HONOURS PROJECT TOPIC
APPLICATIONS OF GENERATING FUNCTIONS

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Prerequisites: WTW 115 and WTW 285.

In combinatorics, a *generating function* is an infinite series of the form $\sum_n a_n x^n$, or sometimes a variation such as $\sum_n (a_n/n!) x^n$ or $\sum_{m,n} a_{m,n} x^m y^n$, where each a_n or $a_{m,n}$ is a number. Often, generating functions allow us to arrive at results more easily than would otherwise be possible. For some results, the only known proofs use generating functions.

This honours project involves discussing several applications of generating functions such as the following.

- Find simple formulas for numbers such as the Fibonacci numbers, Catalan numbers, and Stirling numbers of the second kind using generating functions.
- The *partition function* $p(n)$ counts the number of ways in which n can be written as a sum of positive integers where the order does not matter. (For example, $p(3) = 3$ because $3 = 2 + 1 = 1 + 1 + 1$.) Prove a recurrence relation of the form
$$p(n) = p(n-1) + p(n-2) - p(n-5) - \dots.$$
Surprisingly, pentagonal numbers make an appearance here.
- Prove that if a_n is the number of horizontally convex polyominoes of area n, then the recurrence relation
$$a_n = 5a_{n-1} - 7a_{n-2} + 4a_{n-3}$$
holds. No proof of this relation that proceeds immediately from the definition of a_n without using generating functions is known.
- Use generating functions to relate numbers of things to the numbers of "simple components" in those things. For instance, express the numbers of n-vertex labelled graphs for various n in terms of the numbers of connected n-vertex labelled graphs for various n.
- Use generating functions to determine what happens to sequences (a_n) as $n \to \infty$.
- (Advanced; a background in complex analysis would be very useful for this subtopic) *Lagrange's four-square theorem* says that every positive integer can be written as a sum of four squares. *Jacobi's four-square theorem* gives a formula for the number a_n of ways in which n can be written as a sum of four squares (from which Lagrange's theorem immediately follows). Prove Jacobi's four-square theorem by using the generating function $\sum_n a_n x^n$.