HONOURS PROJECT TOPIC
L-SERIES AND PRIMES IN ARITHMETIC PROGRESSIONS

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Prerequisite: WTW 320. (WTW 381 will help, but is not essential.)

It is well known that there are infinitely many prime numbers. In number theory, *Dirichlet's theorem* on primes in arithmetic progressions says that if a and d are two positive integers that have no common factor other than 1, then the arithmetic sequence $(a, a+d, a+2d, \ldots, a+dn, \ldots)$ contains infinitely many prime numbers.

A simple case of Dirichlet's theorem says that there are infinitely many primes of the form $3+4n$ where n is a positive integer. This particular case can be proved by a straightforward adaptation of Euclid's classic proof that there are infinitely many prime numbers.

In this honours project, the student will rigorously prove Dirichlet's theorem in general, by using tools from complex analysis.

An *L-series* is an infinite series of the form $\sum_{n=1}^{\infty} a_n/n^s$, where s is a complex number and each a_n is a complex number. The concept of an L-series is at the heart of modern number theory. The 1990s proof of Fermat's Last Theorem by Andrew Wiles and Richard Taylor involved showing a special case of what is now called the Modularity Theorem, which says that two different types of mathematical object correspond to the same L-series. The famously unsolved Riemann Hypothesis is about the zeta function $\zeta(s)$, which corresponds to the L-series $\sum_{n=1}^{\infty} 1/n^s$.

An important type of L-series is a *Dirichlet L-series*, that is, an L-series of the form $\sum_{n=1}^{\infty} \chi(n)/n^s$ where the function χ is multiplicative (that is, $\chi(m)\chi(n) = \chi(mn)$ for positive integers m and n), is periodic with some period d, and is such that $\chi(n) = 0$ if and only if n and d have a common factor larger than 1. The fact that χ is multiplicative implies that the L-series can be written as a product rather than a sum:

$$\prod_p \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p)^2}{p^{2s}} + \cdots\right) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the product goes over all prime numbers p; this product is the *Euler product.*

Dirichlet L-series are the central tool in the proof of Dirichlet's theorem. Using Dirichlet L-series, a refinement of the theorem can also be proved: for a given integer $d \geq 2$, the primes are, in a certain sense, "equally distributed" among the arithmetic progressions $(a, a+d, a+2d, \ldots)$ where the positive integer a is less than d and shares no common factor with d other than 1. For example, there are "as many" primes of the form $1+4n$ as there are primes of the form $3+4n$.