The nearest point problem in metric spaces

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1 Project description

(i) **Explanation**

Notations and Conventions Assumed

Let (X, d) be a non-empty metric space, S a subset of X and $x \in X$. Define the distance function from x to S to be $\text{dist}(x, S) = \inf\{d(x, s)|s \in S\}$. This distance depends, of course, on the metric, and if it is necessary to specify the metric in question, we'll write $\text{dist}_d(x, S)$ for the distance function from x to S.

Trivially (and uselessly), except to confirm that $\emptyset$ is a metric space, the only metric on the empty set is the empty function, namely the function with empty domain (and which therefore does nothing). By convention, $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

Let $z \in X$. A point $s \in S$ is said to be a *nearest point* of S to z if, and only if, $d(z, s) = \text{dist}(z, S)$.

The Problem Statement

(a) Under what conditions is there an element of S that is distant exactly $\text{dist}(x, S)$ from x? In particular, what property of S will ensure that such a *nearest point* of S exists independently of the metric superspace X enfolding S and what point x of X are we considering?

(b) If there is such an element of S as alluded to in (a) above, is it unique?

The metric space (X, d) is said to have the *nearest-point property* if, and only if, $X = \emptyset$ or X admits a nearest point to each point in every metric superspace of X.

Various characterizations of this property will be studied in the context of metric spaces; in particular, the normed linear spaces in the case of the normable ones. Important applications will include the finding of best approximations to points in normed linear spaces out of the subsets of these normed linear spaces.

The normed linear space V is said to have a *strictly convex norm* if the closed unit ball $B = \{x \in V| \|x\| \leq 1\}$, is strictly convex (sometimes called rotund). Certainly, B is convex. For it be *strictly convex*, it is required that if $x_1 \neq x_2$,

$\|x_1\| = 1$ and $\|x_2\| = 1$, then $\|\lambda_1 x_1 + \lambda_2 x_2\| < 1$ if $\lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$. (That is, the boundary of B contains no open line seqment).

The question of which normed linear spaces have strictly convex norms will be paramount to the study. Strictly convex norms offer at most one best approximation out of a subspace of a normed linear space to given points of the superspace.

The set of functions continuous on a given closed interval $[a, b]$, which we denote by $C[a, b]$, is a linear space. Given $f \in C[a, b]$, we can define a norm in $C[a, b]$ by

$$\|f\| = \max_{a \leq x \leq b} |f(x)|.$$

The norm is called the *uniform* (or *Chebyshev*) norm (which is not strictly convex). The study of best approximation in the *uniform* norm will ensue, starting with the discussion of how well continuous functions can be approximated by polynomials. This will be followed by the investigation of the properties that characterize a best approximating polynomial.

(ii) **Motivation**

The needs of automatic digital computation have mobilised huge interest in methods of approximating continuous functions by functions which depend only on a finite number of parameters. This project seeks to introduce some of the most significant methods to students, namely, the emphasis on approximation by polynomials.

Banach spaces provide framework for mathematical analysis. In differential calculus, one is often faced with the question of supply of smooth functions on Banach spaces. These functions are usually obtained from smooth norms and smooth norms are in turn often constructed from dual strictly convex (or rotund) norms. The existence of equivalent smooth or strictly convex norms or nontrivial smooth functions on a particular Banach space depends on its structure and has in turn a huge impact on its geometry. The student is introduced to these concepts (namely rotund norms) as they form the foundations of research in this described context.

(iii) **Personal development**

This project, thus, seeks to broaden the scope of the technical expertise of the student in his or her preparation as a future researcher in Banach space theory, in particular, and in mathematical analysis in general.

It seeks to demonstrate to the student the relevance of the theoretical underpinnings of mathematics to concrete situations with the hope of affording him or her the opportunity to indulge in a deeper knowledge system meticulously and rigorously.