Relationship between topology and order

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Let X be non-empty set and $\preceq$ be a relation on X. We study the topologies induced by a relation $\preceq$ on X and also a relation induced by a topology τ on X. For example, let $X = \mathbb{R}$, and consider $(\mathbb{R}, <)$ and $(\mathbb{R}, \prec)$, where $x < y$ means that $x \leq y$ and $y \neq x$ $x, y \in \mathbb{R}$ and $\prec$ is defined by

$$x \prec y \Leftrightarrow |x - y| < 1 \; x, y \in \mathbb{R}.$$

Then $\tau_{<} = \tau_{\prec} = \tau$. That is, these topologies are the same as the usual topology τ on $\mathbb{R}$. Also, given a topology τ on X, we can define a relation $\preceq_\tau$ by

$$x \preceq_\tau y \Leftrightarrow x \in \overline{\{y\}} \; x, y \in X.$$

This is called specialization pre-order. In this study, we discuss the following;

(i) Given a topology τ on X and consider a specialization pre-order $\preceq_\tau$ on X as a binary relation $\mathcal{R} = \preceq_\tau$. Is $\tau_{\mathcal{R}} = \tau$?

(ii) Let $\tau_{\leq}$ and $\tau_{\preceq}$ be two topologies induced by binary relations $\leq$ and $\preceq$ respectively on X. Is $\leq = \preceq$?

(iii) Discuss a characterization of the topology τ induced by a binary relation $\leq$.

(iv) If a topology τ on X is a T_1. How does the specialization order with respect to τ look like?