Honours essay projects in the programme of Applied Mathematics

A. Orthogonal Polynomials and Finances. Supervisor: Dr AS Jooste

A sequence of real polynomials $\{P_n\}_{n=0}^N$, $N \in \mathbb{N} \cup \{\infty\}$, where P_n is of exact degree n, is orthogonal on the interval (a, b), with respect to the weight function $w(x) > 0$, if, for $m, n = 0, 1, \ldots N$,

$$\int_a^b P_n(x)P_m(x)\ w(x)dx = \begin{cases} 0 & \text{if} \quad m \neq n, \\ d_n^2 & \text{if} \quad m = n, \end{cases}$$

where $d_n^2 = \int_a^b P_n^2(x)\ w(x)d(x) \neq 0$.

A stochastic (or random) process is a mathematical object usually defined as a collection of random variables. Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner and random changes in financial markets have motivated the extensive use of stochastic processes in finance. For further reading, see [Schoutens].

Orthogonal polynomials are often connected to stochastic processes in finance, e.g., the Meixner process (a special type of stochastic process) originates from the theory of orthogonal polynomials and is related to a specific family of orthogonal polynomials (the Meixner-Pollaczek polynomials) by a martingale relation.

This study will entail the following:

1. A short introduction on orthogonal polynomials;
2. A short introduction on stochastic processes;
3. A broad literature study in order to identify the different sequences of orthogonal polynomials that can be applied in Finances and some background on their connection with financial processes.

The student will gain some basic (and essential) knowledge in the field of finances when doing this very interesting and relatively easy project.

B. Orthogonal Polynomials and Computer-Algebra. Supervisor: Dr AS Jooste

A sequence of real polynomials $\{P_n\}_{n=0}^N$, $N \in \mathbb{N} \cup \{\infty\}$, where P_n is of exact degree n, is orthogonal on the interval (a, b), with respect to the weight function $w(x) > 0$, if, for $m, n = 0, 1, \ldots N$,

$$\int_a^b P_n(x)P_m(x)\ w(x)dx = \begin{cases} 0 & \text{if} \quad m \neq n, \\ d_n^2 & \text{if} \quad m = n, \end{cases}$$

where $d_n^2 = \int_a^b P_n^2(x)\ w(x)d(x) \neq 0$.

The orthogonal polynomials we consider are part of the Askey scheme of hypergeometric orthogonal polynomials and can be expressed in terms of the ${}_pF_q$ hypergeometric function:

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = 1 + \sum_{k=1}^{\infty} \frac{(a_1)_k(a_2)_k \ldots (a_p)_k}{(b_1)_k(b_2)_k \ldots (b_q)_k} \frac{z^k}{k!},$$

where $a_k, k \in \{1, 2, \ldots, p\}$ and $b_j, j \in \{1, 2, \ldots, q\}$ are complex parameters, $b_j \neq 0, -1, -2, \ldots$ and $(a)_k = a(a+1)(a+2)\ldots(a+k-1)$ is the Pochhammer symbol.

These hypergeometric representations are used in recently developed algorithmic approaches to compute different types of equations by which orthogonal polynomials can be characterized, e.g., three-term recurrence equations, differential equations, Rodrigues' formula, etc.

An essay in this field will typically entail the following:

1. A short introduction on orthogonal polynomials;

2. A discussion of the specific property that used to characterize the polynomials;
3. Using an existing program in a language such as Mathematica or Maple, to obtain the relevant equations (according to the topic chosen) for each orthogonal polynomial system. It will also be expected from the student to understand the algebraic method that lies behind the specific program he uses.

Different projects can be chosen from. An example of a project:

Zeilberger's Algorithm.
Zeilberger's algorithm [Koepf] generates a recurrence equation for a set of orthogonal polynomials, with respect to some discrete variable for a sum under consideration. In this project the student will
1. give a short background on orthogonal polynomials and their properties, focussing on the three-term recurrence relation in general;
2. study the principles used in Zeilberger's algorithm;
3. apply the algorithm in order to find some recurrence relations (not necessarily three-term recurrence relations) in order to show the uses of Zeilbergers's algorithm.

In doing this project, you will gain some basic knowledge in the fields of orthogonal polynomials and Computer-Algebra and the study will contribute to a strong foundation in mathematics. You will get to know the different programs (written in, e.g., Maple or Mathematica) that can be used to obtain results, as well as the extent to which Computer-Algebra can be used to get results in this field.

References (and further reading):
G.E. Andrews, R. Askey and R. Roy. Special Functions. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1999.
T.S. Chihara. An introduction to Orthogonal Polynomials. Gordon and Breach, New York, 1998.
[Koepf] W. Koepf, Hypergeometric Summation, Al algorithmic approach to summation and special function identities (second edition). Springer-Verlag, London, 2014.
[Schoutens] W. Schoutens. Stochastic processes and orthogonal polynomials. Lecture Notes in Statistics, 146. New York, Springer (2000).