Hons essay projects in the programme of Mathematics

The Hermite polynomials and their applications: Dr AS Jooste

A sequence of real polynomials $\{P_n\}_{n=0}^{N}$, where N can be at infinity and each polynomial P_n is of exact degree n, is orthogonal on the interval (a, b), with respect to the weight function $w(x) > 0$, if, for $m, n = 0, 1, \ldots N$,

$$\int_a^b P_n(x)P_m(x)\ w(x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ d_n^2 & \text{if } m = n, \end{cases}$$

and $d_n^2 = \int_a^b P_n^2(x)\ w(x)d(x) \neq 0$.

The Hermite polynomials were defined by Laplace in 1810 and are orthogonal on the real line with respect to the weight function $w(x) = e^{-x^2}$. The polynomials arise in, e.g., probability (such as the Edgeworth series), numerical analysis (as Gaussian quadrature).

An essay on this topic will typically entail the following:

(1) A short introduction on orthogonal polynomials in general, focussing on the properties of their zeros;

(2) A discussion of the specific properties that are used to characterize the Hermite polynomials;

(3) A study on the behaviour of the zeros of the Hermite polynomials;

(4) A section on where these polynomials are used, where we will go deeper into the use of these polynomials in approximation theory or numerical analysis.

Such a project can also be done on any one of the other sequences of orthogonal polynomials in the Askey scheme.

In doing this project, you will gain some basic knowledge in the field of Approximation theory and the study will contribute to a strong foundation in mathematics. We will also use the programs Maple or Mathematica in order to experience how Computer-Algebra can be used to get results in this field.

Further reading:

G.E. Andrews, R. Askey and R. Roy. Special Functions. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1999.

T.S. Chihara. An introduction to Orthogonal Polynomials. Gordon and Breach, New York, 1998.

W. Schoutens. Stochastic processes and orthogonal polynomials. Lecture Notes in Statistics, 146. New York, Springer (2000).