Honours essay topic

Alternative proofs of the Compactness Theorem in first-order model theory

Supervisor: Ruaan Kellerman

The Compactness Theorem, which is one of the fundamental results in first-order model theory, states that a set of first-order sentences Σ has a model if and only if each finite subset of Σ has a model. From this, many important model-theoretical and mathematical results can be obtained, for example, the Upwards Löwenheim-Skolem Theorem, the non-definability by a *finite* set of first-order sentences of the class of division rings of characteristic 0, and the De Bruin-Erdős Theorem in graph theory.

The usual methods for proving the Compactness Theorem in introductory logic texts are either to deduce it from Gödel's Completeness Theorem, or to prove it by using a Henkin construction. This essay will examine two additional methods of proof, one algebraic and one topological, namely using the Łoś Ultraproduct Theorem, and using the Stone space of the Lindenbaum-Tarski algebra.

With the Łoś Ultraproduct Theorem, the logical notion of truth is reduced to algebraic and set-theoretical notions that involve ultraproducts and ultrafilters. Ultraproducts can also be used to prove, in addition to the Łoś Ultraproduct Theorem from which the Compactness Theorem can be obtained, other interesting model-theoretical results of both a concrete, and general, nature. For example, by showing that there exists an ultraproduct of finite structures that is infinite, it can be deduced that the class of finite structures is not first-order definable in the language of equality. Of a more general nature (but well beyond the scope of this essay), the Keisler-Shelah Theorem shows that two structures satisfy the same first-order sentences precisely when they have isomorphic ultrapowers.

Moving to topology, the Stone space $\mathcal{S}_L$ of the Lindenbaum-Tarski algebra is compact in the topological sense, and this compactness can be used to prove the Compactness Theorem of model theory. Other semantic notions also have topological counterparts in $\mathcal{S}_L$, for example, deductively closed sets of sentences correspond to the clopen subsets of $\mathcal{S}_L$, and finitely axiomatisable theories correspond to elements of $\mathcal{S}_L$ that are isolated in $\mathcal{S}_L$.

An essay on this topic will entail proving the Compactness Theorem, and related results, by using the Łoś Ultraproduct Theorem, and separately, the Stone space of the Lindenbaum-Tarski algebra. In doing this, the student will gain exposure to the field of model theory and will also use concepts from topology that they'll encounter in the module WTW 790 (Topology). Applications of the Compactness Theorem, such as the De Bruin-Erdős Theorem in graph theory, may also be looked at.

A student who wishes to write an essay on this topic should have a solid background in the module WTW 381 (Algebra) and should be taking the modules WTW 724 (Axiomatic Set Theory and Logic) and WTW 790 (Topology) for their honours degree. Having taken the module WTW 389 (Geometry) will be an advantage, but is not strictly necessary.