Honours Project
Hamilton–Poisson systems on two-dimensional linear Poisson spaces

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Overview: A Poisson structure allows one to associate a Hamiltonian vector field $\vec{H}$ to any smooth ("energy") function H; such structures can be viewed as generalizing classical Hamiltonian mechanics and account for a wide range of dynamical systems. Poisson structures appear in a large variety of different contexts, ranging from string theory, classical/quantum mechanics and differential geometry to abstract algebra, algebraic geometry and representation theory. In this project, we study linear Poisson structures in two dimensions. We will (1) classify the two-dimensional quadratic systems, (2) study the stability nature of the equilibria of these systems (making use of spectral stability and Lyapunov functions), and finally (3) fully integrate the equations of motion of these systems.

Description: A *Poisson structure* $\{\cdot,\cdot\}$ on a real vector space V is a non-associative, skew-symmetric, bilinear map $\{\cdot,\cdot\} : C^\infty(V) \times C^\infty(V) \to C^\infty(V)$ on the algebra of smooth functions (with point-wise multiplication) such that

1. the Jacobi identity holds, i.e.,
$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0$$
for all $F, G, H \in C^\infty(V)$
2. $\{\cdot,\cdot\}$ is a derivation in each factor, i.e.,
$$\{FG, H\} = \{F, H\}G + F\{G, H\}$$
for all $F, G, H \in C^\infty(V)$.

For any smooth function $H \in C^\infty(\mathfrak{g}^*)$ there exists a unique vector field $\vec{H}$ on V, called *Hamiltonian vector field*, such that $dF(p) \cdot \vec{H}(p) = \{F, H\}(p)$ for any function $F \in C^\infty(\mathfrak{g}^*)$.

A *linear Poisson structure* is a Poisson structure for which the Poisson bracket $\{F, G\}$ of any two linear functions F and G on V is again linear. (It turns out that the category of linear Poisson structures is equivalent to the category of Lie algebras.) A *quadratic Hamilton–Poisson system* (on a vector space equipped with a linear Poisson structure) is simply a system with Hamiltonian $H(x) = x^\top Qx$ being a quadratic form on V. Such systems form a natural setting for a variety of dynamical systems (especially in Mathematical Physics); prevalent examples are Eulers classic equations for the rigid body as well as its extensions and generalizations.

In this project, we study the quadratic Hamilton-Poisson systems in two dimensions. We rely on the close relationship between Lie algebras and linear Poisson structures to justify that there is essentially just one linear Poisson structure on $\mathbb{R}^2$. For this structure, we do the following:

1. We classify all the quadratic Hamilton–Poisson systems on this Poisson space. (We consider two systems to be equivalent if their Hamiltonian vector fields are compatible with a linear bijection.)

2. For each normalized system, we study the non-linear stability nature of the equilibria. This is done by making use of spectral stability as well as Lyapunov functions.

3. For each normalized system, we find explicit expressions for the integral curves of the Hamiltonian vector field.

Prerequisites: Linear algebra; Dynamical systems (and ordinary differential equations); Vector analysis.

Wikis:

`https://en.wikipedia.org/wiki/Hamiltonian_mechanics`
`https://en.wikipedia.org/wiki/Poisson_algebra`
`https://en.wikipedia.org/wiki/Poisson_manifold`