Soliton scattering via the dressing method on coset models of S^5 *

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Abstract

In this thesis we examine the non-trivial behaviour of classical string solutions (solitons) on S^5 obtained from the dressing method. We start off with a vacuum solution on S^5 then embed the solution into a coset model ($SU(3)/SU(2)$ and $SO(6)/SO(5)$). The dressing method is applied and the resulting solutions are examined for non-trivial scattering behaviour where a single soliton at $t \to -\infty$ decays into two solitons at $t \to +\infty$. With vacuum solutions and the chosen embeddings the dressing method did not offer any promising solutions that exhibit the desired behaviour.

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Contents

1 Introduction

The AdS/CFT correspondence [1, 2] is a major area of study concerning the string/gauge duality (duality between quantum field theories and gravity), it is closely related to the study of strongly coupled systems in QCD. The best studied example of this duality is between the 4 dimensional $\mathcal{N} = 4$ super Yang-Mills theory and string theory on $AdS_5 \times S^5$ [3]. The latter of which is a two dimensional non-linear sigma model that is classically integrable [4]. It is this integrability on both sides of the correspondence that has led to further development in the study of the duality [5, 6]. One such aspect of the integrability has led to the discovery of solutions that can be described as solitons on the worldsheet sigma model [7, 8].

Solitons [9], first observed in 1834 by John Scott Russell [10] after observing a solitary wave moving along a canal, are defined as localised travelling waves that maintain their shape as they travel with constant velocity, and also interact trivially with other solitons except for a possible phase shift. Solitons appear in many branches of physics such as field theories, non-linear optics, biophysics, quantum field theories and many more. Indeed many systems are modelled by equations that admit soliton solutions, one such system is the sine-Gordon (SG) equation, a non-linear partial differential equation that is also integrable. One of the interesting discoveries was that string theory on symmetric spaces (such as $AdS_5 \times S^5$) is classically equivalent to sine-Gordon theory [11, 12]. An example of this equivalence to sine-Gordon theory is the discovery of a soliton in the AdS/CFT correspondence by Hofman and Maldacena [13] known as the giant magnon.

A quick way to see the connection between non-linear sigma models and string theory is to take the action describing the motion of classical strings on a background geometry, given by

$$S = \int G_{\mu\nu}(X)\partial^a X^\mu \partial_a X^\nu \tag{1.1}$$

where $G_{\mu\nu}$ is a metric and X^μ the coordinates.
Then if this motion is on a group manifold such as the principal chiral model, then this action can be written as

$$S = \int Tr[(\partial_\eta g g^{-1})(\partial_\xi g g^{-1})] d\eta d\xi \tag{1.2}$$

where g is a matrix and Tr is the trace.
The relation between g and X^μ will then depend on the group. As an example if we take an embedding of a string in $\mathbb{R} \times S^3$ in to the $SU(2)$ principal chiral model (equation (4.1) in [18]), given by

$$g = \begin{bmatrix} Z_1 & -iZ_2 \\ -i\overline{Z_2} & \overline{Z_1} \end{bmatrix}$$

where Z_i are the coordinates on S^3. Now if we compute $(\partial_\eta g g^{-1})(\partial_\xi g g^{-1})$

and take its trace, the result obtained would be of similar form as $G_{\mu\nu}(X)\partial^a X^\mu \partial_a X^\nu$.

An open problem in AdS/CFT theory is the full characterisation of the $\mathcal{N} = 4$ super Yang-Mills theory. Conformal field theory can be fully characterised by its spectrum of operators (which determines its 2-point functions) as well as the 3-point functions of the operators. The former is well studied and understood [14, 15], but not much has been done in terms of 3-point functions. The duality can be exploited to study these 3 point functions on the gravity side in terms of non-trivial scattering solutions (decay of a single soliton into two solitons).

In the paper by Zakharov and Mikhailov [16] the dressing method is proposed for obtaining new soliton solutions of integrable systems starting with an already known soliton solution. In the follow up paper [17] it is shown that non-trivial soliton solutions are obtained when applying the dressing method to the $SU(3)$ principal chiral model (non-linear sigma model with a Lie group $SU(3)$ as the target manifold). These non-trivial soliton solutions are defined as a soliton at $t \to -\infty$ decaying in to two solitons at $t \to +\infty$. In the paper by Spradlin and Volovich [18], the dressing method was used to obtain solutions of giant magnons (string solitons) on the $SO(N)$ vector model as well as the $SU(2)$ principal chiral model. These magnon solutions obtained correspond to 2-point functions of operators in the $\mathcal{N} = 4$ super Yang-Mills theory. However there has been a lot of interest shown recently in 3-point functions of operators, for which the construction of non-trivial scattering solutions of solitons on S^5 would be relevant.

In this paper the classical integrability of string theory on Anti-de Sitter space (AdS) will be exploited in order to apply the dressing method and obtain non-trivial soliton solutions. As was done in the paper by Spradlin and Volovich mentioned above, the dressing method will be applied to an already known vacuum solution (simplest solution). The newly obtained solution will be checked to see for any non-trivial behaviour and then a check will be made to ensure that the solution is still embedded in the original space of the vacuum solution. The spaces that concern this paper are the S^5 (the $\mathcal{N} = 4$ super Yang-Mills theory correspondence to $AdS_5 \times S^5$) and $\mathbb{CP}^3$ ($\mathcal{N} = 6$ superconformal Chern-Simons theory and its gravity dual on $AdS_4 \times \mathbb{CP}^3$ [19]).

The section following will give the background knowledge of the theory utilised in this paper starting with the sine-Gordon equation and its soliton solutions and lastly how the specific coset spaces are equivalent to the spaces appearing in different AdS/CFT setups. Following in section 3, a deeper explanation will be given for the main tool, the dressing method. In section 4 the dressing method will be used on vacuum solutions in the coset spaces described above and the necessary checks will be made. Finally in the last section the paper will be concluded with thoughts of how the results obtained could lay the ground work for further study of 3-point functions in the AdS/CFT correspondence.

2 Sine-Gordon Solitons

Soliton theory is an important subject in modern physics. The soliton occurs as a solution in many different PDE's, where these equations model many varied physical systems [20]. The link between all these vastly different systems that produce soliton solutions, is that they are all linked together by 2-dimensional conformal systems [21]. The interesting feature of solitons to QFT can be seen by how soliton-like particles exhibit a topological charge whereas electrically charged elementary particles have a Noether charge. Solitons are non-perturbative, this could mean that there is a deeper structure to the theory over just perturbative theory. [22, 24]. This non-perturbative theory lends itself to the study of string solitons in AdS/CFT.

One of the most popular of non-linear PDE that has applications to $AdS_5 \times S^5$ string theory is the sine-Gordon equation [25].

2.1 Sine-Gordon equation

The SG equation

$$u_{tt}(x,t) - u_{xx}(x,t) + \sin u(x,t) = 0 \tag{2.1}$$

is the most well known non-linear PDE's that is also relativistically invariant. Originally introduced to study pseudo-spherical surfaces with constant negative curvature, it has found use in many other areas of physics such as the study of magnetic flux [26], crystal dislocations [27], quantum mechanics and many others. An important discovery was that the equation is integrable, meaning that it can be solved exactly. It gets its name from the Klein Gordon equation [28] which is of similar form except that the SG equation has a sine for its potential term. The SG equation is essentially a standard wave equation with a non-linear potential, equation (2.1).

Equation (2.1) represents the SG equation in 2 dimensions or 1 + 1 dimensions, where $u(x,t)$ is a scalar field. It can equivalently be redefined in terms of worldsheet coordinates by letting $\xi \equiv \frac{1}{2}(x-t)$ and $\eta \equiv \frac{1}{2}(x+t)$, this gives

$$u_{\xi\eta}(\xi,\eta) = \sin u(\xi,\eta) \tag{2.2}$$

One of the most fruitful discoveries of the SG equation was that it gave rise to soliton and multi-soliton solutions.

2.2 Soliton solutions

Solitons are essentially localised, 2-dimensional solutions to non-linear wave equations. They are travelling, solitary waves that have the following properties:

- They have constant shape, energy density and travel with constant velocity.
- Interaction with other solitons leave them unchanged except for a possible phase shift, meaning that after collision/interaction the solutions return to their original form asymptotically in time (i.e. $t \to \pm\infty$).

Usually these properties hold well for linear wave solutions but as soon as any non-linear term is added to the wave equation these properties disappear. Either the non-linearity or the dispersion of the wave prevents the solution from maintaining its shape asymptotically. The surprising thing about certain sine-Gordon solutions is that some of them have their dispersion and non-linearity cancel each other out perfectly such that the waves are able to remain unchanged after interaction, which satisfies the above properties thus making these solutions solitons.

The sine-Gordon 1-soliton solution[9, 30] is given by

$$u(x,t) = 4\arctan\left(\exp\left[\frac{(x-vt)}{\pm\sqrt{1-v^2}} - x_0\right]\right) \tag{2.3}$$

where v is the velocity of the travelling wave (i.e. constant) and $|v| < 1$. This solution is also known as a kink when the positive root of $\sqrt{1-v^2}$ is taken and an anti-kink (anti-soliton) when the root is negative. If $v = 0$, then the solution would be a static kink, meaning no propagation of the soliton.

The soliton interacting with another soliton has the solution

$$u(x,t) = 4\arctan\left(\frac{v\sinh\left(\frac{x}{\sqrt{1-v^2}}\right)}{\cosh\left(\frac{vt}{\sqrt{1-v^2}}\right)}\right) \tag{2.4}$$

Another interesting soliton solution of the SG equation, that is also a non-trivial solution is the soliton anti-soliton pair

$$u(x,t) = 4\arctan\left(\frac{\sinh\left(\frac{vt}{\sqrt{1-v^2}}\right)}{v\cosh\left(\frac{x}{\sqrt{1-v^2}}\right)}\right) \tag{2.5}$$

It can be seen from the equations (2.4) and (2.5) that at $t \to -\infty$ both solitons (or soliton and anti-soliton) are separated but approaching each other with velocity v. At $t = 0$ there's an interaction and at $t \to +\infty$ both the solitons (or soliton and anti-soliton) re-emerge.

Lastly there is one more soliton type that emerges from the SG equation. That is the breather solution

$$u(x,t) = 4\arctan\left(\frac{\sin\left(\frac{vt}{\sqrt{1-v^2}}\right)}{v\cosh\left(\frac{x}{\sqrt{1-v^2}}\right)}\right) \tag{2.6}$$

The breather is a localised solution that is periodic and that oscillates. A clearer picture of the time evolution of these different solutions can be seen in figure 1.

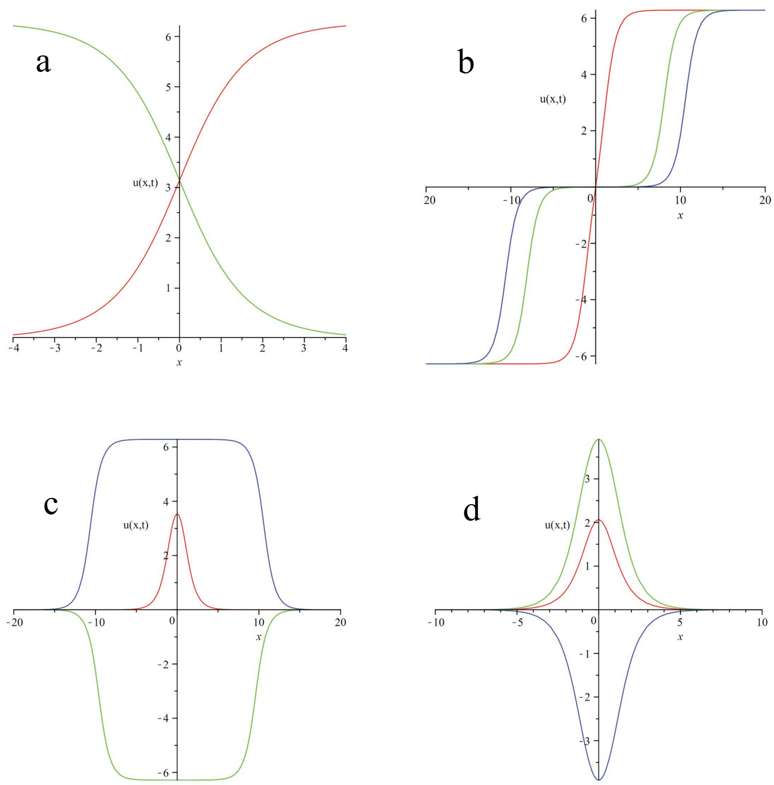

Figure 1: Sine-Gordon soliton solutions as a function of spatial position at fixed time and velocity. (a) Red - single soliton solution or kink, Green - anti-soliton or anti-kink at $t = 0$. (b) Soliton - soliton interaction. Green - as $t \to -\infty$ the solitons are seen to be separate moving with velocity v to each other. Here $t = -15$. Red - where the solitons interact at $t = 0$. Blue - at $t = 20$. As $t \to +\infty$ the solitons re-emerge to their original shape. (c) Soliton - Anti-soliton interaction. Green - $t = -18$. Red - at $t = 1$, after interaction the soliton and anti-soliton reappear. At interaction at $t = 0$ they cancel each other out completely. Blue - at $t = 20$, the soliton - anti-soliton solution reappears but with a phase shift. (d) Static breather solution oscillating in position. Blue - at $t = -4$. Red - at $t = 0.5$. Green - at $t = 4$.

An interesting solution to mention introduced by Hofman and Maldacena, that can be mapped at the classical level to a sine-Gordon soliton is the giant magnon. These are essentially solitons on the worldsheet of $AdS_5 \times S^5$. They sit at a point in AdS with the strings stretched out in the S^5 space. Giant magnons are a solitonic class of classical string solutions. Classical string solutions play an important role in the study of the AdS/CFT correspondence. The technical side of the derivation of these solutions, is not something that will be done in this paper. Broadly speaking giant magnons are string states on the sphere that have infinite energy (E) and angular momentum around the equator (J) such that the dispersion relation is $E - J \neq 0$.

A visualisation of the magnon is that it is a string with both endpoints on the equator of a sphere moving with the speed of light and the string existing in the sphere. In the paper by Spradlin and Volovich the dressing method is applied to the giant magnon and it is shown that N-magnon solutions can be generated by recursion application of the method [18]. For a complete and in-depth explanation of the giant magnon, the following papers [13, 31, 32, 33, 34] are recommended.

3 Cosets

Coset models provide the most general way of constructing conformal field theories thus it is natural to study AdS/CFT correspondence from the string side using coset spaces. The advantage of using coset models is that a coset can be used to describe any homogeneous space (spaces acted on transitively by some group) by having G be a Lie group with H its isotropy group.

In this section the full definition of a coset is given and then the isometry groups of the S^5 and $\mathbb{CP}^3$ spaces are given in terms of cosets. Additional theory will be given as to why these spaces can be written in these coset models.

Of the string backgrounds looked at ($AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$), we are only interested in the bosonic backgrounds S^5 and $\mathbb{CP}^3$ as this will keep the calculations simple and allow us not have to deal with supergroups to account for fermions.

3.1 Definition

A coset or quotient group G/H is a homomorphic image of a group G. In order to define a coset, it requires the definition of a left and right coset. If G is a group and H its subgroup ($H \subset G$) then for any element a in G, aH is the set of all ah where a is fixed and h (an element of H) ranges over all of H. Then aH is the left coset of H in G. Similarly Ha is the right coset of H in G.

Now if G/H is a homomorphic image of G then G/H is also a group and there exists homomorphism from G onto G/H, where a homomorphism is a function $f : G \to G/H$ such that for any two elements a and b in G then $f(ab) = f(a)f(b)$.

For G/H to be a coset, H has to be a normal subgroup of G. Where if H is a normal subgroup of G then $aH = Ha$, for every $a \in G$. This means that there's no difference between left and right cosets.

The coset G/H is abelian if all the commutators of G are in H, where a commutator of G is defined as $abb^{-1}a^{-1}$ where a and b are elements of G. These are called commutators as $abb^{-1}a^{-1} = e$ iff $ab = ba$, where e is the neutral element (a neutral element is defined as $ae = a = ea$). [35]

So now we can define the coset formally [36].

Definition 3.1. A coset is quotient of a group over its non-overlapping subgroups

$$G/H = \frac{G}{H_1 \times H_2 \times ... \times H_n}$$

where H_i is a subgroup of G and $H_i \cap H_j = \emptyset$ when $i \neq j$.

Now that the coset has been defined, a look at how the spaces S^5 and $\mathbb{CP}^3$ can be equivalently described in terms of cosets.

3.2 Spheres and S^5 case

The $(n-1)$ sphere S^{n-1} in $\mathbb{R}^n$ is defined as

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\} \tag{3.1}$$

where $|x|$ is the usual distance $|(x_1, x_2, ..., x_n)| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$

The sphere is a highly symmetric object, in fact the sphere looks the same from every point on the sphere. Spaces with this property are known as homogeneous spaces. A homogeneous space is a smooth manifold that has a Lie group G such that for any two point x and y in the homogeneous space there is an element $g \in G$ such that $gx = y$.

Any homogeneous space can be seen as a coset space, this can be shown by considering a point x in the homogeneous space and H a subgroup of G such that $hx = x$ for all $h \in H$. Then H is called the isotropy group of G at x. Thus the coset G/H can be quipped with a smooth structure such that for any x in the homogeneous space $gH \mapsto gx$ or G/H is diffeomorphic to the homogeneous space. A diffeomorphism between two manifolds M and N is a differential map $f : M \mapsto N$ that is a bijection [37].

The sphere can therefore be expressed in many different ways in terms of homogeneous spaces. One of which is given by the coset of special orthogonal groups (groups of angle preserving rotations on $\mathbb{R}$) [38]

$$S^n \cong \frac{SO(n+1)}{SO(n)} \tag{3.2}$$

where $\cong$ stands for diffeomorphic.

$SO(n+1)/SO(n)$ can be shown to be a homogeneous space (i.e. show that $SO(n)$ is an isotropy group of $SO(n+1)$). Seeing as $SO(n+1)$ is the group of all rotations on the sphere S^n then if we take a point (call it a pole) and rotate around that point leaving the point and its opposite pole unchanged (for example the north and south pole of a two sphere S^2). Then this type of rotation corresponds to the action of the group $SO(n)$. Thus $SO(n)$ is the isotropy group of $SO(n+1)$, therefore the coset $SO(n+1)/SO(n)$ is a homogeneous space.

Another way to look at it is the isotropy group around a point $(1,0,0,...,0) \in S^n$ is the collection of matrices

$$\begin{bmatrix} 1 & \\ & SO(n) \end{bmatrix} \subset SO(n+1)$$

The above matrix is isomorphic to $SO(n)$. Thus $SO(n)$ is an isotropy group to $SO(n+1)$.

Now if we consider $\mathbb{R}^{2n}$ as being $\mathbb{C}^n$ then able to look at spheres as $S^{2n-1} \subset \mathbb{C}^n \approx \mathbb{R}^{2n}$, defined by

$$S^{2n-1} = \{z \in \mathbb{C}^n : \langle z, z \rangle = 1\} \tag{3.3}$$

where $\langle , \rangle$ is the usual hermitian inner product. With rotations that are linear in $\mathbb{C}$, thus holomorphic rotations. These rotations that preserve the angle in $\mathbb{C}$ are the special unitary groups $SU(n)$.

Now $SU(n)/SU(n-1)$ is a homogeneous space as well. This can be proved by first showing that $SU(n)$ is transitive on S^{2n-1} in $\mathbb{C}^n$ (for an $x \in S^{2n-1}$ there exists a $g \in SU(n)$ such that $gx = y$ for all $y \in S^{2n-1}$). Thus $SU(n)/SU(n-1)$ is a homogeneous space [39]. Therefore

$$S^{2n-1} \cong \frac{SU(n)}{SU(n-1)} \tag{3.4}$$

In a similar way to the case of $SO(n+1)$ above, the isotropy group around a point is the following collection of matrices

$$\begin{bmatrix} 1 & \\ & SU(n-1) \end{bmatrix} \subset SU(n)$$

So $SU(n)$ is an isotropy group of $SU(n+1)$.

Finally in the case of the five sphere S^5 we have

$$S^5 \cong \frac{SO(6)}{SO(5)} \quad and \quad S^5 \cong \frac{SU(3)}{SU(2)} \tag{3.5}$$

These coset models have the right dimensions. S^5 is 5 dimensional, with $SO(n)$ having dimensions of $\frac{n(n-1)}{2}$ and $SU(n)$ having dimensions given by

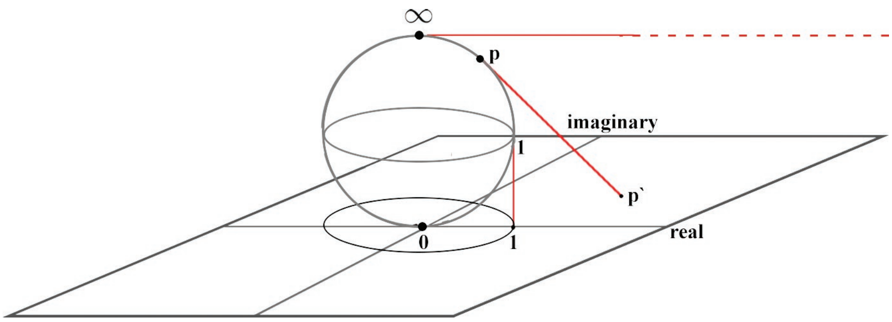

Figure 2: Riemann sphere visualised as the complex plane wrapped around as a sphere.

$n^2 - 1$. So for $SO(6)/SO(5)$ there are 15 dimensions for $SO(6)$ with the 10 dimensions of $SO(5)$ being "factored out" by the coset, leaving 5 dimensions for the coset. Same goes for $SU(3)/SU(2)$ with 8 dimensions for $SU(3)$ and $SU(2)$ with 3 dimensions being "factored out", so this leaves 5 dimensions. All the same dimensions as the five sphere S^5.

3.3 Complex projective space and $\mathbb{CP}^3$ case

The complex projective space n-space, $\mathbb{CP}^n$ is just the space of complex lines (planes in real vector space) in $\mathbb{C}^{n+1}$, or all complex lines through the origin in $\mathbb{C}^{n+1}$.

A good example to obtain a clearer idea of complex projective space is $\mathbb{CP}^1$, the complex projective line [40]. $\mathbb{CP}^1$ is simply the complex plane with a point at infinity. This can be seen by starting with all the numbers in $\mathbb{C}$ and then associating each $z \in \mathbb{C}$ with a vector $(z, 1)^T$ and identify all the non-zero scalar multiples. So by this associate all vectors of the form $(a, b)^T$ with $b \neq 0$ to a number in $\mathbb{C}$. All that will be left is $(1, 0)^T$ and all of its non-zero multiples, which together represents the point at infinity. One can visualise this complex projective space as a stereographic projection, which is a mapping from a sphere onto a plane. For $\mathbb{CP}^1$ this mapping gives the Riemann sphere. This sphere has a pole with the value of 0 and its opposite pole with a value of ∞. If a sphere were to be placed at the origin of the complex plane thus giving the point at which the sphere meets the origin the value 0. Then if a tangent line is drawn at every point on the sphere, the point where that tangent intersects with the complex plane is the value of the point on the sphere. Continuing to draw these tangent lines, the closer to the opposite pole of 0 the larger the number on the complex plane. Thus the opposite pole has a value of ∞ (see figure 2).

Now to describe complex projective space in terms of a coset. $\mathbb{CP}^n$ has the isometry group of $PU(n + 1)$ (projective unitary group). For a Hilbert space $\mathcal{H}$, the projective unitary group $PU(\mathcal{H})$ is a quotient of the unitary group $U(\mathcal{H})$ with its center $U(1)$. If $\mathcal{H}$ is of finite dimension n, then $PU(n) = U(n)/U(1)$. The stabilizer of a point in $PU(n)$ (the mapping that maps an element from $PU(n)$ to itself [41]) is given by $P(1 \times U(n)) \cong PU(n)$. Thus

$$\mathbb{CP}^n \cong PU(n+1)/PU(n)$$

For the case of $\mathbb{CP}^3$, we have $PU(4)/PU(3)$. This coset is equivalent to the coset $SU(4)/U(3)$ [42]. Therefore

$$\mathbb{CP}^3 \cong \frac{SU(4)}{U(3)} \tag{3.6}$$

Again the dimensions can be checked on both sides. $\mathbb{CP}^n$ has $2n$ dimensions therefore $\mathbb{CP}^3$ has 6 dimensions. Now $SU(n)$ has n^2-1 dimensions and $U(n)$ has dimensions of n^2, thus for $SU(4)$ there are 15 dimensions and for $U(3)$ there are 9 dimensions factored out, leaving 6 dimensions. The same as for $\mathbb{CP}^3$.

4 Dressing Method

In this section the dressing method of Zakharov and Mikhailov [16, 17] will be described and the algorithm will be given of the dressing method applied to each of the coset spaces of S^5 and $\mathbb{CP}^3$ mentioned in the last section.

4.1 Review

The dressing method is an algorithm proposed by Zakharov and Mikhailov to construct soliton solutions for non-linear integrable partial differential equations. The dressing method is useful as it takes a 2$^{\text{nd}}$ order differential equation and reduces it to two 1$^{\text{st}}$ order differential equations. Another advantage is that if only a single solution is known then the method can be applied recursively in order to generate more, complicated solutions.

The procedure will be described in terms of the principal chiral field on $SU(N)$ where $N \geq 3$.

Consider $g(\eta, \xi)$ where $\eta = \frac{1}{2}(x-t)$ and $\xi = \frac{1}{2}(x+t)$ (where x and t are worldsheet coordinates), and that is subject to the equation of motion

$$\partial_\xi(\partial_\eta g g^{-1}) + \partial_\eta(\partial_\xi g g^{-1}) = 0 \tag{4.1}$$

The above 2$^{\text{nd}}$ order partial differential equation can be written as a linear 1$^{\text{st}}$ order system for a different filed Ψ, by introducing a new complex variable λ (called the spectral parameter).

$$i\partial_\xi \Psi(\lambda) = \frac{A\Psi}{1+\lambda} \qquad i\partial_\eta \Psi(\lambda) = \frac{B\Psi}{1-\lambda} \tag{4.2}$$

where

$$A = i\partial_\xi g g^{-1} \quad and \quad B = i\partial_\eta g g^{-1} \tag{4.3}$$

So if a solution g is known then A and B can be obtained and also by imposing the initial condition of $\Psi(0) = g$ then $\Psi(\lambda)$ can be solved.

This will give

$$\Psi(\lambda) = \exp\left[-i\int \frac{A}{\lambda+1}d\xi + i\int \frac{B}{\lambda-1}d\eta\right] \tag{4.4}$$

where $\Psi(\lambda)$ must satisfy the unitary condition $\Psi^{\dagger}(\overline{\lambda})\Psi(\lambda) = I$.

Now in order to find a new soliton solution, take an analogue of a gauge transform of Ψ for the linear system. This will give the new solution Ψ' in the form

$$\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda) \tag{4.5}$$

With the subsequent A' and B' given by

$$A' = \chi A\chi^{-1} + i(1+\lambda)\partial_{\xi}\chi\chi^{-1} \qquad B' = \chi B\chi^{-1} + i(1-\lambda)\partial_{\eta}\chi\chi^{-1} \tag{4.6}$$

Now if A' and B' are independent of λ then (4.5) will be a legitimate solution to (4.2). The new generated solution is provided when $\lambda = 0$, this gives

$$g' = \chi(0)g \tag{4.7}$$

Solving for A' and B' and ensuring they're independent of λ, it is found that χ, the dressing factor must have a pole at λ. This dressing factor will then be given by

$$\chi(\lambda) = I + \frac{\lambda_1 - \overline{\lambda_1}}{\lambda - \lambda_1}P \tag{4.8}$$

Note that $\chi(\lambda)$ must satisfy $\chi^{\dagger}(\overline{\lambda})\chi(\lambda) = I$, this ensures that the new solution also satisfies the unitary condition. The parameter λ_1 is an arbitrary complex parameter and P is the projection operator ($P^2 = P$) onto the subspace spanned by $\Psi(\overline{\lambda_1})e$ for a constant complex vector e.

The projection operator is written explicitly as

$$P = \frac{\Psi(\overline{\lambda_1})ee^{\dagger}[\Psi(\lambda_1)]^{\dagger}}{e^{\dagger}[\Psi(\lambda_1)]^{\dagger}\Psi(\overline{\lambda_1})e} \tag{4.9}$$

One possible problem is that depending on the embedding (for example an embedding into a coset), the dressing factor $\chi(\lambda)$ could need more than a single pole (λ_1) in order for any imposed constraints on the solution to be satisfied.

For example if an additional pole ($\frac{1}{\lambda_1}$) is introduced to the dressing factor (4.8), it changes as follows

$$\chi(\lambda) = I + \frac{\lambda_1 - \overline{\lambda_1}}{\lambda - \lambda_1}P + \frac{\frac{1}{\lambda_1} - \frac{1}{\overline{\lambda_1}}}{\lambda - \frac{1}{\lambda_1}}Q \tag{4.10}$$

where Q is the projection operator whose image is spanned by $\Psi(\frac{1}{\overline{\lambda_1}})w$, with w a constant complex vector. Q will be given by

$$Q = \frac{\Psi(\frac{1}{\overline{\lambda_1}})ww^\dagger\Psi^{-1}(\frac{1}{\lambda_1})}{w^\dagger\Psi^{-1}(\frac{1}{\lambda_1})\Psi(\frac{1}{\overline{\lambda_1}})w} \tag{4.11}$$

4.2 Dressing method on $SU(N)$

For the case of $SU(N)/SU(N-1)$, the dressing method for $SU(N)$ can be applied with the added constraint that the initial solution and generated solutions must be embedded in $SU(N)/SU(N-1)$.

For $SU(N)$ the method will be to take an initial solution g satisfying the equation of motion (4.1) and use it solve for A and B which can then be used to solve the linear system (4.2) to get $\Psi(\lambda)$. This $\Psi(\lambda)$ must then satisfy the unitary constraint that $\Psi^\dagger(\overline{\lambda})\Psi(\lambda) = I$ and also satisfy the constraint $det(\Psi(0)) = 1$ in order for Ψ to be in $SU(N)$. The solved linear system (4.4) can then be used in (4.5) along with (4.9) in order to obtain the new solution $\Psi'(\lambda)$. Note that $\chi(\lambda)$ must also satisfy the unitary condition as well as have determinant of one in order for the new solution to satisfy these conditions. In this regard for $SU(N)$ the determinant of $\chi(\lambda)$ is equal to $\frac{\lambda-\overline{\lambda_1}}{\lambda-\lambda_1}$. To compensate for this a factor of $\sqrt{\frac{\overline{\lambda_1}}{\lambda_1}}$ is included in the dressed solution of g'.

4.3 Dressing method on $SO(N)$

To apply the dressing method for this model, the method needs to be embedded into the principal chiral model. The embedding is given by [18] where more details can be found on using the dressing method in the $SO(N)$ vector model.

The equations of motion for the $SO(N)$ vector model is given by

$$\partial_\xi\partial_\eta X_i + (\partial_\eta X_j \partial_\xi X_j)X_i = 0 \quad and \quad X_iX_i = 1 \tag{4.12}$$

With the added constraint of

$$\partial_\eta X_i \partial_\eta X_i = \partial_\xi X_i \partial_\xi X_i = 1 \tag{4.13}$$

Now to embed the vector X_i into the $SO(N)$ principal chiral model

$$\{X_i : X_iX_i = 1\} \quad \leftrightarrow \quad g = \alpha(2XX^T - 1) \in SO(N) \tag{4.14}$$

where α is an $N\times N$ diagonal matrix of the form $\alpha = diag(+1, -1, -1, ..., -1)$. Also noting that g satisfies $g\alpha g\alpha = I$ and $g^\dagger g = I$.

The dressing method continues from this point on, in exactly the same manner as the for $SU(N)$, except with the added constraints

$$\overline{\Psi(\overline{\lambda})} = \Psi(\lambda) \quad and \quad \Psi(\lambda) = \Psi(0)\alpha\Psi(\frac{1}{\lambda})\alpha \tag{4.15}$$

Which means the new solution obtained must also meet these constraints (i.e. $\chi(\lambda)\Psi(\lambda)$ must satisfy the above constraint). This leads to the constraint that

$$\overline{\chi(\overline{\lambda})} = \chi(\lambda) \quad and \quad \chi(\lambda) = \chi(0)\Psi(0)\alpha\chi(\frac{1}{\lambda})\Psi(\frac{1}{\lambda})\alpha \tag{4.16}$$

With these added constraints $\chi(\lambda)$ cannot have a single pole. The most simple option is to have two poles

$$\lambda_1 \quad and \quad \overline{\lambda_1} = \frac{1}{\lambda_1} \tag{4.17}$$

This gives the dressing factor χ as

$$\chi(\lambda) = I + \frac{\lambda_1 - \overline{\lambda_1}}{\lambda - \lambda_1}P + \frac{\overline{\lambda_1} - \lambda_1}{\lambda - \overline{\lambda_1}}\overline{P} \tag{4.18}$$

Where the projector P is the same as before, but the constant vector e in (4.9) must also satisfy

$$e^T e = 0 \quad and \quad \overline{e} = \alpha e \tag{4.19}$$

Where $e \in \mathbb{C}^N$.

4.4 Non-trivial solutions from the dressing method

Before applying the dressing method one last piece of theory is needed; that of identifying the non-trivial interaction of solitons (3-point functions) that this paper sets out to find. This interaction can be seen in [17] where it is shown that the dressing method applied to a field with values in $SU(3)$ results in a decay of a soliton of one component of the field in to two solitons of other components in the field. This was shown with the projector operator P, where the general P obtained from the dressing method for a field with values in $SU(3)$ is

$$|P_{pq}| = \frac{1}{2\cosh\left(\alpha_{pq}(x - v_{pq}t - x^0_{pq})\right) + \exp\left(\Gamma_{pq}t - \kappa_{pq}\right)} \quad where \quad p \neq q \tag{4.20}$$

Essentially, without going into all the details of the above result, when in the non-degenerate case for matrices A and B (obtained from 4.3) then every Γ_{pq} will not have the same sign. So for instance if $\Gamma_{12} > 0 > \Gamma_{13}, \Gamma_{23}$ (determined by the choice of spectral parameter λ), then as $t \to -\infty$ then

$$|P_{12}| = \frac{1}{2\cosh\left(\alpha_{12}(x - v_{12}t - x^0_{12})\right)} \qquad P_{13}, P_{23} \to 0 \tag{4.21}$$

It must be noted that this is the standard $SU(2)$ soliton (equation (4.15) in [18].

Now as t increases when $t \to +\infty$ the solution undergoes a smooth transition to the $|P_{pq}|$ in the form given by

$$P_{12} \to 0 \quad and \quad |P_{13}| = \frac{1}{2\cosh\left(\alpha_{13}(x - v_{13}t - x^0_{13})\right)}, \qquad |P_{23}| = \frac{1}{2\cosh\left(\alpha_{23}(x - v_{23}t - x^0_{23})\right)} \tag{4.22}$$

If the opposite is true for the signs of Γ_{pq} then it will be seen that two solitons combine into a single soliton. This is true for the $SU(3)$ model. In the next section it will be checked if this non-trivial interaction occurs for coset models.

The non-trivial solutions will be checked for in this paper by examining the solutions obtained from the dressing method applied to the vacuum solutions embedded in each coset, at $t \to \pm\infty$. Hopefully the new solution will exhibit a 3-point function where at $t \to -\infty$ a certain behaviour is examined and at $t \to +\infty$ two different behaviours are observed. For example

$$t \to -\infty \begin{bmatrix} * & * & \bigcirc \\ * & * & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc \end{bmatrix}, \quad t \to +\infty \begin{bmatrix} * & \bigcirc & * \\ \bigcirc & \bigcirc & \bigcirc \\ * & \bigcirc & * \end{bmatrix} \; and \; \begin{bmatrix} \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & * & * \\ \bigcirc & * & * \end{bmatrix}$$

where $*$ is non-zero and $\bigcirc$ represents a zero value.
If the above or any evolution of a similar nature is examined in any of the obtained solutions embedded in the cosets proposed for S^5 then this would be an example of non-trivial interactions of solitons in S^5.

5 Application of the Dressing Method

Now to apply the dressing method to vacuum solutions embedded in each of the coset models and solve for new solutions.

5.1 $SU(3)/SU(2)$ coset model

Before the dressing method can be applied, the solution will need to be embedded in to the coset model. This will be done using the embedding given below (the technical details of the embedding are provided in Appendix A), where $\alpha = \frac{-1+\sqrt{1+r^2}}{r^2}$. This is only one of many other embeddings of which some may be worth considering in possible future studies.

$$g = \frac{1}{\sqrt{1+r^2}} \begin{bmatrix} e^{i\phi} & -e^{i(\theta-\phi)}\overline{x_1} & -e^{-i\theta}\overline{x_2} \\ e^{i\phi}x_1 & e^{i(\theta-\phi)}(\sqrt{1+r^2}-\alpha|x_1|^2) & -\alpha e^{-i\theta}x_1\overline{x_2} \\ e^{i\phi}x_2 & -\alpha e^{i(\theta-\phi)}x_2\overline{x_1} & e^{-i\theta}(\sqrt{1+r^2}-\alpha|x_2|^2) \end{bmatrix} \tag{5.1}$$

Where x_1 and x_2 are complex non-homogeneous coordinates of $\mathbb{CP}^2$, so the corresponding homogeneous coordinates of S^5 can be written as

$$(Z_1, Z_2, Z_3) = \sqrt{1+r^2}e^{i\phi}(1, x_1, x_2)$$

Note, the above embedding relies on 6 parameters which is one too many to describe th 5-sphere, but it could be that one of the parameters could possibly be dependent on other parameters.

Beginning with the vacuum solution coordinates of S^5 (particle moving along the equator at the speed of light)

$$Z_1 = e^{it} \quad Z_2 = 0 \quad Z_3 = 0 \tag{5.2}$$

where Z_i is complex and $Z_i\overline{Z_i} = 1$.
The embedding gives the element g as

$$g_0 = \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & e^{-it} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.3}$$

Here $x_1 = x_2 = \theta = 0$ and $\phi = t$. The parameter θ has been ignored in the above application of the embedding but another possible application of the embedding on the vacuum solution that does notice the angle θ is given by

$$g_1 = \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & e^{i(x-t)} & 0 \\ 0 & 0 & e^{-ix} \end{bmatrix} \tag{5.4}$$

Where $x_1 = x_2 = 0, \theta = x$ and $\phi = t$.

5.1.1 $\theta = 0$

Now to apply the dressing method, firstly to (5.3). Using the following substitution of coordinates $\eta = \frac{1}{2}(x - t)$ and $\xi = \frac{1}{2}(x + t)$ the element g_0 looks like

$$g_0 = \begin{bmatrix} e^{i(\xi-\eta)} & 0 & 0 \\ 0 & e^{i(\eta-\xi)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.5}$$

Then the matrices A and B are given by

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using (4.4), we obtain

$$\Psi(\lambda) = \begin{bmatrix} e^{iZ(\lambda)} & 0 & 0 \\ 0 & e^{-iZ(\lambda)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.6}$$

Where $Z(\lambda) = \frac{\xi}{\lambda+1} + \frac{\eta}{\lambda-1} = \frac{x\lambda-t}{\lambda^2-1}$.

In order to determine the projector P, the arbitrary constant vector $e \in \mathbb{C}^3$ must be chosen. So we choose $e = (c_1, c_2, 1)$ to be a constant element of $\mathbb{P}^2$. Since e only enters into P through

$$\Psi(\overline{\lambda})e = \begin{bmatrix} c_1 e^{iZ(\lambda)} \\ c_2 e^{-iZ(\lambda)} \\ 1 \end{bmatrix}$$

therefore c_1 and c_2 can be absorbed by simply transforming $Z(\lambda) \to Z(\lambda) + i \log c_j$. Since any translational transformation of $Z(\lambda)$ is just a shift

in x and t, we are thus able to set $c_1 = c_2 = 1$.
The projector operator is then given by

$$P = \frac{1}{1 + e^{-i2(Z'-Z)} + e^{-i(Z'-Z)}} \begin{bmatrix} 1 & e^{i2Z} & e^{iZ} \\ e^{-i2Z'} & e^{-i2(Z'-Z)} & e^{i(Z-2Z')} \\ e^{-iZ'} & e^{-i(Z'-2Z)} & e^{-i(Z'-Z)} \end{bmatrix} \tag{5.7}$$

where Z' is used in place of $Z(\overline{\lambda_1})$ and Z for $Z(\lambda_1)$ to simplify the notation.

Now the dressing solution will only have a single pole since the coset constraint is already included in the embedding thus we have no need to alter χ. So the resultant solution g_0' is obtained from $g_0' = \chi(0)g_0$ and not forgetting the compensating factor of $\sqrt{\frac{\overline{\lambda_1}}{\lambda_1}}$, we obtain

$$g_0' = (\frac{\overline{\lambda_1}}{\lambda_1})^{1/2} R \begin{bmatrix} e^{it} & -e^{-it}e^{i2Z}M & -e^{iZ}M \\ -e^{it}e^{-i2Z'}M & e^{-it}(L - e^{-i2(Z'-Z)}M) & -e^{i(Z-2Z')}M \\ -e^{it}e^{-iZ'}M & -e^{-it}e^{-i(Z'-2Z)}M & L - e^{-i(Z'-Z)}M \end{bmatrix}$$

where $R = (\frac{1+e^{-i2(Z'-Z)}+e^{-i(Z'-Z)}-c}{1+e^{-i2(Z'-Z)}+e^{-i(Z'-Z)}})$, $M = \frac{c}{1+e^{-i2(Z'-Z)}+e^{-i(Z'-Z)}-c}$, $L = \frac{1+e^{-i2(Z'-Z)}+e^{-i(Z'-Z)}}{1+e^{-i2(Z'-Z)}+e^{-i(Z'-Z)}-c}$ and $c = \frac{\lambda_1 - \overline{\lambda_1}}{\lambda_1}$.

Now to check the time evolution of the new solution g_0' at $\pm\infty$.

$$t \to -\infty \begin{bmatrix} * & \bigcirc & \bigcirc \\ \bigcirc & * & \bigcirc \\ \bigcirc & \bigcirc & * \end{bmatrix} \qquad t \to +\infty \begin{bmatrix} * & \bigcirc & \bigcirc \\ \bigcirc & * & \bigcirc \\ \bigcirc & \bigcirc & * \end{bmatrix} \tag{5.8}$$

where $*$ represents a non-zero value and $\bigcirc$ represents a zero value.

It can be seen from the above that the new solution does not show any non-trivial interaction. The solution's behaviour at $t \to -\infty$ and $t \to +\infty$ is the same.

5.1.2 $\theta = x$

Moving to the vacuum solution with $\theta = x$ (5.4) and using coordinates $\eta = \frac{1}{2}(x - t)$ and $\xi = \frac{1}{2}(x + t)$, g_1 will look like

$$g_1 = \begin{bmatrix} e^{i(\xi-\eta)} & 0 & 0 \\ 0 & e^{i2\eta} & 0 \\ 0 & 0 & e^{-i(\xi+\eta)} \end{bmatrix} \tag{5.9}$$

This gives the matrices A and B as

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So Ψ will be given as

$$\Psi(\lambda) = \begin{bmatrix} e^{iF(\lambda)} & 0 & 0 \\ 0 & e^{iG(\lambda)} & 0 \\ 0 & 0 & e^{iH(\lambda)} \end{bmatrix} \tag{5.10}$$

where $F(\lambda) = \frac{\xi}{1+\lambda} - \frac{\eta}{1-\lambda}$, $G(\lambda) = \frac{2\eta}{1-\lambda}$ and $H(\lambda) = -\frac{\xi}{1+\lambda} - \frac{\eta}{1-\lambda}$.

Taking the constant vector e to again be of the form $e = (1, 1, 1)$, the projector is then given by

$$P = \frac{1}{1 + e^{i(\overline{G}-G-\overline{F}+F)} + e^{i(\overline{H}-H-\overline{F}+F)}} \begin{bmatrix} 1 & e^{i(F-G)} & e^{i(F-H)} \\ e^{-i(\overline{F}-\overline{G})} & e^{i(\overline{G}-G-\overline{F}+F)} & e^{i(\overline{G}-H-\overline{F}+F)} \\ e^{-i(\overline{F}-\overline{H})} & e^{-i(G-\overline{H}-F+\overline{F})} & e^{i(\overline{H}-H-\overline{F}+F)} \end{bmatrix}$$

where $\overline{F} = F(\overline{\lambda_1})$, $F = F(\lambda_1)$, $\overline{G} = G(\overline{\lambda_1})$, $G = G(\lambda_1)$, $\overline{H} = H(\overline{\lambda_1})$ and $H = H(\lambda_1)$.

Again only a single pole is needed for the dressing factor $\chi(\lambda)$. Therefore the solution $g_1' = \chi(0)g_1$ will be given by

$$g_1' = T \begin{bmatrix} e^{it} & -e^{i(x-t)}e^{i(F-G)}U & -e^{-ix}e^{i(F-H)}U \\ -e^{it}e^{-i(\overline{F}-\overline{G})}U & e^{i(x-t)}(W - e^{i(\overline{G}-G-\overline{F}+F)}U) & -e^{-ix}e^{i(G-H-\overline{F}+F)}U \\ -e^{it}e^{-i(\overline{F}-\overline{H})}U & -e^{i(x-t)}e^{-i(G-\overline{H}-F+\overline{F})}U & e^{-ix}(W - e^{i(\overline{H}-H=\overline{F}+F)}U) \end{bmatrix} \tag{5.11}$$

where $T = 1 - \frac{c}{1+e^{i(\overline{G}-G-\overline{F}+F)}+e^{i(\overline{H}-H-\overline{F}+F)}}$, $U = \frac{c}{1+e^{i(\overline{G}-G-\overline{F}+F)}+e^{i(\overline{H}-H-\overline{F}+F)}-c}$, $W = (\frac{1+e^{i(\overline{G}-G-\overline{F}+F)}+e^{i(\overline{H}-H-\overline{F}+F)}-c}{1+e^{i(\overline{G}-G-\overline{F}+F)}+e^{i(\overline{H}-H-\overline{F}+F)}})$ and $c = \frac{\lambda_1 - \overline{\lambda_1}}{\lambda_1}$.

Again after investigating the limits of $t \to \pm\infty$ of the solution, the solution exhibits the evolution of a 2-point function with the same behaviour at $t \to -\infty$ and $t \to +\infty$.

$$t \to -\infty \begin{bmatrix} * & \bigcirc & \bigcirc \\ \bigcirc & * & \bigcirc \\ \bigcirc & \bigcirc & * \end{bmatrix} \qquad t \to +\infty \begin{bmatrix} * & \bigcirc & \bigcirc \\ \bigcirc & * & \bigcirc \\ \bigcirc & \bigcirc & * \end{bmatrix} \tag{5.12}$$

where $*$ indicates a non-zero value and $\bigcirc$ represents a zero value.

5.2 $SO(6)/SO(5)$ coset model

For the $SO(6)/SO(5)$ coset model, we first begin with a vacuum solution of a particle moving along the equator of S^5 at the speed of light. This is given by

$$\mathbf{X} = (\cos t, \sin t, 0, 0, 0, 0)$$

After embedding using (4.14), the solution g is given by

$$g = \begin{bmatrix} \cos 2t & \sin 2t & 0 & 0 & 0 & 0 \\ -\sin 2t & \cos 2t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.13}$$

This embedding satisfies $g\alpha g\alpha = I$ where $\alpha = diag(1, -1, -1, -1, -1, -1)$, which means that the solution is embedded into the $SO(6)/SO(5)$ coset model.

Now matrices A and B are given by

$$A = \begin{bmatrix} 0 & i & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Taking into consideration the constraints of (4.16), and using (4.4), the form of $\Psi(\lambda)$ becomes

$$\Psi(\lambda) = \begin{bmatrix} \cos 2Z(\lambda) & \sin 2Z(\lambda) & 0 & 0 & 0 & 0 \\ -\sin 2Z(\lambda) & \cos 2Z(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.14}$$

To obtain the projector the constant vector will be chosen to have the form $e = (1, i\sin w, i\vec{v}\cos w)$ where $\vec{v}$ is an arbitrary unit vector of $N-2$ components, in this case it is chosen to be $\vec{v} = (1,0,0,0)$. Again since the parameter w can be absorbed by a translation of x and t, it is safe to set $w = 0$. Thus $e = (1, 0, -i, 0, 0, 0)$ which meets the requirements of (4.19).

This gives the projector as

$$P = \begin{bmatrix} \cos(2Z')\cos(2Z) & \cos(2Z')\sin(2Z) & i\cos(2Z') & 0 & 0 & 0 \\ -\sin(2Z')\cos(2Z) & \sin(2Z')\sin((2Z) & -i\sin(2Z') & 0 & 0 & 0 \\ -i\cos(2Z) & i\sin(2Z) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $Z' = Z(\overline{\lambda_1})$, $Z = Z(\lambda_1)$ and $\overline{\lambda_1} = \frac{1}{\lambda_1}$.

With P we can now obtain $\chi(\lambda)$, given by (4.18). Thus the solution $g' = \chi(0)g$ is given by

$$g' = \begin{bmatrix} \cos(2t)J - \sin(2t)K & \sin(2t)J + \cos(2t)K & -iN & 0 & 0 & 0 \\ \cos(2t)L - \sin(2t)M & \sin(2t)L + \cos(2t)M & iO & 0 & 0 & 0 \\ -i(-\cos(2t)N - \sin(2t)O) & -i(-\sin(2t)N + \cos(2t)O) & 1-a-b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $J = 1 - a\cos(2Z')\cos(2Z) - b\cos(2Z')\cos(2Z)$,
$K = a\cos(2Z')\sin(2Z) + b\cos(2Z)\sin(2Z')$,
$L = a\sin(2Z')\cos(2Z) + b\cos(2Z')\sin(2Z)$,

$M = 1 - a\sin(2Z')\sin(2Z) - b\sin(2Z')\sin(2z)$,
$N = a\cos(2Z) - b\cos(2Z')$, $O = a\sin(2Z) - b\sin(2Z')$,
$a = \frac{\lambda_1 - \frac{1}{\lambda_1}}{\lambda_1}$ and $b = \frac{\frac{1}{\lambda_1} - \lambda_1}{\frac{1}{\lambda_1}}$.
It can be seen that the solution above gives a harmonic solution for t thus it's behaviour at $t \to -\infty$ and at $t \to +\infty$ will be the same. Again we have no non-trivial scattering behaviour for the solution obtained.

6 Conclusion

From the results we see that no non-trivial solutions were observed with the vacuum solutions embeddings. Perhaps the missing element was the initial solution, seeing as the initial solution affects the form of the projector therefore the generated solution. It could be that using a vacuum solution does not give rise to any non-trivial behaviour. Another factor could be the embeddings themselves as in the case of the $SU(3)/SU(2)$ coset model where the embedding could possibly be restricting the form of the obtained solutions thus limiting the type of solutions. To add there is always the possibility of errors in the calculations themselves, for example the dressing factor might not have been in the correct form to satisfy some of the constraints or possibly the constant vector used to calculate the projector, was naively set to a trivial vector when possible, unforeseen constraints should have been applied.

A promising avenue that could lead to a positive result would be to try more general solutions for the motion of strings in S^5, as initial solutions thus giving a new general solution where the desired non-trivial behaviour could be more apparent. It is possible that with further thought a different embedding could be found, for example initially embedding an S^5 element in to $SU(3)$ and then imposing constraints on the embedded solution to ensure it lies in the sub-manifold $SU(3)/SU(2)$. To add, the dressing method can be applied recursively, so it could be possible that if the dressing method is applied to the solutions obtained in this paper then the new solutions could exhibit the desired non-trivial behaviour.

Due to time constraints and no suitable embedding for $\mathbb{CP}^3$ found, this case was not examined. Perhaps with further study, an embedding could be found which would then make altering the dressing method to fit this case possible and allowing for investigation of non-trivial solitons in $\mathbb{CP}^3$.

The results obtained can be used as a starting point to further study with the general idea and framework of finding non-trivial soliton solutions in S^5 already set up in this paper.

Appendix A SU(3)/SU(2) embedding

In this appendix we explain the origins of the embedding used in section 5.1 based on the notes [46].

We begin with the $GL(3)$ element from [45]

$$z = \begin{bmatrix} 1 & 0 \\ X & \mathbb{1} \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \mathbf{v} \end{bmatrix} \begin{bmatrix} 1 & Y \\ 0 & \mathbb{1} \end{bmatrix} \tag{A.1}$$

where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$.

The unitary condition then gives

$$|u|^2 = \frac{1}{1+|X|^2}, \quad Y = -\overline{u}X^\dagger \mathbf{v} \mathbf{v}\mathbf{v}^\dagger = 1 + XX^\dagger = \begin{bmatrix} 1+|x_1|^2 & x_1\overline{x_2} \\ \overline{x_1}x_2 & 1+|x_2|^2 \end{bmatrix} \tag{A.2}$$

which was solved as

$$\mathbf{v} = \begin{bmatrix} 1+\alpha|x_1|^2 & \alpha x_1\overline{x_2} \\ \alpha\overline{x_1}x_2 & 1+\alpha|x_2|^2 \end{bmatrix} \cdot h \tag{A.3}$$

where $\alpha = \frac{-1+\sqrt{1+r^2}}{r^2}$, $r^2 = |x_1|^2 + |x_2|^2$ and h an arbitrary 2×2 unitary matrix.
u and Y can then be written as

$$u = \frac{e^{i\phi}}{\sqrt{1+r^2}} \quad \textit{and} \quad Y = -\overline{u}X^\dagger \mathbf{v} \tag{A.4}$$

Now the general $SU(3)$ element can be parametrised by 8 parameters; the complex parameters x_1 and x_2, ϕ and the 3 parameters of h (where h unitary).
Now the isotropy subgroup $SU(2)$ can be taken to act in the lower right block where the reduction to $SU(3)/SU(2)$ corresponds to using $SU(2)$ to determine a fixed form of h.

A suggested choice of h is given by

$$h = \begin{bmatrix} e^{i(\theta-\phi} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \tag{A.5}$$

Finally if we substitute everything back into (A.1), we obtain

$$z = \frac{1}{\sqrt{1+r^2}} \begin{bmatrix} e^{i\phi} & -e^{i(\theta-\phi)}\overline{x_1} & -e^{-i\theta}\overline{x_2} \\ e^{i\phi}x_1 & e^{i(\theta-\phi)}(\sqrt{1+r^2}-\alpha|x_1|^2) & -\alpha e^{-i\theta}x_1\overline{x_2} \\ e^{i\phi}x_2 & -\alpha e^{i(\theta-\phi)}x_2\overline{x_1} & e^{-i\theta}(\sqrt{1+r^2}-\alpha|x_2|^2 \end{bmatrix}$$

The above is an $SU(3)$ element but depends on only 6 parameters; complex x_1 and x_2, θ and ϕ. There is one parameter too much to describe the coset $SU(3)/SU(2)$ but it could be that one of the parameters could be expressed in terms of the other parameters.

Note that z also satisfies the constraint of the 5-sphere. The S^5 coordinates can be written in terms of the parameters of z by taking x_1 and x_2 to be non-homogeneous coordinates of $\mathbb{CP}^2$. Thus we have

$$(Z_1, Z_2, Z_3) = \sqrt{1+r^2}e^{i\phi} \cdot (1, x_1, x_2) \tag{A.6}$$

This gives the embedding for an S^5 element into the $SU(3)/SU(2)$ coset.

References

[1] J. M. Maldacena. *The Large N Limit of Superconformal field theories and supergravity.* Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113], [arXiv:hep-th/9711200]

[2] J. A. Minahan, K. Zarembo. *The Bethe-ansatz for $N = 4$ super Yang-Mills.* JHEP 0303, 013 (2003), [arXiv:hep-th/0212208v3]

[3] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz. *Large N Field Theories, String Theory and Gravity.* Phys. Rept. 323, 183-386 (2000), [arXiv:hep-th/9905111v3]

[4] I. Bena, J. Polchinski, R. Roiban. *Hidden symmetries of the $AdS_5 \times S^5$ superstring.* Phys. Rev. D 69, 046002 (2004), [arXiv:hep-th/0305116v2]

[5] A. Rej. *Integrability and the AdS/CFT correspondence.* J. Phys. A42, 254002 (2009), [arXiv:0907.3468v2]

[6] D. Serban *Integrability and the AdS/CFT correspondence.* J. Phys. A44, 124001 (2011), [arXiv:1003.4214v3]

[7] A. A. Tseytlin. *Semiclassical quantization of superstrings: $AdS_5 \times S^5$ and beyond.* Int. J. Mod. Phys. A18 (2003) 981, [arXiv:hep-th/0209116v1]

[8] G. Mandal, N. V. Suryanarayana, S. R. Wadia. *Aspects of semiclassical strings in AdS_5.* Phys. Lett. B 543, 81 (2002), [arXiv:hep-th/0206103v2]

[9] R. Rajaraman. *Solitons and Instantons.* North–Holland, Amsterdam, (1982)

[10] J. Scott Russell. *Report on waves.* Fourteenth meeting of the British Association for the Advancement of Science, (1844)

[11] A. Mikhailov. *An action variable of the sine-Gordon model.* J. Geom. Phys. 56 (2006) 2429, [arXiv:hep-th/0504035]

[12] A. Mikhailov. *A non-local Poisson bracket of the sine-Gordon model.* J. Geom. Phys. 61 (2011) 85, [arXiv:hep-th/0511069v3]

[13] D. M. Hofman, J. Maldacena. *Giant Magnons.* J. Phys. A39, 13095-13118 (2006), ([arXiv:hep-th/0604135v2]

[14] A. Tsuji. *Holography of Two-point Functions in the Spinning String/Spin Chain Correspondence.* Prog. Theor .Phys. 117, 557-568 (2007), [arXiv:hep-th/0606030v2]

[15] N. Gromov, V. Kazakov, P. Vieira. *Integrability for the Full Spectrum of Planar AdS/CFT.* Phys. Rev. Lett. 103, 131601 (2009), [arXiv:0901.3753v3

[16] V. E. Zakharov, A. V. Mikhailov. *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method.* Sov. Phys. JETP 05 (1978) 47, [e047061017]

[17] V. E. Zakharov, A. V. Mikhailov. *Example of nontrivial interaction of solitons in two-dimensional classical field theory.* Sov. Phys. JETP Lett. (1978) 27.

[18] M. Spradlin, A. Volovich. *Dressing the Giant Magnon.* JHEP 10, 012 (2006), [arXiv:hep-th/0607009v3]

[19] O. Aharony, O. Bergman, D. L. Jafferis, J. Maldacena. *$\mathcal{N}=6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals.* JHEP 0810, 091 (2008), [arXiv:0806.1218v4]

[20] C. Gu. *Soliton Theory and Its Applications.* Springer Science and Business Media,(2013).

[21] M. Kaku. *Strings, Conformal Fields, and M-Theory, (2nd Ed.)* Springer Science and Business Media, (2012).

[22] C.G.Callan Jr., J.A.Harvey, A.E.Strominger. *Super-symmetric String Solitons.* [arXiv:hep-th/9112030v1]

[23] M. Dunajski. *Integrable Systems.* Department of Applied Mathematics and Theoretical Physics, University of Cambridge (2012), [ISlecture.notes.2012]

[24] M. J. Duff, Ramzi R. Khuri, J. X. Lu. *String Solitons.* Phys. Rept. 259, 213-326 (1995), [arXiv:hep-th/9412184v1]

[25] T. J. Hollywood, J. Luis Miramontes. *The $AdS_5 \times S^5$ Semi-Symmetric Space Sine-Gordon Theory.* JHEP 1105, 136 (2011), [arXiv:1104.2429v2]

[26] P. K. Atanasova, E. V. Zemlyanaya, Y. Shukrinov. *Numerical Study of Fluxon Solutions of Sine-Gordon Equation under the Influence of the Boundary Conditions.* LNCS 7125, pp. 201–206 (2012).

[27] T. M. Elzaki, H. Kim. *The solutions of Burger's equation by Elzaki homotopy perturbation method.* Appl. Math. Sci. 8, 2931-2940 (2014).

[28] J. Shatah. *Normal forms and quadratic non-linear Klein-Gordon equations.* Appl. Math. 38, 685-696 (1985).

[29] V. G. Ivancevic, T. T. Ivancevic. *Sine-Gordon Solitons, Kinks and Breathers as Physical Models of Non-linear Excitations in Living Cellular Structures.* [arXiv:1305.0613v1]

[30] S. Cuenda, N. R. Quintero, A. Sanchez. *Sine-Gordon Wobbles Through Backlund Transformations.* Discrete and Continuous Dynamical Systems Series S, Vol. 4, No. 5, 1047–1056 (2011)

[31] K. Zarembo. *Quantum Giant Magnons.* JHEP 0805, 047 (2008), [arXiv:0802.3681v2]

[32] C. Kalousios, G. Papathanasiou. *Giant Magnons in Symmetric Spaces: Explicit N-soliton solutions for $\mathbb{CP}^n$, $SU(n)$ and S^n* JHEP 1007, 068 (2010), [arXiv:1005.1066v2]

[33] M. Kruczenski, J. Russo, A.A. Tseytlin. *Spiky strings and giant magnons on* S^5 JHEP 0610, 002 (2006), [arXiv:hep-th/0607044v3]

[34] N.P. Bobev, R.C. Rashkov. *Multispin Giant Magnons.* Phys.Rev. D74, 046011 (2006), [arXiv:hep-th/0607018v3]

[35] C. C. Pinter. *A Book of Abstract Algebra.* 2nd Ed. McGraw-Hill, New York (1990).

[36] M. Dukalski *A comparative study of coset* σ*-model formulations of superstring theory on* $AdS_5 \times S^5$ *and* $AdS_4 \times \mathbb{CP}^3$ *backgrounds.* MA thesis. Institute for Theoretical Physics, Utrecht University (2009), [MDukalski.pdf]

[37] A. Banyaga. *The structure of classical diffeomorphism groups, Volume 400 of Mathematics and its applications.* Kluwer Academic, (1997).

[38] M. Kerin and D. Wraith. *Homogeneous Metrics on Spheres.* Irish Math. Soc. Bulletin 51, 59-71 (2003), [M5102]

[39] P. Garrett. *Classical Homogeneous Spaces.* School of Mathematics, University of Minnesota (2010), [08 Homogeneous]

[40] P. Garrett. *Compactification: Riemann sphere, projective space.* School of Mathematics, University of Minnesota (2014), [06 Riemann Sphere]

[41] D. A. Holton, J. Sheehan, *The Petersen Graph.* Cambridge University Press, Cambridge, England. (1993).

[42] S. Kobayashi, K. Nomizu. *Foundations of Differential Geometry, Volume II. Wiley Classics Library Ed.* Wiley, New York (1996).

[43] S. J. van Tongeren. *Integrable superstrings in* $AdS_4 \times \mathbb{CP}^3$. MA thesis. Institute for Theoretical Physics and Spinoza Institute, Utrecht University (2009), [Van Tongeren.pdf]

[44] C. Kalousios, M. Spradlin, A. Volovich. *Dressing the Giant Magnon II.* JHEP 03, 020 (2007), [arXiv:hep-th/0611033v1]

[45] M. Hatsuda, K. Kamimura. *Projective Coordinates and Projective Space Limit.* Nucl. Phys. B798, 310-322 (2008), [arXiv:0707.0326v2]

[46] K. Zoubos. *Notes on the* $SU(3)$ *embedding* Personal consultation notes, Dated: 27/10/2010, Received: 2015.