Honours Project

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Project: The Riesz Representation Theorem

In your third year analysis course, you encountered the Riemann-Stieltjes integral. Recall the definition:

Definition 1 (Riemann-Stieltjes integral) *If $f, g : [a, b] \to \mathbb{R}$ are bounded functions, then f is* integrable with respect to g on J *if there exists a real number I so that the following is true: For every $\varepsilon > 0$ there exists a partition P_ε of J so that if P is a partition of J that is finer than P_ε, then*

$$|S(P; f, g) - I| < \varepsilon$$

for all Riemann-Stieltjes sums for f with respect to g and corresponding to P. The number I is called the Riemann-Stieltjes integral *of f with respect to g on $[a, b]$, and is denoted by $I = \int_a^b f dg$.*

The main result on the existence of the integral, from your third year analysis course, is as follows:

Theorem 2 (Integrability of continuous functions) *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is monotone increasing, then f is integrable with respect to g on $[a, b]$.*

An immediate consequence of this result is the following:

Theorem 3 *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is can be written as the difference of two increasing functions, then f is integrable with respect to g on $[a, b]$.*

One of the key insights of 20th century mathematics is that functions can be treated as vectors. Consider, for instance, the set

$$C[0, 1] = \{f : [0, 1] \to \mathbb{R} \ : \ f \text{ is continuous on } [0, 1]\}.$$

It is easily verified that if $f, g \in C[0, 1]$ and $\alpha, \beta \in \mathbb{R}$, then the function $\alpha f + \beta g$ defined as

$$[\alpha f + \beta g](x) = \alpha f + \beta g(x), \quad x \in [0, 1]$$

is an element of $C[0, 1]$. In fact, $C[0, 1]$ is a vector space over the reals. Furthermore, we can define norms on $C[0, 1]$. The most natural one is a version of the ∞-norm on $\mathbb{R}^n$. For each $f \in C[0, 1]$,

$$\|f\|_\infty = \sup\{|f(x)| \ : \ x \in [0, 1]\}.$$

The familiar operations from elementary calculus can now be viewed from a new perspective. Consider, for instance, the ordinary Riemann integral. Every continuous function on $[0, 1]$ is Riemann integrable, so with a continuous function $f \in C[0, 1]$ we can associate a unique real number $\int_0^1 f(x)dx$. We therefore have a function

$$\Lambda : C[0, 1] \ni f \mapsto \int_0^1 f(x)dx \in \mathbb{R}.$$

That is, the Riemann integral is now a function from the set $C[0, 1]$ into the real numbers. This function has some special properties: It is linear, meaning

$$\Lambda(\alpha f + \beta g) = \alpha\Lambda(f) + \beta\Lambda(g)$$

for all $f, g \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Furthermore, Λ is continuous: If (f_n) converges to f with respect to the norm $\|\cdot\|_\infty$, then $(\Lambda(f_n))$ converges to $\Lambda(f)$ in $\mathbb{R}$. Note that the continuity of Λ is just another way of stating the familiar theorem from real analysis: If $(\Lambda(f_n))$ converges uniformly to f on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$

Considering Theorem 3, we can take this a step further. Suppose that $g : [0,1] \to \mathbb{R}$ is the difference of two monotone increasing functions. Then we can define a function

$$\Lambda_g : C[0,1] \ni f \mapsto \int_0^1 f dg \in \mathbb{R}.$$

The function Λ_g is also linear and continuous. The *Riesz Representation Theorem* states that the functions Λ_g are the only continuous, linear functions from $C[0,1]$ into $\mathbb{R}$. In particular, if

$$\Psi : C[0,1] \to \mathbb{R}$$

is linear and continuous, then there exists exactly one function g which is the difference of monotone increasing functions such that

$$\Psi(f) = \int_0^1 f dg, \quad f \in C[0,1].$$

This result is one of the first major results in Functional Analysis. It has been generalised to functions defined on spaces more general than the interval $[0,1]$, and has many major applications in Functional Analysis, and to other branches of mathematics.