \leq x + y - \inf\{x, y\}$; that is,

$$x + y \geq \sup\{x, y\} + \inf\{x, y\}.$$

On the other hand, because $y \leq \sup\{x, y\}$, in a similar way we obtain $x + y - \sup\{x, y\} \leq x$ and also $x + y - \sup\{x, y\} \leq y$ so that

$$x + y \leq \sup\{x, y\} + \inf\{x, y\}$$

and the identity in property 1 follows.

2. Clearly, $x+y \leq x+\sup\{y, z\}$ and $x+z \leq x+\sup\{y, z\}$ and thus $\sup\{x+y, x+z\} \leq x+\sup\{y, z\}$. On the other hand, $y = -x+(x+y) \leq -x+ \sup\{x+y, x+z\}$ and similarly $z = -x+(x+z) \leq -x+\sup\{x+y, x+z\}$ so that $\sup\{y, z\} \leq -x + \sup\{x + y, x + z\}$ or, equivalently $x + \sup\{y, z\} \leq \sup\{x + y, x + z\}$. Together, we obtain $x + \sup\{y, z\} = \sup\{x + y, x + z\}$. The other identity can be proved in the same manner.

3. Because $x, y \leq \sup\{x, y\}$, we obtain that $-\sup\{x, y\} \leq -x$ and $-\sup\{x, y\} \leq -y$ and so $-\sup\{x, y\} \leq \inf\{-x, -y\}$. On the other hand, if $-x \geq z$ and $-y \geq z$, then $x, y \leq -z$ and hence $-z \geq \sup\{x, y\}$. Replacing z by $\inf\{-x, -y\}$ shows $-\sup\{x, y\} = \inf\{-x, -y\}$. To get the second identity we replace x by $-x$ and y by $-y$ in the first one.

4. Let $\alpha > 0$. Clearly, by Definition 2.6 (5), $\sup\{\alpha x, \alpha y\} \leq \alpha \sup\{x, y\}$. Since $\sup\{\alpha x, \alpha y\} \geq \alpha x, \alpha y$, $\alpha^{-1} \sup\{\alpha x, \alpha y\} \geq x, y$, hence $\sup\{x, y\} \leq \alpha^{-1} \sup\{\alpha x, \alpha y\}$ which implies $\alpha \sup\{x, y\} = \sup\{\alpha x, \alpha y\}$. The second one is proved in the same way.

5. Since x, y, z are positive, $y + z \geq z$ and hence $(y + z) \wedge z = z$. Thus, by Proposition 2.3 and Proposition 2.8 2., we have

$$\begin{aligned}(x + y) \wedge z &= (x + y) \wedge ((y + z) \wedge z) = ((x + y) \wedge (y + z)) \wedge z = (y + x \wedge z) \wedge z \\ &\leq (y + x \wedge z) \wedge (z + x \wedge z) = x \wedge z + y \wedge z.\end{aligned}$$

□

Remark 2.9. Proposition 2.8 3. shows that to show that an ordered vector lattice X is a Riesz space it suffices only to prove that $x \vee y$ exists for any two $x, y \in X$. Indeed, if we only know that $u \vee v$ exists for any $u, v \in X$, for a given x, y, we define $x \wedge y = -(-x) \vee (-y)$. It is indeed the infimum of x and y since, repeating the argument above, if $z \leq x, z \leq y$, then $-z \geq -x, -z \geq -y$, thus $-z \geq (-x) \vee (-y)$ and $z \leq - \geq (-x) \vee (-y)$. On the other hand, since $(-x) \vee (-y) \geq -x, -y$, we find $(-x) \vee (-y) \leq x, y$.

Even more, Proposition 2.8 2. shows that it is sufficient if $x \vee 0$ exists for any $x \in X$ as then $x \vee y = x + (y - x) \vee 0$. Indeed, as shown in 2., $x + y \vee z$ is an upper bound for $x + y$ and $x + z$. On the other hand, if ξ is any other upper bound, then $y \le \xi - x, z \le \xi - x$ and thus $y \vee z \le \xi - x$ so $\xi \ge x + y \vee z$ and thus the supremum of $\{x + y, x + z\}$ exists and equals $x + y \vee z$. Hence only the existence of one supremum is required in property 2.

The element $x \vee 0$ plays a special role discussed below.

For an element x in a Riesz space X we can define its positive and negative part, and its absolute value, respectively, by

$$x_+ = \sup\{x, 0\}, \quad x_- = \sup\{-x, 0\}, \quad |x| = \sup\{x, -x\}.$$

The functions $(x, y) \to \sup\{x, y\}, (x, y) \to \inf\{x, y\}, x \to x_\pm$ and $x \to |x|$ are collectively referred to as the *lattice operations* of a Riesz space. The relation between them is given in the next proposition.

Proposition 2.10. *If x is an element of a Riesz space, then*

$$x = x_+ - x_-, \qquad |x| = x_+ + x_-. \tag{2.1.1}$$

Thus, in particular, the positive cone in a Riesz space is generating.

Proof. By Proposition 2.8(1) and (3) we have

$$x = x + 0 = \sup\{x, 0\} + \inf\{x, 0\} = \sup\{x, 0\} - \sup\{-x, 0\} = x_+ - x_-.$$

Furthermore, from Theorem 2.8(2) and (4), and the previous result we get

$$\begin{aligned} |x| &= \sup\{x, -x\} = \sup\{2x, 0\} - x = 2\sup\{x, 0\} - x = 2x_+ - x \\ &= 2x_+ - (x_+ - x_-) = x_+ + x_-. \end{aligned}$$

□

The absolute value has a number of useful properties that are reminiscent of the properties of the scalar absolute value; that is, for example, $|x| = 0$ if and only if $x = 0$, $|\alpha x| = |\alpha||x|$ for any $x \in X$ and any scalar α, as well as some others which are proved below.

For a subset S of a Riesz space we write

$$\begin{aligned} \sup\{x, S\} &= x \vee S := \{\sup\{x, s\};\ s \in S\}, \\ \inf\{x, S\} &= x \wedge S := \{\inf\{x, s\};\ s \in S\}. \end{aligned}$$

The following infinite distributive laws are used later.

Proposition 2.11. *Let S be a nonempty subset of a Riesz space X. If $\sup S$ exists, then $\sup\{\inf\{x, S\}\}$ and $\sup\{\sup\{x, S\}\}$ exist for each $x \in X$ and*

$$\begin{aligned} \sup\{\inf\{x, S\}\} &= \inf\{x, \sup S\}, \\ \sup\{\sup\{x, S\}\} &= \sup\{x, \sup S\}. \end{aligned} \tag{2.1.2}$$

Similarly, if $\inf S$ exists, then $\inf\{\sup\{x, S\}\}$, $\inf\{\inf\{x, S\}\}$ exist for each $x \in X$ and

$$\begin{aligned} \inf\{\sup\{x, S\}\} &= \sup\{x, \inf S\}, \\ \inf\{\inf\{x, S\}\} &= \inf\{x, \inf S\}. \end{aligned} \tag{2.1.3}$$

In particular, if $S = \{y, z\}$, then

$$\begin{aligned} (x \wedge y) \vee (x \wedge z) &= x \wedge (y \vee z), \\ (x \vee y) \vee (x \vee z) &= x \vee (y \vee z), \\ (x \vee y) \wedge (x \vee z) &= x \vee (y \wedge z), \\ (x \wedge y) \wedge (x \wedge z) &= x \wedge (y \wedge z). \end{aligned} \tag{2.1.4}$$

Proof. Let us assume $y = \sup S$ exists and let $x \in X$. Because for any $s \in S$ we have $\inf\{x, s\} \leq \inf\{x, y\}$, we can write

$$\sup\{\inf\{x, S\}\} \leq \inf\{x, \sup S\}$$

provided the left-hand side exists. To prove the existence and the equality, we should prove that if $z \geq \inf\{x, s\}$ for any $s \in S$, then $z \geq \inf\{x, \sup S\}$. Using property 2 of Proposition 2.8, we have

$$s = \inf\{x, s\} + \sup\{x, s\} - x \leq z + \sup\{x, s\} - x \leq z + \sup\{x, y\} - x$$

for any $s \in S$ so that taking the supremum over S we get

$$y \leq z + \sup\{x, y\} - x;$$

that is,

$$y + x - \sup\{x, y\} \leq z.$$

Again using Proposition 2.8, $x + y - \sup\{x, y\} = \inf\{x, y\}$ and therefore

$$\inf\{x, \sup S\} = \inf\{x, y\} \leq z$$

which proves the first equation of (2.1.2).

To prove the second identity, again let $y = \sup S$ exist and note that $\sup\{x, y\}$ is an upper bound for the set $\sup\{x, S\}$. If z is another upper bound for this set we have $z \geq \sup\{x, s\} \geq s$ for all $s \in S$. Hence $z \geq y$. Because also $z \geq x$, we get $z \geq \sup\{x, y\}$. Thus $\sup\{x, y\} = \sup\{\sup\{x, S\}\}$.

Identities (2.1.3) can be proved in the same way. □

The following inequalities are essential in proving the relations between order and norm in the later sections.

Proposition 2.12. *For arbitrary elements x, y, z of a Riesz space X, the following inequalities hold.*

1. $||x| - |y|| \leq |x + y| \leq |x| + |y|$;
2. *For $x, y, z \in X$ we have*

$$\begin{aligned} |x - y| &= \sup\{x, y\} - \inf\{x, y\}, \\ |x - y| &= |\sup\{x, z\} - \sup\{y, z\}| + |\inf\{x, z\} - \inf\{y, z\}|. \end{aligned} \tag{2.1.5}$$

3. *(Birkhoff's inequality)*

$$\begin{aligned} |\sup\{x, z\} - \sup\{y, z\}| &\leq |x - y| \\ |\inf\{x, z\} - \inf\{y, z\}| &\leq |x - y|. \end{aligned} \tag{2.1.6}$$

Proof. 1. Clearly, we have $x+y \leq |x|+|y|$ and $-x-y \leq |x|+|y|$ so that $|x+y| = \sup\{x+y, -x-y\} \leq |x|+|y|$. From this we see that $|x| = |(x + y) - y| \leq |x + y| + |y|$ and in the same way $|y| \leq |x + y| + |x|$. Hence, by the same argument, $||x| - |y|| \leq |x + y|$.

2. We observe that for any $u, v \in X$, by Proposition 2.8 2. and 3.,

$$\begin{aligned} v + (u - v)_+ &= v + \sup\{u - v, 0\} = \sup\{u, v\}, \\ -v + (u - v)_- &= -v + \sup\{v - u, 0\} = \sup\{-u, -v\} = -\inf\{u, v\}. \end{aligned}$$

Hence

$$|u - v| = (u - v)_+ + (u - v)_- = \sup\{u, v\} - \inf\{u, v\}.$$

To prove the second part we write, using in turn (2.1.5), Proposition 2.8 2., (2.1.4) and (2.1.5) again,

$$\begin{aligned}|x-y| &= (x-y)_+ + (x-y)_- = \sup\{x,y\} - \inf\{x,y\}\\ &= (\sup\{x,y\}+z) - (\inf\{x,y\}+z) = (x\vee y)\vee z + (x\vee y)\wedge z - (x\wedge y)\vee z - (x\wedge y)\wedge z\\ &= (x\vee z)\vee(y\vee z) + (x\wedge z)\vee(y\wedge z) - (x\vee z)\wedge(y\vee z) - (x\wedge z)\wedge(y\wedge z)\\ &= |x\vee z - y\vee z| + |x\wedge z - y\wedge z|\end{aligned}$$

3. Follows immediately from 2. □

Definition 2.13. *We say that $x,y\in X$ are disjoint (and denote it by $x\perp y$) if* $\inf\{|x|,|y|\}=0$. *For any set $D\subset X$ we define*

$$D^d = \{x\in X;\ x\perp y \text{ for any } y\in D\} \tag{2.1.7}$$

Proposition 2.14. *Let $x,y\in X$. Then*

1. *$x_+\perp x_-$ and the decomposition of x into the difference of disjoint positive elements is unique;*
2. *$x\perp y$ is equivalent to $|x|\vee|y| = |x|+|y|$ and then*

$$|x+y| = |x|+|y|.$$

Proof. 1. We have

$$x_+\wedge x_- = x_- - x_- + x_+\wedge x_- = x_- + (x_+ - x_-)\wedge 0 = x_- - (-(x_+ + x_-)\vee 0 = x_- - (-x)\vee 0 = x_- - x_- = 0.$$

Further, let $u,v\geq 0$ satisfy $u\perp v$ and $x=u-v$. Since $x = x_+ - x_-$, we have $u - x_+ = v - x_-$. Clearly, $u\geq x$ and $x_+ = x\vee 0\leq u\vee 0 = u$. Similarly, $-x\leq v$ implies $x_- = (-x)\vee 0\leq v\vee 0 = v$. Hence

$$0\leq u - x_+ = v - x_- = (u-x_+)\wedge(v-x_-)\leq u\wedge v = 0$$

by the disjointness.

2. The first part follows by Proposition 2.8 1., as

$$|x|+|y| = |x|\vee|y| + |x|\wedge|y|,$$

and the definition of the disjointness. To prove the second part, we observe that

$$|x+y| = (x+y)_+ + (x+y)_-$$

so the result will follow if we could prove $(x+y)_\pm = x_\pm + y_\pm$. Since obviously $x_+ + x_- + y_+ + y_- = (x+y) = (x+y)_+ + (x+y)_-$, in view of 1. we need to prove that $(x_+ + y_+)\perp(x_- + y_-)$. Since both are nonnegative,

$$(x_+ + y_+)\wedge(x_- + y_-)\leq x_+\wedge x_- + x_+\wedge y_- + y_+\wedge x_- + y_+\wedge y_- = 0,$$

where we also used $x_+\wedge y_-\leq |x|\wedge|y| = 0$ and $y_+\wedge x_-\leq |y|\wedge|x| = 0$ as well as the distributive law from Proposition 2.8 5. □

Proposition 2.15. *(Riesz decomposition property) If $x,y,z\in X_+$ and $z\leq x+y$, then there exist $u,v\in X_+$ such that $u\leq x$, $v\leq y$ and $z=u+v$.*

Proof. Let us take $u = x\wedge z$ and $v = z-u$. Obviously, $0\leq u,v$ and $u\leq x$. Further, by Proposition 2.8 2.,

$$y - v = y - z + x\wedge z = (y-z+x)\wedge y\geq 0.$$

□

Remark 2.16. The Riesz decomposition property gives an easy proof of the distributive law from Proposition 2.8 5. Indeed,

$$x\wedge(z+y)\leq y+z.$$

Thus, from the Riesz decomposition property, there are $0\leq u\leq y, 0\leq v\leq z$ such that $x\wedge(z+y) = u+v$. But then $u\leq x\wedge(z+y)\leq x$ and so $u\leq y\wedge x$. Similarly $v\leq x$ and hence $v\leq x\wedge z$ and

$$x\wedge(z+y)\leq y\wedge x + z\wedge x.$$

2 Order and Norm

As the next step, we investigate the relation between the lattice structure and the norm, when X is both a normed and an ordered vector space.

Definition 2.17. *A norm on a vector lattice X is called a lattice norm if*

$$|x| \leq |y| \quad \text{implies} \quad \|x\| \leq \|y\|. \tag{2.2.8}$$

A vector lattice X, complete under a lattice norm, is called a Banach lattice.

Property (2.2.8) gives the important identity:

$$\|x\| = \||x|\|, \qquad x \in X. \tag{2.2.9}$$

Indeed, this follows as taking first x and $y = |x|$ we have $|x| \leq |(|x|)|$ and hence $\|x\| \leq \||x|\|$. On taking $|x|$ and $y = x$, we also have $|(|x|)| \leq |x|$ and hence $\||x|\| \leq \|x\|$.

Proposition 2.18. *Any normed lattice is Archimedean.*

Proof. Let $0 \leq y \leq \{n^{-1}x\}$, $x \geq 0$, $n \in \mathbb{N}$. By the lattice norm property, we have

$$\|y\| \leq \|n^{-1}x\| = n^{-1}\|x\|$$

for any n and hence $\|y\| = 0$, yielding $y = 0$. Hence $\inf\{n^{-1}x\} = 0$. $\square$

For a non-increasing sequence $(x_n)_{n\in\mathbb{N}}$ we write $x_n \downarrow x$ if $\inf\{x_n;\ n \in \mathbb{N}\} = x$. For a non-decreasing sequence $(x_n)_{n\in\mathbb{N}}$ the symbol $x_n \uparrow x$ have an analogous meaning. One of the basic results is:

Proposition 2.19. *Let X be a Banach lattice. Then:*

(1) All lattice operations are continuous.

(2) The positive cone X_+ is closed.

(3) The positive cone X_+ is weakly closed.

(4) If $(x_n)_{n\in\mathbb{N}}$ is nondecreasing and $\lim_{n\to\infty} x_n = x$ in the norm of X, then

$$x = \sup\{x_n;\ n \in \mathbb{N}\}.$$

Analogous statement holds for nonincreasing sequences.

(5) Order intervals are norm bounded and closed.

Proof. Ad. (1). Consider $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ of elements of X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Using the Birkhoff inequality, (2.1.6), and Proposition 2.12 1., we have

$$|x_n \wedge y_n - x \wedge y| \leq |x_n \wedge y_n - x_n \wedge y| + |x_n \wedge y - x \wedge y| \leq |y_n - y| + |x_n - x|$$

and hence, by the definition of a lattice norm,

$$\|x_n \wedge y_n - x \wedge y\| \leq \|y_n - y\| + \|x_n - x\|.$$

Analogously we prove the continuity of $\vee$ and, by (2.1.1), of $|\cdot|$.

Ad. (2). Since $X_+ = \{x \in X;\ x_- = 0\}$ and the lattice operation $X \ni x \to x_- \in X$ is continuous, X_+ is closed.

Ad. (3). Since X_+ is convex, it is closed if and only if it is weakly closed.

Ad. (4) For any fixed $k \in \mathbb{N}$ we have

$$\lim_{n\to\infty}(x_n - x_k) = x - x_k$$

in norm and $x_n - x_k \in X_+$ for $n \geq k$ so that $x - x_k \in X_+$ for any $k \in \mathbb{N}$ by point (1). Thus x is an upper bound for $\{x_n\}_{n\in\mathbb{N}}$. On the other hand, if $x_n \leq y$ for all n, then $0 \leq y - x_n \to y - x$ so that, again by point (1), we have $y \geq x$ and hence $x = \sup_{n\in\mathbb{N}}\{x_n\}$.

The proof of (3) is analogous.

Ad. (5). Consider $f \in [x, y]$. Then from $x \leq f \leq y$ it follows $0 \leq f - x \leq y - x$ and hence $\|f - x\| \leq \|y - x\|$. Thus,

$$\|f\| = \|f - x + x\| \leq \|x\| + \|y - x\|.$$

To prove that $[x, y]$ is closed we use the closedness of the positive cone. If $(h_n)_{n\in\mathbb{N}} \subset [x, y]$ and $h_n \to h$ in X. Then $y - x_n \geq 0$ and $h_n \geq x \geq 0$ and hence $y - h \geq 0$ and $h - x \geq 0$. □

In general, the converse of Proposition 2.19 (3) is false; that is, we may have $x_n \uparrow x$ but $(x_n)_{n\in\mathbb{N}}$ does not converge in norm. Indeed, consider $x_n = (1, 1, 1 \ldots, 1, 0, 0, \ldots) \in l_\infty$, where 1 occupies only the n first positions. Clearly, $\sup_{n\in\mathbb{N}} x_n = x := (1, 1, \ldots, 1, \ldots)$ but $\|x_n - x\|_\infty = 1$. This can be remedied by adding an additional condition on the sequence or on the space. In the latter case, the converse of Proposition 2.19(3) holds in a special class of Banach lattices, called Banach lattices with order continuous norm. A (σ-complete) Banach lattice is said to have an order continuous norm if any (sequence) net monotonically decreasing to 0 is norm convergent to 0. In such Banach lattices we have, in particular, that if $0 \leq x_n \uparrow x$ and $x_n \leq x$ for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Such spaces will be (possibly) discussed later. In the former case, we have for instance,

Theorem 2.20. *[3, Proposition 10.9] If X is a Banach lattice, then every weakly convergent increasing sequence is norm convergent.*

Proof. We can restrict our attention to nonnegative sequences by e.g. considering $(-f_1 + f_n)_{n\geq 1}$. Let $\{f(t)\}_{t\geq 0}$n weakly converges to f. We have $f_n \leq f$ for all $n \in \mathbb{N}$. Indeed, fix arbitrary n, then $f_{n+p} - f_n \geq 0$ for all $p \geq 0$. Taking $p \to \infty$, by Theorem 2.19 (3), $f - f_n \geq 0$ and the claim follows since n is arbitrary. Using the Mazur theorem, there is a sequence $(g_n)_{n\in\mathbb{N}}$ of convex combinations of $\{f_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} g_n = f$. In other words, for any $\epsilon > 0$, there is $g_n = \alpha_1 f_1 + \ldots + \alpha_{m_n} f_{m_n}$ with $\alpha_i \geq 0, i = 1, \ldots, m_n$, and $\alpha_1 + \cdots \alpha_{m_n} = 1$ such that $\|f - g_n\| \leq \epsilon$. However, since

$$g_n = \alpha_1 f_1 + \ldots + \alpha_{m_n} f_{m_n} \leq (\alpha_1 + \ldots + \alpha_{m_n}) f_{m_n} \leq f_{m_n} \leq f,$$

we have, for $r \geq m_n$,

$$\|f - f_r\| \leq \|f - f_{m_n}\| \leq \|f - g_n\| \leq \epsilon$$

which shows the norm convergence.

It is important to rule out certain properties of general Banach lattices. For instance, $\mathbb{R}$ has the property that any element is either nonnegative or nonpositive. If a general ordered space has this property, we say that it is totally ordered. It turns out, however, that among Banach lattices, $\mathbb{R}$ is essentially the only one with this property.

Proposition 2.21. *Any totally ordered Banach lattice X is at most one-dimensional.*

Proof. Let $e \in X_+$ and $f \in X$ and consider

$$A_+ = \{\alpha \in \mathbb{R} \ : \ \alpha e \geq f\}, \quad A_- = \{\alpha \in \mathbb{R} \ : \ \alpha e \leq f\}.$$

Each set is nonempty and closed and, since $\alpha e - f$ is either nonpositive or nonnegative, $A_+ \cup A_- = \mathbb{R}$. Since $\mathbb{R}$ is connected, there is $\alpha \in A_+ \cap A_-$ and so $f = \alpha e$. □

Example 2.22. Consider $\mathbb{R}^n$. A standard order in $\mathbb{R}^n$ is given by $\boldsymbol{x} \geq \boldsymbol{y}$ if and only if $x_i \geq y_i$ for any $i = 1, \ldots, n$. It is a Riesz space with $\boldsymbol{x} \vee \boldsymbol{y} = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$. It is also a Banach lattice with all standard norms.

Another often used order in $\mathbb{R}^n$ is called the lexicographical order and is defined by $\boldsymbol{x} \geq \boldsymbol{y}$ if and only if there exists $k \in \{0, \ldots, n\}$ such that $x_1 = y_1, \ldots, x_k = y_k$ and $x_{k+1} > x_k$. It is a totally ordered Riesz space with $\boldsymbol{x} \vee \boldsymbol{y}$ being the bigger element. The positive cone e.g. in $\mathbb{R}^2$ consists of the open right half-plane $\{(x_1, x_2);\ x_1 > 0\}$ and the semiaxis $\{(x_1, x_2);\ x_1 = 0, x_2 > 0\}$. We observe that this is not an Archimedean space. Indeed, since $(n^{-1}, 0) \geq (0, 1)$ for any n, $\inf_n\{(n^{-1}, 0)\} \neq 0$.

Example 2.23. Typical examples of Riesz spaces are provided by *function spaces.* If X is a vector space of real-valued functions on a set Ω, then we can introduce a pointwise order in X by saying that $f \leq g$ in X if $f(x) \leq g(x)$ for any $x \in S$. Equipped with such an order, X becomes an ordered vector space. Let us define on $X \times X$ the operations $f \vee g$ and $f \wedge g$ by taking point-wise maxima and minima; that is, for any $f, g \in X$,

$$(f \vee g)(x) := \max\{f(x), g(x)\},$$
$$(f \wedge g)(x) := \min\{f(x), g(x)\}.$$

We say that X is a *function space* if $f \vee g, f \wedge g \in X$, whenever $f, g \in X$. Clearly, a function space with pointwise ordering is a Riesz space. Examples of function spaces are offered by the spaces of all real functions $\mathbb{R}^\Omega$ or all real bounded functions $M(\Omega)$ on a set Ω, and by, defined earlier, spaces $C(\Omega)$, $C(\overline{\Omega})$, or l_p, $1 \leq p \leq \infty$.

If Ω is a measure space, then all above considerations are valid when the pointwise order is replaced by $f \leq g$ if $f(x) \leq g(x)$ almost everywhere. With this understanding, $L_0(\Omega)$ and $L_p(\Omega)$ spaces with $1 \leq p \leq \infty$ become function spaces and are thus Riesz spaces.

These are also Banach lattices under standard norms.

Example 2.24. Consider the Banach space $C^1([0, 1])$ with the norm

$$\|f\| = \max_{s \in [0,1]} |f(s)| + \max_{s \in [0,1]} |f'(s)|$$

and the natural order inherited from $C([0, 1])$. Since $\sup\{t, 1 - t\} \notin C^1([0, 1])$, this is not a vector lattice. Moreover, the norm is not compatible with the order. Indeed, if we take $f(s) = 1$ and $g(s) = \sin 2s$ for $s \in [0, 1]$, we have $g \leq f$ but $\|g\| \geq |g'(0)| = 2 > 1 = \|f\|$.

Example 2.25. On the other hand, consider the space $W_2^1(0, 1)$. It follows that the absolute value of a function from $W_2^1(0, 1)$ is still in $W_2^1(0, 1)$, so $W_2^1(0, 1)$ with the order inherited from $L_1(0, 1)$ is a vector lattice. However, as in the above example, the norm is not compatible with the order. Indeed, $\|f\| = 1$ but

$$\|g\|^2 = \int_0^1 \sin^2(2s)ds + 2\int_0^1 \cos^2(2s)ds = 1 + \int_0^1 \cos^2(2s)ds > 1.$$

Example 2.26. Two classes of Banach lattices play an important role in our considerations: AL- and AM-spaces. We say that a Banach lattice X is

(i) an AL-space if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X_+$,

(ii) an AM-space if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X_+$.

Standard examples of AM-spaces are offered by the spaces $C(\overline{\Omega})$, where $\overline{\Omega}$ is either a bounded subset of $\mathbb{R}^n$, or in general, a compact topological space. Also the space $L_\infty(\Omega)$ is an AM-space. On the other hand, most known examples of AL-spaces are the spaces $L_1(\Omega, d\mu)$.

The importance of AL-spaces stems from the fact that increasing and norm bounded sequences $(x_n)_{n\in\mathbb{N}}$ are norm convergent. As usual, we can restrict our considerations to positive sequences. Since $(\|x_n\|)_{n\in\mathbb{N}}$ is also increasing, its boundedness implies that it converges. Then for $m \geq n$ we have

$$\|x_m - x_n\| = \|x_m - x_n\| + \|x_n\| - \|x_n\| = \|x_m\| - \|x_n\|$$

and hence $(x_n)_{n\in\mathbb{N}}$ is a Cauch sequence. These examples exhaust all (up to a lattice isometry; that is, topological isomorphism that preserves lattice operations) cases of *AM*- and *AL*-spaces. However, particular representations of these spaces can be very different.

We note that the existence of suprema or infima of finite sets, ensured by the definition of a Riesz space, does not extend to infinite sets. This warrants introducing a more restrictive class of spaces.

Definition 2.27. *We say that a Riesz space X is Dedekind (or order) complete if every nonempty and bounded from above subset of X has a least upper bound. X is said to be σ-Dedekind or (σ-order) complete, if every bounded from above nonempty countable subset of X has a least upper bound.*

Remark 2.28. In some definitions, [2, p. 12], for a Riesz space X to be order complete, it is enough if any directed upward set of nonnegative elements has a supremum in X. Here, a set $S \subset X$ is called *directed upward* if for any $x, y \in S$ there is $z \in S$ such that $x \leq z$ and $y \leq z$. We prove that the supremum of any set (if it exists) can be obtained through a directed set of nonnegative elements so that both definitions are equivalent.

Let S be a nonempty subset of X. First, we show that $\sup S$ can be replaced by $\sup \boldsymbol{S}$, where $\boldsymbol{S}$ is the set of all suprema of finite collections of elements from S. It suffices to show that the sets of upper bounds for both sets are the same. If u is an upper bound for S, then $u \geq s$ for any $s \in S$ but then, from the definition of supremum, $u \geq x$ for any $x \in \boldsymbol{S}$. Conversely, if u is an upper bound for $\boldsymbol{S}$, then, because the supremum of a set is not smaller than any of its elements, we obtain $u \geq s$ for any $s \in S$. Hence both suprema exist or do not exist at the same time and are equal in the former case. By the second equation of (2.1.2) we see that the set $\boldsymbol{S}$ is directed. Note that we have proved even more: for any $x, y \in \boldsymbol{S}$ we can take $z = \sup\{x, y\} \in \boldsymbol{S}$.

Next, let $x_0 \in \boldsymbol{S}$. Then $\sup \boldsymbol{S}$ and $\sup \boldsymbol{S}_1 := \sup\{x \in \boldsymbol{S};\ x \geq x_0\}$ either both exist and are equal, or do not exist. In fact, clearly any upper bound for $\boldsymbol{S}$ is also an upper bound for $\boldsymbol{S}_1$. Conversely, if u is an upper bound for $\boldsymbol{S}_1$, then for any $x \in \boldsymbol{S}$, $\sup\{x_0, x\} \in S$ and thus $\sup\{x_0, x\} \in \boldsymbol{S}$ so that $u \geq \sup\{x_0, x\} \geq x$. Hence we always can replace $\boldsymbol{S}$ by a set of nonnegative elements using the shift

$$\sup \boldsymbol{S} = \sup\{x \in \boldsymbol{S};\ x \geq x_0\} = \sup\{x - x_0;\ x \in \boldsymbol{S}, x - x_0 \geq 0\} + x_0.$$

Example 2.29. Order complete Riesz spaces are Archimedean. To show this, let X be an order complete Riesz space and assume that $x \leq n^{-1}y$ for some $x, y \in X_+$ and any $n \in \mathbb{N}$. Because $u = \sup\{nx;\ n \in \mathbb{N}\}$ exists in X, we can write $nx = (n+1)x - x \leq u - x$. Taking the supremum of both sides, we find $u \leq u - x$ which yields $x \leq 0$. Because x is positive, we have $x = 0$.

Example 2.30. The space $C([0, 1])$ is not σ-order complete (and thus also not order complete). To see this, consider the sequence of functions given by

$$f_n(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \frac{1}{2} - \frac{1}{n}, \\ n\left(\frac{1}{2} - x\right) & \text{for } \frac{1}{2} - \frac{1}{n} < x \leq \frac{1}{2}, \\ 0 & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

This is clearly an increasing sequence bounded from above by $g(x) \equiv 1$. However, it converges pointwise to a discontinuous function $f(x) = 1$ for $x \in [0, 1/2)$ and $f(x) = 0$ for $x \in [1/2, 0]$. In general, spaces $C(\Omega)$ are not σ-order complete unless Ω consists of isolated points.

On the other hand, the spaces l_p, $1 \leq p \leq \infty$, are clearly order complete, as taking the coordinatewise suprema of sequences bounded from above by an l_p sequence produces a sequence which is in l_p.

If we move to the spaces $L_p(\Omega), p \in \{0\} \cup [1, \infty]$, then the problem becomes more complicated. Because the measure is σ-finite, the supremum and the infimum of a countable subset of measurable functions are measurable, $L_0(\Omega)$ and $L_\infty(\Omega)$ are σ-order complete by definition, and the spaces $L_p(\Omega)$ also are σ-order complete by the dominated convergence theorem for Lebesgue integrals.

The proof that they are also order complete is much more delicate; see [1, Problem 1.6.5]. We recall that μ is assumed to be σ-finite and $S \subset L_0(\Omega)$. By Remark 2.28 we can assume that S consists of nonnegative elements satisfying $\sup\{f,g\} \in S$ whenever $f,g \in S$. Let $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $0 < \mu(\Omega_n) < +\infty$ and define $\rho : L_{0,+}(\Omega) \to [0,\infty)$ by

$$\rho(f) = \sum_{n=1}^{\infty} \frac{1}{2^n \mu(\Omega_n)} \int_{\Omega_n} \frac{f}{1+f} d\mu.$$

It is clear that ρ has the following properties: (a) $f \in L_{1,+}(\Omega)$ satisfies $\rho(f) = 0$ if and only if $f = 0$; (b) if $0 \le f \le g$ and $\rho(f) = \rho(g)$, then $f = g$; and (c) if $(f_n)_{n\in\mathbb{N}} \subset L_{0,+}(\Omega)$ converges to f in an increasing way, then $\rho(f_n) \to \rho(f)$.

The function ρ is bounded on $L_{0,+}(\Omega)$, therefore we can set $m := \sup_{g\in S} \rho(g) < +\infty$ and choose a sequence $(f_n)_{n\in\mathbb{N}} \subset S$ such that $\rho(f_n)$ converges to m. Because S was assumed to be a directed set, we can construct this sequence to be increasing. Furthermore, S is bounded from above and $(f_n)_{n\in\mathbb{N}}$ is countable, thus it follows that there is $f \in L_{0,+}(\Omega)$ such that f_n converges to f in an increasing way. By property (c), we also have $\rho(f_n) \nearrow \rho(f)$.

We show that $f = \sup S$. First, f is an upper bound for S. In fact, let $g \in S$. Then $\sup\{g, f_n\} \in S$ for any $n \in \mathbb{N}$ and by (2.1.2) we get $\sup_n\{\sup\{g, f_n\}\} = \sup\{g, f\}$. From $f_n \le \sup\{g, f_n\}$ and $\rho(\sup\{g, f_n\}) \le m$ we obtain by (c) that $\rho(\sup\{g, f\}) = m$. Because $0 \le f \le \sup\{g, f\}$, property (b) gives $f = \sup\{g, f\}$, hence $f \ge g$ and f is an upper bound for S. Let $h \in L_0(\Omega)$ be another upper bound. Then $f_n \le h$, but because f is the pointwise limit almost everywhere of $(f_n)_{n\in\mathbb{N}}$, we have $f \le h$ and thus $f = \sup S \in L_{0,+}(\Omega)$.

The fact that $L_p(\Omega)$ are also order complete for $1 \le p \le \infty$ then follows from the Lebesgue dominated convergence theorem for $p < +\infty$ and directly from the definition for $p = \infty$.

3 Sublattices, Ideals and Bands

We observe that a vector subspace Y of a vector lattice X, which is ordered by the order inherited from X, may fail to be a vector sublattice of X in the sense that Y may be not closed under lattice operations. For instance, the subspace

$$Y := \{f \in L_1(\mathbb{R}) \ : \ \int_{-\infty}^{\infty} f(t)dt = 0\}$$

does not contain any nontrivial nonnegative function, and thus it is not closed under the operations of taking $f_\pm$ or $|f|$.

Accordingly, we call Y a *vector sublattice* if Y is closed under lattice operations. Actually, it is sufficient (and necessary) if it is closed under one lattice operation; that is, Y is a vector sublattice if one of the following conditions holds: (i) $|x| \in Y$; (ii) $x_\pm \in Y$, whenever $x \in Y$.

A subspace I of a vector lattice is called an *ideal* if for any $x, y \in X$, $y \in I$ implies $|y| \in I$ and $0 \le x \le y$ implies $x \in I$; ideals are automatically vector sublattices. A *band* is an ideal that contains suprema of all its subsets.

Since vector sublattices, ideals and bands are closed under intersections, a subset $S \subset X$ uniquely determines the smallest (in the inclusion sense) vector sublattice (respectively, an ideal, a band) in X containing S, called the *vector sublattice* (respectively, *ideal, band*) *generated by* S.

Proposition 2.31. *If $S = \{x\}$, $x \ge 0$, consists of a single point, then the ideal generated by it, called the* principal ideal *generated by x, is given by*

$$E_x = \{y \in X \ : \ \text{there exists } \lambda \ge 0 \text{ such that } |y| \le \lambda x\} = \bigcup_{k\in\mathbb{N}} k[-x, x].$$

Proof. Let $I = \{y \in X \ : \ \text{there exists } \lambda \geq 0 \text{ such that } |y| \leq \lambda x\}$; of course the second equality is obvious. I is an ideal containing x. Indeed, if $y_1, y_2 \in E_x$, then there are $k_1, k_2 \geq 0$ such that $-k_1 x \leq y_1 \leq k_1 y_1$ and $-k_2 x \leq y_2 \leq k_2 y_2$. Thus $\alpha y_1 + \beta y_2 \in (|\alpha| k_1 + |\beta| k_2)[-x, x] \subset I$. Also, if $0 \leq y \in I$, then $y \leq \lambda x$ for some $\lambda \geq 0$. Hence, if $0 \leq z \leq y$, then $0 \leq z \leq \lambda x$ and so $x \in I$. Thus, we have $E_x \in I$ by definition. On the other hand, let $y \in I$, then $|y| \leq \lambda x$ for some $\lambda \geq 0$, but then $|y/\lambda| \leq x$. This means y/λ belongs to any ideal containing x and hence $y \in E_x$. □

If for some vector $e \in X$ we have $E_e = X$, then e is called an *order unit.*

Example 2.32. Any strictly positive function is an order unit in $C(\Omega)$, Ω compact.

Theorem 2.33. *Let X be a Banach lattice, $e \in X_+$ and let E_e be the principal ideal generated by e and define*

$$\|y\|_e = \inf\{\lambda > 0;\ y \in \lambda[-e, e]\}, \quad y \in E_e. \tag{2.3.10}$$

Then $(E_e, \|\cdot\|_e)$ is an AM-space having e as an order unit.

Proof. To prove that $\|\cdot\|_e$ is a norm, first let $\|y\| = 0$. This shows that $-\lambda e \leq y \leq \lambda e$ for any $\lambda > 0$ and hence, since X is Archimedean, $|y| = 0$ yielding $y = 0$. Homogeneity follows from

$$\|\alpha y\|_e = \inf\{\lambda > 0;\ \alpha y \in \lambda[-e, e]\} = \inf\{\lambda > 0;\ |\alpha| y \in \lambda[-e, e]\} = |\alpha| \inf\left\{\frac{\lambda}{|\alpha|} > 0;\ y \in \frac{\lambda}{|\alpha|}[-e, e]\right\}.$$

Similarly, if $-\lambda_1 e \leq x \leq \lambda_1 e$ and $-\lambda_2 e \leq y \leq \lambda_2 e$ for any $\lambda_1 \geq \|x\|_e$ and $\lambda_2 \geq \|y\|_e$, then

$$-\lambda_1 e - \lambda_2 e \leq x + y \leq \lambda_1 e + \lambda_2 e.$$

Hence, $\inf\{\lambda > 0;\ x + y \in \lambda[-e, e]\} \leq \lambda_1 + \lambda_2$ for any $\lambda_1 \geq \|x\|_e$ and $\lambda_2 \geq \|y\|_e$ and so

$$\|x + y\|_e \leq \|x\|_e + \|y\|_e.$$

Further, it is clearly a lattice norm satisfying $\|x\|_e = \||x|\|_e$.

Let $0 \leq x, y \in E_e$. We may assume $\|x\|_e \leq \|y\|_e$. Let $\|y\|_e < c$ for some $c > 0$. Then $x, y \leq ce$ and hence $0 \leq x \vee y \leq ce$ but this implies $\|x \vee y\|_e \leq c$ for any $c > \|y\|_e$ and hence $\|x \vee y\|_e \leq \|y\|_e = \max\{\|x\|_e, \|y\|_e\}$. On the other hand, since $\|\cdot\|_e$ is a lattice norm and $x, y \leq x \vee y$, $\max\{\|x\|_e, \|y\|_e\} \leq \|x \vee y\|_e$.

Finally, let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the norm $\|\cdot\|_e$. By selecting a subsequence, we can assume that $\|x_{n+1} - x_n\|_e < 2^{-n}$. This means, however, that $|x_{n+1} - x_n| \leq 2^{-n} e$. Using the fact that X is a Banach lattice, we have $\|x_{n+1} - x_n\| \leq 2^{-n}\|e\|$ and thus $\lim_{n\to\infty} x_n = x \in X$. Furthermore, for any n and $m = n + p \geq n$ we have

$$|x_{n+p} - x_n| \leq \left|\sum_{i=1}^{p} (x_{n+i} - x_{n+p-1})\right| \leq e 2^{-n} \sum_{i=0}^{p-1} 2^{-i} \leq 2^{1-n} e.$$

Since the modulus is continuous in the norm of X, we find

$$|x - x_n| \leq 2^{1-n} e \tag{2.3.11}$$

for any n. From the definition of infimum, $|x| \leq \|x\|_e e$. This shows, in particular,

$$|x| \leq |x_1| + e \leq (1 + \|x_1\|_e) e$$

and thus $x \in E_e$ and therefore (2.3.11) shows that

$$\lim_{n\to\infty} \|x - x_e\|_e = 0.$$

That e is an order unit of E_e follows directly from Proposition 2.31. □

As a corollary we note

Corollary 2.34. *Let X be a Banach lattice and $e \in X_+$. If $A \subset E_e$ is relatively compact in $(E_e, \|\cdot\|_e)$, then* $\sup A \in E_e$.

Proof. Since $[-e,e]$ is the unit ball in E_e, for any n there is $\{x_1^n, \ldots, x_{r_n}^n\} \subset A$ such that

$$A \subset \bigcup_{i=1}^{r_n} \{x \in E;\ x_i^n + n^{-1}[-e,e]\}.$$

Define $z_n = x_1^n \vee \cdots \vee x_{r_n}^n$. For any $x \in A$, since $x \leq x_i^n + n^{-1}e$ for some $i \in \{1, \ldots, r_n\}$, we have $x \leq z_n + n^{-1}e$. Such a z_n can be created for any n. Let $y_n = z_1 \vee \ldots \vee z_n$. Let us fix n and arbitrary $p \in \mathbb{N}_0$. Then $y_{n+p} = z_1 \vee \ldots \vee z_{n+p} = (x_1^1 \vee \ldots \vee x_{r_1}^1) \vee \ldots \vee (x_1^{n+p} \vee \ldots \vee x_{r_{n+p}}^{n+p})$ and since any $x_i^k, i = 1, \ldots, r_k, k = 1, \ldots, n+p$ belongs to A, we have

$$z_n \leq y_n \leq y_{n+p} \leq z_n + n^{-1}e,$$

or

$$0 \leq y_{n+p} - y_n \leq n^{-1}e$$

for all $p \geq 0$. This shows that for any $\epsilon > 0$ there is $n = \lfloor \epsilon^{-1} \rfloor + 1$ such that $\|y_k - y_n\|_e \leq \epsilon$ for any $k \geq n$. Thus $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and there is $E_e \ni y = \lim_{n\to\infty} y_n$.

By construction, for any $x \in A$ and for any n there is $x_i^n \in A$ such that

$$x \leq x_i^n + n^{-1}e \leq y + n^{-1}e$$

and since n is arbitrary, y is an upper bound for A. On the other hand, if $z \geq x$ for any $x \in A$, then $z_n \leq z$ for any n and hence

$$y \leq z_n + n^{-1}e \leq z + n^{-1}e$$

for any n, leading to $y \leq z$. Thus $y = \sup A$. □

3.1 Complexification

In some cases, especially when we want to use spectral theory, we need to move the problem to a complex space. This is done by the procedure called *complexification.*

Definition 2.35. *Let X be a real vector lattice. The complexification X_C of X is the set of pairs $(x,y) \in X \times X$ where, following the scalar convention, we write $(x,y) = x + iy$. Vector operations are defined as in scalar case*

$$\begin{aligned} x_1 + iy_1 + x_2 + iy_2 &= x_1 + x_2 + i(y_1 + y_2), \\ (\alpha + i\beta)(x + iy) &= \alpha x - \beta y + i(\beta x + \alpha y). \end{aligned}$$

The partial order in X_C is defined by

$$x_0 + iy_0 \leq x_1 + iy_1 \quad \text{if and only if} \quad x_0 \leq x_1 \text{ and } y_0 = y_1. \tag{2.3.12}$$

The operators of the complex adjoint, real part, and imaginary part of $z = x + iy$ are defined through:

$$\begin{aligned} \bar{z} &= \overline{x + iy} = x - iy, \\ \Re z &= \frac{z + \bar{z}}{2} = x, \\ \Im z &= \frac{z - \bar{z}}{2i} = y. \end{aligned}$$

Remark 2.36. Note, that from the definition, it follows that $x \geq 0$ in X_C is equivalent to $x \in X$ and $x \geq 0$ in X. In particular, X_C with partial order (2.3.12) is not a lattice.

It is a more complicated task to introduce a norm on X_C because standard product norms, in general, fail to preserve the homogeneity of the norm.

Example 2.37. Let us norm $X_C = X \times X$ by the Euclidean norm. Then,

$$\|(1+i)(x+iy)\|^2 = 2(\|x\|^2 + \|y\|^2),$$

and on the other hand,

$$\|(1+i)(x+iy)\|^2 = \|(x-y) + i(x+y)\|^2 = \|x-y\|^2 + \|x+y\|^2$$

which gives the parallelogram identity in X yielding X to be a Hilbert space.

The simplest norm, compatible with multiplication by complex scalars, is

$$\|x+iy\|_C = \sup_{\theta\in[0,2\pi]} \|x\cos\theta + y\sin\theta\|. \tag{2.3.13}$$

It can be proved, [1, Problem 1.1.7], that this is a norm satisfying

$$\frac{1}{2}(\|x\| + \|y\|) \leq \|x+iy\|_C \leq \|x\| + \|y\|$$

so that topological properties of X_C and X are the same.

The disadvantage of (2.3.13) is that $(X_C, \|\cdot\|_C)$ will usually not inherit the lattice structure from X. Thus it is important to find a norm on X_C which is compatible with the order in X_C. This is done by first introducing the modulus on X_C. In the scalar case we obviously have

$$\begin{aligned}
&\sup_{\theta\in[0,2\pi]} (\alpha\cos\theta + \beta\sin\theta) \\
&= |\alpha + i\beta| \sup_{\theta\in[0,2\pi]} \left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\cos\theta + \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\sin\theta\right) \\
&= |\alpha + i\beta| \sup_{\theta\in[0,2\pi]} \cos(\theta - \theta_0) = |\alpha + i\beta|,
\end{aligned} \tag{2.3.14}$$

where $\cos\theta_0 = \alpha/\sqrt{\alpha^2+\beta^2}$ and $\sin\theta_0 = \beta/\sqrt{\alpha^2+\beta^2}$. Mimicking this,we have

Proposition 2.38. *For $x + iy \in X_C$, the modulus*

$$|x+iy| = \sup_{\theta\in[0,2\pi]} (x\cos\theta + y\sin\theta)$$

exists.

Proof. Consider $e = |x| + |y| > 0$ and E_e. Since X is a Banach lattice, E_e is an AM-space with the unit e. Then $x\cos\theta + y\sin\theta \in [-e, e] \subset E_e$ for any $\theta \in [0, 2\pi]$. Further, since linear operations are continuous (in E_e as well) and $[0, 2\pi]$ is compact, we see that

$$A = \{x\cos\theta + y\sin\theta;\ \theta \in [0, 2\pi]$$

is compact in E_e. Thus, by Corollary 2.34, $\sup_{\theta\in[0,2\pi]} (x\cos\theta + y\sin\theta)$ exists. □

Such a defined modulus has all standard properties of the scalar complex modulus, [1, Problem 3.2.2]: for any $z, z_1, z_2 \in X_C$ and $\lambda \in C$,

(a) $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$,

(b) $|\lambda z| = |\lambda||z|$,

(c) $|z_1 + z_2| \leq |z_1| + |z_2|$ (triangle inequality),

and thus one can define another norm on the complexification X_C by

$$\|z\|_c = \|x + iy\|_c = \||x + iy|\|. \tag{2.3.15}$$

Properties (a)–(c) and $|x| \leq |z|$, $|y| \leq |z|$ imply

$$\frac{1}{2}(\|x\| + \|y\|) \leq \|z\|_c \leq \|x\| + \|y\|,$$

hence $\|\cdot\|_c$ is a norm on X_C which is equivalent to $\|\cdot\|_C$. As the norm $\|\cdot\|$ is a lattice norm, we have $\|z_1\|_c \leq \|z_2\|_c$, whenever $|z_1| \leq |z_2|$, and $\|\cdot\|_c$ becomes a lattice norm on X_C. We observe that $z \to |z|$ is continuous with repect to $\|\cdot\|_c$.

Definition 2.39. *A complex Banach lattice is an ordered complex Banach space X_C that arises as the complexification of a real Banach lattice X, according to Definition 2.35, equipped with the norm (2.3.15).*

4 Some other stuff

The *principal band* generated by $x \in X$ is given by

$$B_x = \{y \in X \ : \ \sup_{n\in\mathbb{N}}\{|y| \wedge n|x|\} = |y|\}.$$

An element $e \in X$ is said to be a *weak unit* if $B_e = X$. It follows that $e \geq 0$ is a weak unit in a vector lattice X if and only if for any $x \in X$, $|x| \wedge e = 0$ implies $x = 0$. Every order unit is a weak unit. If $X = C(\Omega)$, where Ω is compact, then any strictly positive function is an order unit. On the other hand, L_p spaces, $1 \leq p < \infty$, will not typically have order units, as they include functions that could be unbounded. However, any strictly positive a.e. L_p function is a weak order unit.

An intermediate notion between order unit and weak order unit is played by *quasi-interior points*. We say that $0 \neq u \in X_+$ is a quasi-interior point of X if $\overline{E_u} = X$. We have

Lemma 2.40. *[1, Lemma 4.15] For $u > 0$ the following are equivalent.*

(a) u is a quasi-interior point of X;

(b) For each $x \in X_+$ we have $\lim_{n\to\infty} \|x \wedge nu - x\| = 0$;

(c) If $0 < x^* \in X_+^*$, then $\langle x^*, u\rangle > 0$.

It is clear that $f \in L_p(\Omega, \mu), 1 \leq p < \infty$, where μ is σ-finite, is a quasi-interior point if and only if $f(s) > 0$ for almost all $s > 0$.

3

Operators on Banach Lattices

1 Types of operators

Definition 3.1. *A linear operator A from a Banach lattice X into a Banach lattice Y is called positive, denoted $A \geq 0$, if $Ax \geq 0$ for any $x \geq 0$. An operator A is called strictly positive if $Ax > 0$ for any $x > 0$.*

Proposition 3.2. *An operator A is positive if and only if $|Ax| \leq A|x|$.*

Proof. The proposition follows easily from $-|x| \leq x \leq |x|$ so, if A is positive, then $-A|x| \leq Ax \leq A|x|$. Conversely, taking $x \geq 0$, we obtain $0 \leq |Ax| \leq A|x| = Ax$. □

Proposition 3.3. *If A is positive, then*

$$\|A\| = \sup_{x \geq 0,\, \|x\| \leq 1} \|Ax\|.$$

Proof. Because $\|A\| = \sup_{\|x\| \leq 1} \|Ax\| \geq \sup_{x \geq 0, \|x\| \leq 1} \|Ax\|$, it is enough to prove the opposite inequality. For each x with $\|x\| \leq 1$ we have $|x| = x_+ + x_- \geq 0$ with $\|x\| = \||x|\| \leq 1$. On the other hand, $A|x| \geq |Ax|$, hence $\|A|x|\| \geq \||Ax|\| = \|Ax\|$. Thus $\sup_{\|x\| \leq 1} \|Ax\| \leq \sup_{x \geq 0, \|x\| \leq 1} \|Ax\|$ and the statement is proved. □

Remark 3.4. As a consequence, we note that if $0 \leq A \leq B$, then $\|A\| \leq \|B\|$. Moreover, it is worthwhile to emphasize that if there exists K such that $\|Ax\| \leq K\|x\|$ for $x \geq 0$, then this inequality holds for any $x \in X$. Indeed, by Proposition 3.3 we have $\|A\| \leq K$ and using the definition of the operator norm, we obtain the desired statement.

The space of linear operators $\mathcal{L}(X,Y)$ (between real ordered vector spaces) is an ordered vector space becomes an ordered vector space by defining $A_1 \geq A_2$ if and only if $A_1 - A_2 \geq 0$. We introduce two further classes of operators.

Definition 3.5.

The operator $A \in \mathcal{L}(X,Y)$ is called regular if A can be written as $A = A_1 - A_2$, where A_1, A_2 are positive operators. The space of regular operators is denoted by $\mathcal{L}_r(X,Y)$.

The operator $A \in \mathcal{L}(X,Y)$ is called order bounded whenever it maps any order interval of X into an order interval in Y. The space of order bounded operators is denoted by $\mathcal{L}_b(X,Y)$.

We observe that A is regular if and only if it is dominated by a positive operator that is, there is a positive B such that $A \leq B$. Further, A is order bounded if and only if it maps intervals of the form $[0,x]$ in X into intervals in Y. Hence, in particular, positive operators are both regular and order bounded. Similarly, any regular operator is order bounded.

Positive operators are fully determined by their behaviour on the positive cone. Precisely speaking, we have the following theorem.

Theorem 3.6. *[2, Theorem 1.10] If $A : X_+ \to Y_+$ is additive, then A extends uniquely to a positive linear operator from X to Y. Keeping the notation A for the extension, we have, for each $x \in X$,*

$$Ax = Ax_+ - Ax_-. \tag{3.1.1}$$

Proof. Because the operation of taking positive and negative part is not linear, it is not a priori clear that $Ax := Ax_+ - Ax_-$ is an additive operator. However, by taking two representations of x: $x = x_+ - x_- = x_1 - x_2$ with $x_+, x_-, x_1, x_2 \geq 0$, we see that $x_+ + x_2 = x_- + x_1$ so that $Ax_+ - Ax_- = Ax_1 - Ax_2$ and Ax is independent of the representation of x. As $x + y = x_+ + y_+ - (x_- + y_-)$ is a representation of $x + y$ we see that $A(x + y) = A(x_+ - x_-) + A(y_+ - y_-) = Ax + Ay$.

To prove homogeneity of A, we first observe that if $0 \leq y \leq x$, then $Ay \leq Ax$. Obviously, from the additivity, it follows that A is finitely additive and satisfies $A(-x) = -A(x)$; thus it is homogeneous with respect to rational numbers. Indeed, taking $r = p/q$, where p and q are integers, we have

$$pA(x) = A(px) = A\left(q\frac{p}{q}x\right) = qA\left(\frac{p}{q}x\right).$$

Now, let $x \in X_+$, $\lambda \geq 0$, and choose sequences of rational numbers $(r_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ satisfying $0 \leq r_n \leq \lambda \leq t_n$ for all $n \in \mathbb{N}$ and monotonically converging to λ. From the homogeneity for rational numbers we obtain

$$r_n A(x) = A(r_n x) \leq A(\lambda x) \leq A(t_n x) = t_n A(x),$$

from where, using the fact that X is Archimedean, we obtain $A(\lambda x) = \lambda Ax$. Finally, by taking arbitrary $x \in X$ and $\lambda \geq 0$ we have

$$A(\lambda x) = A(\lambda x_+) - A(\lambda x_-) = \lambda(A(x_+ - x_-)) = \lambda Ax,$$

and for $\lambda < 0$ the thesis follows by

$$A(\lambda x) = -A(-\lambda x) = \lambda Ax.$$

Finally, let us denote by B any other linear extension of A. It must be a positive operator and because it is linear it must satisfy

$$Bx = B(x_+ - x_-) = Bx_+ - Bx_- = Ax_+ - Ax_- = Ax,$$

and hence the extension is unique. □

Another frequently used property of positive operators is given in the following theorem.

Theorem 3.7. *If A is an order bounded operator from a Banach lattice to a normed Riesz space, then A is bounded.*

Proof. By Proposition 2.19 (5), the order interval is norm bounded.

If A were not bounded, then we would have a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $\|x_n\| = 1$ and $\|Ax_n\| \geq n^3$, $n \in \mathbb{N}$. Since $\|Ax_n\| \leq \|Ax_{n,+}\| + \|Ax_{n,-}\|$, we may assume that one of the sequences, $(\|Ax_{n,+}\|)_{n\geq 1}$ or $(\|Ax_{n,-}\|)_{n\geq 1}$ is greater than n^3. So, we can assume that we have

$$x_n \geq 0, \quad \|x_n\| \leq 1, \quad \|Ax_n\| \geq n^3.$$

Because X is a Banach space,

$$x := \sum_{n=1}^{\infty} n^{-2} x_n \in X.$$

Because $0 \leq x_n/n^2 \leq x$, all elements $\{x/n^2\}_{n\geq 1}$ are contained in the order interval $[0, x]$. On the other hand, $\|A(x_n/n^2)\| \geq n$ which contradicts the definition of an order bounded operator. □

Corollary 3.8. *Any positive operator A defined on the whole space is norm bounded.*

Example 3.9. The assumption that X in Theorem 3.7 is a complete space is essential. Indeed, let X be a space of all real sequences which have only a finite number of nonzero terms. It is a normed Riesz space under the norm $\|\boldsymbol{x}\| = \sup_n\{|x_n|\}$, where $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}}$. Consider the functional

$$f(\boldsymbol{x}) = \sum_{n=1}^{\infty} x_n.$$

It is a positive everywhere defined linear functional. However, taking the sequence of elements $\boldsymbol{x}_n = (1, 1, \ldots, 1, 0, 0, \ldots)$, where 0 appears starting from the $n+1$st place, we see that $\|\boldsymbol{x}_n\| = 1$ and $f(\boldsymbol{x}_n) = n$ for each $n \in \mathbb{N}$ so that f is not bounded.

A striking consequence of this fact is that all norms, under which X is a Banach lattice, are equivalent as the identity map must be continuously invertible, [2, Corollary 12.4].

Theorem 3.10. *Let X and Y be Banach lattices with Y being Dedekind complete. A linear operator $A : X \to Y$ is order bounded if and only if it is regular. Furthermore, $\mathcal{L}_b(X,Y)$ is Dedekind complete with the set of positive operators from X to Y being the positive cone.*

Proof. We noted earlier that regular operators are order bounded. For the converse, let $A : X \to Y$ be order bounded. Then, for any given $x \in X_+$, the set $\{Av;\ 0 \leq v \leq x\}$ is contained in an order interval in Y. Since Y is Dedekind complete,

$$\alpha(x) := \sup\{Av;\ 0 \leq v \leq x\}$$

exists in Y. We observe that

$$\alpha(x) \geq 0, \qquad \alpha(x) \geq Ax, \quad x \in X_+. \tag{3.1.2}$$

We prove that α is additive on X_+. Consider $x_1, x_2 \in X_+$. Then

$$\alpha(x_1) + \alpha(x_2) = \sup\{A(v_1 + v_2);\ 0 \leq v_1 \leq x_1, 0 \leq v_1 \leq x_2\} \leq \alpha(x_1 + x_2).$$

To prove the reverse inequality, let $0 \leq v \leq x_1 + x_2$. By the Riesz decomposition property, Proposition 2.15, there are $0 \leq v_1 \leq x_1$ and $0 \leq v_2 \leq x_2$ such that $v = v_1 + v_2$ and hence

$$Av = Av_1 + Av_2 \leq \alpha(x_1) + \alpha(x_2).$$

Taking the supremum gives $\alpha(x_1 + x_2) = \alpha(x_1) + \alpha(x_2)$. So, α is an additive mapping from X_+ to Y_+ and thus, by Theorem 3.6, it extends to a linear positive operator on X. Denote it by A_1 and define $A_2 = A_1 - A$. By (3.1.2), $A_1, A_2 \geq 0$ and thus $A = A_1 - A_2$ implies that A is regular.

Next we prove that $\mathcal{L}_b(X,Y) = \mathcal{L}_r(X,Y)$ is Dedekind complete. Let $A \in \mathcal{L}_b(X,Y)$ and

$$A_1 x = \sup\{Av;\ 0 \leq v \leq x\}, \quad x \in X_+.$$

Then, as noted above, A_1 is an upper bound of both A and the nul operator. If A' is another upper bound, then $A'x \geq A'v \geq Av$ for any $0 \leq v \leq x$, since A' is a positive operator (being an upper bound for the zero operator). Hence

$$A'x \geq \sup\{Av;\ 0 \leq v \leq x\} = A_1 x.$$

Hence A_1 is the least upper bound, or supremum, of A and 0, $A_1 = A \vee 0 = A^+$. If we take now arbitrary $A, B \in \mathcal{L}_b(X,Y)$, then

$$A + (B - A)_+ = A + (B - A) \vee 0 = A \vee B,$$

as in Proposition 2.8 (2).

To prove the Dedekind completeness, let D be an upwards directed and bounded from above set in $\mathcal{L}_b(X,Y)_+$. Let

$$\tau(x) = \sup\{Tx;\ T \in D\}, \quad x \in X_+.$$

We observe that $\tau(x)$ exists in Y_+ since D is bounded and Y is Dedekind complete. By the definition of upwards directed set, for any $A_1, A_2 \in D$, there is $A_3 \in D$ such that $A_3 \geq A_1, A_3 \geq A_2$. Taking $x_1, x_2 \in X_+$, we thus have

$$A_1x_1 + A_2x_2 \leq A_3(x_1 + x_2) \leq \tau(x_1 + x_2).$$

Taking suprema, we find

$$\tau(x_1) + \tau(x_2) \leq \tau(x_1 + x_2).$$

The inequality in the opposite direction follows as for any $A \in D, x_1, x_2 \in X_+$,

$$A(x_1 + x_2) = A(x_1) + A(x_2) \leq \tau(x_1) + \tau(x_2)$$

and taking supremum on the left hand side. Hence the mapping $\tau : X_+ \to Y_+$ is additive and thus it extends to a positive operator $A_0 : X \to Y$. Since $A_0x = \sup\{Ax;\ A \in D\}, x \in X_+$ it must satisfy $A_0 = \sup D$. □

Theorem 3.11. *As before, let X and Y be Riesz speces with Y being Dedekind complete, and let $A, B \in \mathcal{L}_b(X,Y)$. Then, for any $x \in X_+$, we have*

1. $A_+x = \sup\{Av;\ 0 \leq v \leq x\}$,
2. $-A_-x = \inf\{Av;\ 0 \leq v \leq x\}$,
3. $(A \vee B)(x) = \sup\{Av + Bw;\ v \geq 0, w \geq 0, v + w = x\}$,
4. $(A \wedge B)(x) = \inf\{Av + Bw;\ v \geq 0, w \geq 0, v + w = x\}$,
5. $|A|(x) = \sup\{Av;\ |v| \leq x\} = \sup\{|Av|;\ |v| \leq x\}$,
   and for any $x \in X$
6. $|Ax| \leq |A|(|f|)$.

Proof. The operator A_+, as defined above, coincides with the operator A_1 defined in the previous proof, where it was also proved that $A_1 = A \vee 0$ and hence A_+ indeed is the positive part of A. Then, by Proposition 2.10 and Proposition 2.8 2. & 3.,

$$\begin{aligned} -A_-x &= Ax - A_+x = Ax - \sup\{Av;\ 0 \leq v \leq x\} = Ax + \inf\{-Av;\ 0 \leq v \leq x\} \\ &= \inf\{A(x - v);\ 0 \leq v \leq x\} = \inf\{Az;\ 0 \leq z \leq x\}, \end{aligned}$$

which proves 2.

To prove (3), we define $A_3 = A_1 \vee A_2$ so that, as in Remark 2.9, $A_3 = A_2 + (A_1 - A_2)_+$. Thus, by Proposition 2.8 2.,

$$A_3x = \sup\{(A_1 - A_2)y;\ 0 \leq y \leq x\} + A_2x = \sup\{(A_1y + A_2(x - y);\ 0 \leq y \leq x\}$$

which is the desired result. The proof of (4) is analogous.

To prove (5), we have

$$|A|(x) = A_+x + A_-x = \sup\{Ay;\ 0 \leq y \leq x\} + \sup\{-Az;\ 0 \leq z \leq x\} = \sup\{A(y - z);\ 0 \leq y \leq x, 0 \leq z \leq x\}.$$

Since $0 \leq y \leq x, 0 \leq z \leq x$ implies $|y - z| \leq x$, we have further

$$|A|(x) = \sup\{Af;\ |f| \leq x\}.$$

On the other hand, $|f| \leq x$ implies

$$Af \leq |Af| \leq |A|(|f|) \leq |A|(x)$$

and so

$$\sup\{Af;\ |f| \leq x\} \leq \sup\{|Af|;\ |f| \leq x\} \leq |A|(x)$$

from where (5) follows.

1.1 Positive functionals

A particular role among linear operators is played by linear functionals. As in the case of operators, functionals can be order bounded, regular, positive. In particular, a functional x^* is said to be positive if $\langle x^*, x\rangle > 0$ for any $x \in X_+$.

Definition 3.12. *The space of all order bounded functional on X is denoted by $X^\sim$ and is called the order dual of X.*

The set of all (order bounded) positive functionals is denoted by $X_+^\sim$.

Since $\mathbb{R}$ is Dedekind complete, we have

Corollary 3.13. *If X is a Riesz space, a linear functional on E is order bounded if and only if it is regular. The space of all order bounded funcytionals $X^\sim$ is a Dedekind complete Riesz space.*

Proposition 3.14. *For every $x \in X$, $x \in X_+$ if and only if $\langle x^*, x\rangle > 0$ for any $x^* \in X_+^*$.*

Proof. If $x \in X_+$, then $\langle x^*, x\rangle > 0$ for any $x^* \in X_+^*$ by definition.

To prove the proposition in the opposite direction, consider $x \in X$ such that $\langle x^*, x\rangle > 0$ for any $x^* \in X_+^*$ and assume $x \notin X_+$. Since $\{x\}$ is compact and X_+^* is closed, the Hahn–Banach theorem (geometric form) asserts that there is $x^* \in X^*$ such that

$$\langle x^*, x\rangle < \inf_{y \in X_+} \langle x^*, y\rangle.$$

We prove that $x^* \in X_+*$. Indeed, if it was not true, then there would be an element $y_0 \in X_+$ such that $\langle x^*, y_0\rangle < 0$. Since $ty_0 \in X_+$ for any $t \geq 0$

$$\langle x^*, x\rangle < \inf_{y \in X_+} \langle x^*, y\rangle \leq \inf_{t \geq 0} \langle x^*, ty_0\rangle \lim_{t \to \infty} t\langle x^*, y_0\rangle = -\infty$$

that is a contradiction. Hence $x \in X_+$. □

It turns out that the adjoint X^* of a vector lattice with a lattice norm is a Banach lattice, [2, Theorem 4.1]. The positive cone X_+^* in X^* is precisely the cone of positive functionals in the sense of Definition 3.1. In particular, if $\langle x^*, x\rangle > 0$ for any $x > 0$, then x^* is called strictly positive.

1.2 Positive operators on complex Banach lattices

We introduced the concept of complex Banach lattice X_C in Section 3.1. We begin with a simple observation.

Proposition 3.15. *Any positive linear operator A on X_C is a real operator; that is, $A : X \to X$.*

Proof. Let $X \ni x = x_+ - x_-$. By definition, $Ax_+ \geq 0$ and $Ax_- \geq 0$ so $Ax_+, Ax_- \in X$ and thus $Ax = Ax_+ - Ax_- \in X$. □

If A is a linear operator on X, then it can be extended to X_C according to the formula

$$A_C(x+iy) = Ax + iAy.$$

If we use the norm (2.3.13),

$$\|x+iy\|_C = \sup_{\theta\in[0,2\pi]} \|x\cos\theta + y\sin\theta\|.$$

Clearly, we have $\|A\| \leq \|A_C\|_C$. Moreover,

$$\|(Ax)\cos\theta + (Ay)\sin\theta\| \leq \|A\|\|x\cos\theta + y\sin\theta)\| \leq \|A\|\|x+iy\|,$$

thus taking supremum over θ we obtain $\|A_C\| \leq \|A\|$ and finally

$$\|A_C\|_C = \|A\|. \tag{3.1.3}$$

As we noted earlier, the disadvantage of (2.3.13) is that $(X_C, \|\cdot\|_C)$ is not consistent with complex operations on X_C. Thus, we defined another norm, (2.3.15),

$$\|z\|_c = \|x+iy\|_c = \||x+iy|\|.$$

We observe that if A is a positive operator between real Banach lattices X and Y then, for $z = x+iy \in X_C$, we have

$$(Ax)\cos\theta + (Ay)\sin\theta = A(x\cos\theta + y\sin\theta) \leq A|z|$$

and therefore $|A_C z| \leq A|z|$. Hence for positive operators

$$\|A_C\|_c = \|A\|. \tag{3.1.4}$$

There are examples, where $\|A\| < \|A_C\|_c$, contrary to the previous complexification norm (see [1, Problem 3.2.9]).

Note that the standard $L_p(\Omega)$ and $C(\Omega)$ norms are of the type (2.3.15). These spaces have a nice property of preserving the operator norm even for operators which are not necessarily positive. To show this for $L_p(\Omega)$, let us note that, in a similar way to (2.3.14),

$$\int_{-\pi}^{\pi} |\alpha\cos\theta + \beta\sin\theta|^p d\theta = |\alpha + i\beta|^p \int_{-\pi}^{\pi} |\cos(\theta-\theta_0)|^p d\theta = \Theta|\alpha+i\beta|^p,$$

where $\Theta = \int_{-\pi}^{\pi} |\cos\theta|^p d\theta$. Therefore

$$\begin{aligned}
\|A_C z\|_c^p &= \int_\Omega |(Ax)(\omega) + i(Ay)(\omega)|^p d\omega \\
&= \Theta^{-1} \int_\Omega \int_{-\pi}^{\pi} |(Ax)(\omega)\cos\theta + (Ay)(\omega)\sin\theta|^p d\theta d\omega \\
&= \Theta^{-1} \int_{-\pi}^{\pi} \int_\Omega |(A(x\cos\theta + y\sin\theta)(\omega)|^p d\omega d\theta \\
&\leq \|A\|^p \int_\Omega \left(\Theta^{-1} \int_{-\pi}^{\pi} |x(\omega)\cos\theta + y(\omega)\sin\theta|^p d\theta\right) d\omega = \|A\|^p\|z\|_c^p.
\end{aligned}$$

For $C(\Omega)$ this follows by (2.3.14) as we can interchange the order of taking suprema.

2 Perron–Frobenius theorems

2.1 Basics of spectral theory

Let $A \in \mathcal{L}(X)$. The *resolvent set* of A is defined as

$$\rho(A) = \{\lambda \in \mathbb{C};\ \lambda I - A : X \to X \text{ is invertible}\}. \tag{3.2.5}$$

We call $(\lambda I - A)^{-1}$ the resolvent of A and denote it by $R(\lambda, A) = (\lambda I - A)^{-1}$, for $\lambda \in \rho(A)$. The complement of $\rho(A)$ in $\mathbb{C}$ is called the *spectrum* of A and denoted by $\sigma(A)$. In general, it is possible that either $\rho(A)$ or $\sigma(A)$ is empty. In what follows we always assume that the resolvent set is non-empty. The spectrum is usually subdivided into several subsets. First,

- *Point spectrum* $\sigma_p(A)$ is the set of $\lambda \in \sigma(A)$ for which the operator $\lambda I - A$ is not one-to-one. In other words, $\sigma_p(A)$ is the set of all eigenvalues of A.
- *Continuous spectrum* $\sigma_c(A)$ is the set of $\lambda \in \sigma(A)$ for which the operator $\lambda I - A$ is one-to-one and its range is dense in, but not equal to, X
- *Residual spectrum* $\sigma_r(A)$ is the set of $\lambda \in \sigma(A)$ for which $Im\ (\lambda I - A)$ is not dense in X.

If A is bounded, then the number

$$r(A) = \limsup_{n\to\infty} \sqrt[n]{\|A^n\|} \tag{3.2.6}$$

is called *the spectral radius.*

Theorem 3.16. *Let X be a Banach space and let A be a linear operator with domain $D(A) \subseteq X$. Then the following assertions are true.*

1. *The resolvent set $\rho(A)$ is open, hence the spectrum is closed.*
2. *The mapping $\rho(A) \to R(\lambda, A) \in \mathcal{L}(X)$ is complex differentiable. Moreover, for $k \in \mathbb{N}$ we have*

$$\frac{d^k}{d\lambda^k} R(\lambda, A) = (-1)^k k! R(\lambda, A)^{k+1}. \tag{3.2.7}$$

3. *If $A \in \mathcal{L}(X)$, then for every $\lambda \in \mathbb{C}$ with $|\lambda| > r(A)$ we have $\lambda \in \rho(A)$ and*

$$R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n, \tag{3.2.8}$$

where the series converges in the operator norm for $|\lambda| > r(A)$ and diverges for $|\lambda| < r(A)$. Moreover

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|. \tag{3.2.9}$$

Proof. Ad 1. & 2.) Let $\mu \in \rho(A)$. Then

$$\lambda I - A = (I - (\mu - \lambda) R(\mu, A))(\mu I - A)$$

and

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1} \tag{3.2.10}$$

for $|\mu - \lambda| < \|R(\mu, A)\|^{-1}$. This shows that $\rho(A)$ is open and hence $\sigma(A)$ is closed. Moreover, $R(\cdot, A)$ is analytic in $\rho(A)$ and (3.2.7) follows by comparing (3.2.10) with the Taylor expansion of $R(\lambda, A)$.

Ad 3.) If $A \in \mathcal{L}(X)$, then, by writing

$$\lambda I - A = \lambda\left(I - \frac{A}{\lambda}\right)$$

and using the Neumann series representation, we obtain (3.2.8) that, by Cauchy-Hadamard criterion, converges for $|\lambda| > \limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} = r(A)$. This shows that $r(A) \geq \sup_{\lambda\in\sigma(A)}\{|\lambda|\}$. To prove the converse, we observe that $R(\cdot, A)$ is analytic in $|\lambda| > \sup_{\lambda\in\sigma(A)}\{|\lambda|\}$ and hence it has a uniquely defined Laurent expansion there. Thus, this Laurent expansion must coincide with (3.2.8). Hence, in particular, $\lim_{n\to\infty} \|\lambda^{-n}T^n\| = 0$ for $|\lambda| \geq \sup_{\lambda\in\sigma(A)}\{|\lambda|\}$. For any ϵ there is λ such that $\sup_{\lambda\in\sigma(A)}\{|\lambda|\} \leq |\lambda| \leq \epsilon + \sup_{\lambda\in\sigma(A)}\{|\lambda|\}$ and hance for sufficiently large n we have

$$\|A^n\| \leq |\lambda|^n \leq \left(\epsilon + \sup_{\lambda\in\sigma(A)}\{|\lambda|\}\right)^n$$

and this shows $r(A) \leq \sup_{\lambda\in\sigma(A)}\{|\lambda|\}$. □

The resolvent of any operator A satisfies the *resolvent identity*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \qquad \lambda, \mu \in \rho(A), \tag{3.2.11}$$

from which it follows, in particular, that $R(\lambda, A)$ and $R(\mu, A)$ commute.

For any bounded operator the spectrum is a compact subset of $\mathbb{C}$ so that $\rho(A) \neq \emptyset$. Clearly, $r(A) \leq \|A\|$. To show that $\lambda \in \mathbb{C}$ belongs to the spectrum we often use the following result.

Theorem 3.17. *Let A be a closed operator. If $\lambda_n \to \lambda$, $\lambda_n \in \rho(A)$, then $\lambda \in \sigma(A)$ if and only if $\{\|R(\lambda_n, A)\|\}_{n\in\mathbb{N}}$ is unbounded. Furthermore, if $\lambda \in \rho(A)$, then $dist(\lambda, \sigma(A)) = 1/r(R(\lambda, A)) \geq 1/\|R(\lambda, A)\|$.*

Proof. To prove the first part, let $\lambda \in \rho(A)$. Then, by continuity of $R(\cdot, A)$ on $\rho(A)$, $R(\lambda, A)$ is finite. On the other hand, if $\|R(\lambda_n, A)\| \leq M$ for all n, then by (3.2.10) we see that each λ_n is a centre of a disc $|\mu - \lambda_n| < 1/M$, where the series converges and therefore defines a resolvent. Because the radii of these discs do not depend on n, λ belongs to some of them, thus $\lambda \in \rho(A)$.

To prove the second part, first we observe that for $\lambda \in \rho(A)$ we have

$$\sigma(R(\lambda, A)) \setminus \{0\} = \left\{\frac{1}{\lambda - \mu};\ \mu \in \sigma(A)\right\}. \tag{3.2.12}$$

Indeed, for $\alpha \neq 0$

$$\begin{aligned}(\alpha I - R(\lambda, A))f &= \alpha\left(\left(\lambda - \frac{1}{\alpha}\right)I - A\right)R(\lambda, A)f\\ &= \alpha R(\lambda, A)\left(\left(\lambda - \frac{1}{\alpha}\right)I - A\right)f,\end{aligned}$$

where in the second line $f \in D(A)$ if A is unbounded. Hence, $0 \neq \alpha \in \sigma(R(\lambda, A))$ if and only if $\lambda - \frac{1}{\alpha} \in \sigma(A)$. Hence, for $\lambda \in \rho(A)$,

$$\begin{aligned}\mathrm{dist}(\lambda, \sigma(A)) = \inf\{|\lambda - \mu|;\ \mu \in \sigma(A)\} &= \left(\sup\left\{\frac{1}{|\lambda - \mu|};\ \mu \in \sigma(A)\right\}\right)^{-1}\\ &= (\max\{|\alpha|;\ \alpha \in \sigma(R(\lambda, A))\})^{-1} = \frac{1}{r(R(\lambda, A))} \geq \frac{1}{\|R(\lambda, A)\|}.\end{aligned}$$

□

The *peripheral spectrum* of a bounded operator A is the set

$$\sigma_{per,r(A)} = \{\lambda \in \sigma(A);\ |\lambda| = r(A)\}. \tag{3.2.13}$$

Clearly, $\sigma_{per,r(A)}(A)$ is compact and, by (3.2.9), non-empty. Also, $r(A) \in \{|\lambda|;\ \lambda \in \sigma(A)\}$. This follows from the compactness of $\sigma(A)$.

As a more serious application of the theory of Banach lattices, we prove the abstract version of the Perron-Frobenius theorem. First we note that we can carry the considerations in the complexification of X, if necessary. Since all operators are positive, the operator norms in the real lattice and its complexification are equal, see (3.1.4), and we shall not distinguish them in the proofs.

First, we need some preliminary results.

Proposition 3.18. *Let $0 \leq A \in \mathcal{L}(X)$ and $r(A)$ be its spectral radius. Then*

1. *The resolvent $R(\lambda, A)$ is positive for $\lambda > r(A)$.*
2. *If $|\lambda| > r(A)$, then*
$$|R(\lambda, A)f| \leq R(|\lambda|, A)|f|, \quad f \in X. \tag{3.2.14}$$

Proof. To prove the first statement, we use the Neumann series representation (3.2.8):

$$R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n,$$

valid for $\lambda > r(A)$. Hence, if $A \geq 0$, then the statement follows from the closedness of the positive cone.

The second statement follows similarly, by the triangle inequality for the modulus and its continuity:

$$\begin{aligned}|R(\lambda, A)f| &= \left|\lim_{N\to\infty} \sum_{n=0}^{N} \lambda^{-(n+1)} A^n f\right| \leq \lim_{N\to\infty} \sum_{n=0}^{N} \left|\lambda^{-(n+1)} A^n f\right| \\ &\leq \lim_{N\to\infty} \sum_{n=0}^{N} |\lambda|^{-(n+1)} A^n |f| = \sum_{n=0}^{\infty} |\lambda|^{-(n+1)} A^n |f| = R(|\lambda|, A)|f|.\end{aligned}$$

Theorem 3.19. *Let $r(A)$ be the spectral radius of a positive operator A on a Banach lattice X. Then $r(A) \in \sigma(A)$.*

Proof. Let $\lambda_n = r(A) + 1/n$, then $\lambda_n \in \rho(A)$ for any n. Since $\lambda_n \to r(A)$, $r(A) \in \sigma(A)$ will follow, by Theorem 3.17, if $\lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty$.

Since the peripheral spectrum is non-empty, let $\alpha \in \sigma(A)$ with $|\alpha| = r(A)$ and define $\mu_n = \alpha\lambda_n/|\alpha|$. We have $\mu_n \in \rho(A)$ and $\mu_n \to \alpha$ so that, invoking Theorem 3.17 again, $\lim_{n\to\infty} \|R(\mu_n, A)\| = \infty$. Next, for each n we pick a unit vector $z_n \geq 0$, see Proposition 3.3, satisfying

$$\|R(\mu_n, A)z_n\| \geq \frac{1}{2}\|R(\mu_n, A)\|.$$

Using (3.2.14) we have

$$|R(\lambda, A)z| \leq R(|\lambda|, A)|z|$$

so that $|R(\mu_n, A)z_n| \leq R(\lambda_n, A)|z_n|$ and consequently

$$\|R(\lambda_n, A)\| \geq \|R(\lambda_n, A)|z_n|\| \geq \|R(\mu_n, A)z_n\| \geq \frac{1}{2}\|R(\mu_n, A)\|$$

which proves the thesis. □

Theorem 3.20. *If $A : X \to X$ is a compact positive operator on a Banach lattice X with $r(A) > 0$, then $r(A)$ is an eigenvalue with positive eigenvector.*

Proof. Let $r(A) > 0$. As above, we put $\lambda_n = r(A) + 1/n$ so that $\lambda_n \downarrow r(A)$ and $\|R(\lambda_n, A)\| \to \infty$ as $n \to \infty$. Furthermore, for each n there is z_n with $\|z_n\| = 1$ satisfying

$$\|R(\lambda_n, A)z_n\| \geq \frac{1}{2}\|R(\lambda_n, A)\|.$$

We define $x_n = R(\lambda_n, A)z_n/\|R(\lambda_n, A)z_n\|$ and note that x_n is a vector with $\|x_n\| \geq 1/2$. From

$$Ax_n - r(A)x_n = (\lambda_n - r(A))x_n + Ax_n - \lambda_n x_n = \frac{x_n}{n} - \frac{z_n}{\|R(\lambda_n, A)z_n\|}$$

we obtain

$$\|Ax_n - r(A)x_n\| \to 0, \quad n \to \infty.$$

Since A is compact, the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a convergent subsequence which we denote by $(Ax_n)_{n\in\mathbb{N}}$ again. Since $r(A) > 0$ and $\|x_n\| \geq 1$, the above implies that $\lim_{n\to\infty} x_n = x > 0$ satisfying $Ax = r(A)x$. □

Corollary 3.21. *The thesis of Theorem 3.20 remains valid if the positive operator A only is power compact.*

Proof. If $r = r(A) > 0$ and A is power compact, then from the Spectral Mapping Theorem we have $A^k x = r^k x$ for some $x > 0$. The element $y = \sum_{i=0}^{k-1} r^i A^{k-1-i}x > 0$ (from positivity of A, x and r), hence

$$Ay - ry = A^k x - r^k x = 0.$$

□

References

1. Y. A. Abramovich and C. D. Aliprantis. *An invitation to operator theory*, volume 50 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.
2. C. D. Aliprantis and O. Burkinshaw. *Positive operators*. Springer, Dordrecht, 2006.
3. A. Bátkai, M. K. Fijavž, and A. Rhandi. *Positive Operator Semigroups: from Finite to Infinite Dimensions*, volume 257. Birkhäuser, 2017.
4. P. Meyer-Nieberg. *Banach lattices*. Universitext. Springer-Verlag, Berlin, 1991.
5. B. Z. Vulikh. *Introduction to the theory of partially ordered spaces*, volume 1967. Wolters-Noordhoff Groningen, 1967.
6. A. C. Zaanen. *Introduction to operator theory in Riesz spaces*. Springer-Verlag, Berlin, 1997.

Index

AL-space, 15
AM-space, 15
$\Im z$-imaginary part, 20
$\Re z$-real part, 20
$\bar{z}$-complex adjoint, 20
σ-Dedekind complete Riesz space, 16
σ-order complete Riesz space, 16
$\sigma(A)$ – spectrum of A, 29

Archimedean Riesz spaces, 9

Banach lattice, 13
band, 17

complex Banach lattice, 21
complexification, 19
continuous spectrum, 29

Dedekind complete Riesz space, 16
directed upward set, 16

generating cone, 8

ideal, 17
infimum, 7

lattice, 8
lattice norm, 13
lattice operations, 10

maximal element, 7
minimal element, 7

order bounded operator, 23
order complete Riesz space, 16
order unit, 18
ordered vector spaces, 8

partial order, 7
point spectrum, 29
positive cone, 7, 8
positive element, 7
positive operator, 23
principal band, 21
principal ideal, 17

quasi-interior point, 21

regular operator, 23
residual spectrum, 29
resolvent identity, 30
Riesz space, 8

spectrum, 29
strictly positive functional, 27
strictly positive operator, 23
supremum, 7

the spectral radius, 29

vector lattice, 8
vector sublattice, 17

weak unit, 21