Honours Project

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Project: Prof Baire's Marvelous Category Theorem

A Baire Category Theorem is a topological theorem that guarantees that a particular space X is 'large', in the sense that it cannot expresses as the union of countably many 'small' pieces. Despite its apparently simple formulation, this is a deep and powerful theorem in Mathematics, with many applications in Functional Analysis, Topology, Measure Theory and Set Theory.

What do we mean by 'large' and 'small'? Let's consider the example of $X = \mathbb{R}$.

Definition 1 *A closed subset Γ of $\mathbb{R}$ is* nowhere dense *if it has no interior point; that is, for every $x \in \Gamma$ and every $\delta > 0$ there exists a point $y \in \mathbb{R} \setminus \Gamma$ such that $|x - y| < \delta$.*

For instance $\mathbb{Z}$ is a closed and nowhere dense subset of $\mathbb{R}$. We consider such a set to be (topologically) small, even though it may be infinite; a closed and nowhere dense set does not contain a nonempty interval (a, b), so it doesn't take up much 'room' in $\mathbb{R}$.

The Baire Category Theorem for $\mathbb{R}$ can be stated as follows.

Theorem 1 *For every $n \in \mathbb{N}$, let Γ_n be a closed and nowhere dense subset of $\mathbb{R}$. Then*

$$\mathbb{R} \setminus \bigcup_{n=1}^{\infty} \Gamma_n \neq \emptyset.$$

In fact, $\mathbb{R} \setminus \bigcup_{n=1}^{\infty} \Gamma_n$ is dense in $\mathbb{R}$.

Versions of this result hold in more general settings. Every complete metric space and every locally compact Hausdorff topological space satisfies a Baire Category Theorem.

In this project, we will consider **different formulations of Baire Category Theorems**, as well as some **applications** of this powerful results. Depending on the interests of the student, applications could come from the following list, or other applications could be considered.

1. The existence of nowhere-differentiable functions (one of the main motivations for the development of much of modern analysis).
2. A linear partial differential equation without any solutions (a big shock at the time of its discovery in the 1950s).
3. The Principle of Dependent Choice (in Set Theory) is equivalent to the Baire Category Theorem.
4. Existence of a wining strategy for the Banach-Mazure game.