Honours Project
Classification of low-dimensional real Lie algebras

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Overview: A real Lie (pronounced "Lee") algebra consists of a real vector space equipped with a special kind of bilinear map (called the Lie bracket). For instance, $\mathbb{R}^3$ with the cross product forms a Lie algebra. Any real Lie algebra is the linearization (in a certain sense) of a smooth group of transformations. For instance, $\mathbb{R}^3$ with the cross product is the "linearization" of the group of rotations in three dimensions. In this project, we classify the real Lie algebras up to (and including) dimension three.

Description: A Lie algebra $\mathfrak{g}$ is a vector space equipped with a non-associative, skew-symmetric bilinear map (called the Lie bracket)

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (A, B) \mapsto [A, B],$$

satisfying the Jacobi identity

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0, \qquad A, B, C \in \mathfrak{g}.$$

Any real Lie algebra can be regarded as the "linearization" of a real Lie group (i.e., a smooth group of transformations). For example the group of rotations in two dimensions

$$\mathsf{SO}(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \,:\, \theta \in \mathbb{R} \right\}$$

with group product being matrix multiplication, has Lie algebra (i.e., "linearization") consisting of skew-symmetric matrices

$$\mathfrak{so}(2) = \left\{ \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix} \,:\, x \in \mathbb{R} \right\}.$$

Here the Lie bracket is given by the matrix commutator (i.e., $[A, B] = AB - BA$). Similarly, the group of rotations in three dimensions

$$\mathsf{SO}(3) = \left\{ M \in \mathbb{R}^{3\times 3} \,:\, M^\top M = I,\, \det M = 1 \right\}$$

has (three-dimensional) Lie algebra consisting of skew symmetric matrices

$$\mathfrak{so}(3) = \left\{ \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \,:\, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Note that any element of the group $\mathsf{SO}(2)$ (resp. $\mathsf{SO}(3)$) can be written as a finite product of the matrix exponentials of elements of the Lie algebra $\mathfrak{so}(2)$ (resp. $\mathfrak{so}(3)$). The correspondence between Lie groups and Lie algebras allows one to study Lie groups in terms of Lie algebras.

Lie groups are fundamental objects for continuous symmetries of mathematical objects and structures, making them fundamental tools for many parts of contemporary mathematics as well as for modern theoretical physics.

In this project, we consider the equivalence of low-dimensional real Lie algebras. Two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are considered isomorphic (i.e., equivalent) if there exists a linear bijection $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ between them which preserves the respective Lie brackets, i.e., $\psi \cdot [A, B]_{\mathfrak{g}_1} = [\psi \cdot A, \psi \cdot B]_{\mathfrak{g}_2}$ for all $A, B \in \mathfrak{g}_1$. We classify the real Lie algebras of dimension ≤ 3. Such a classification would be the first step in classifying the real Lie groups of dimension ≤ 3.

Prerequisites: Linear algebra.

Further reading: Stillwell, John. *Naive Lie theory.* Springer, 2008.

Wikis:

`https://en.wikipedia.org/wiki/Lie_algebra`
`https://en.wikipedia.org/wiki/Lie_theory`
`https://en.wikipedia.org/wiki/Lie_group`
`https://en.wikipedia.org/wiki/List_of_Lie_groups_topics`