REPRESENTATION THEORY - HONOUR TOPIC 2017

TUNG LE

Cenarios: Let G be a finite group. An *ordinary reprensentation* of degree n of G is a homomorphism $\phi : G \to \mathrm{GL}(n, \mathbb{C})$ where $\mathrm{GL}(n, \mathbb{C})$ is the general linear group of degree n over the complex field $\mathbb{C}$.

A representation ϕ of degree n is *reducible* if there are a matrix $A \in \mathrm{GL}(n, \mathbb{C})$ and $k \in [2, n-1]$ such that for all $g \in G$,

$$\phi(g) = A^{-1} \begin{pmatrix} S(g) & * \\ 0 & R(g) \end{pmatrix} A$$

where $S(g) \in \mathrm{GL}(k, \mathbb{C})$ and $R(g) \in \mathrm{GL}(n-k, \mathbb{C})$.

A representation ϕ of degree n is *irreducible* if it is not reducible.

Let ϕ be a represnetation of degree n of G. An *ordinary characrater* χ of G afforded by ϕ is defined by

$$\chi(g) := \mathrm{Tr}(\phi(g)) \text{ for all } g \in G,$$

where $\mathrm{Tr}(\phi(g))$ is the trace of $\phi(g)$. Here, the *degree* of χ is n, which is also equal to $\mathrm{Tr}(\phi(e_G))$ where e_G is the identity of G.

A character χ afforded by ϕ is *irreducible* (reducible) if ϕ is irreducible (reducible, respectively).

Let $\mathrm{Irr}(G)$ be the set of all ordinary irreducible characters of G and $\mathrm{Ch}(G)$ be the set of all odinary characters of G.

For $\chi, \gamma \in \mathrm{Ch}(G)$, the tensor product $\chi \otimes \gamma$ of χ and γ is defined as follows

$$(\chi \otimes \gamma)(g) := \chi(g)\gamma(g) \text{ for all } g \in G.$$

It is well-known that a character is a sum of irreducible characters and the product of two characters is also a character, i.e. $\chi \otimes \gamma \in \mathrm{Ch}(G)$. However, a product of two irreducible characters may be not irreducible. Thus, a product of two irreducible characters can be decomposed into a sum of irreducible characters, which are called *irreducible constituents.*

Problem: Let $\mathbb{F}_q$ be a general finite field of q elements and $\mathrm{SL}(2, q)$ be the special linear group of degree 2 over $\mathbb{F}_q$. Find and construct all ordinary irreducible characters of $\mathrm{SL}(2, q)$. Furthermore, try to decompose products of two irreducible characters into irreducible constituents.

Required: Linear Algebra, Complex Analysis, Group Theory, (Galois) Field Theory...

Motivation: This is the first step into the fields: Finite Group Theory/ Representation Theory/ Character Theory/ Coding Theory/ Cryptography Theory.

Date: February 27, 2017.