# Coordinate Bethe Ansatz and Quantum Group Symmetry of the Spin-$\frac{1}{2}$ XXZ Heisenberg Spin Chain *

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Abstract: The closed spin-$\frac{1}{2}$ isotropic (XXX) spin chain is a quantum many body problem that is exactly solvable, or integrable. With the introduction of an anisotropy parameter, it can be deformed to the XXZ spin chain, which remains integrable. The technique pioneered by Hans Bethe in 1931, called the Coordinate Bethe ansatz is applied to solve the energy eigenvalue equation. We also see that instead of the simple SU(2) symmetry of the XXX chain, in the limit of infinite sites, the symmetry of the closed XXZ chain is described by a Hopf algebra, the quantum group $U_q(sl(2))$.

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## Contents



# 1 Introduction

The Heisenberg spin chain is a quantum many body system that is exactly solvable, or *integrable*. Like the harmonic oscillator in quantum mechanics, it serves as a very good introduction to theory of integrable models, and appears in a large number of different topics. It was originally devised as a model of 1-dimensional ferromagnetic metals, which have been realised in some way in laboratories (See for example [1–3]). Hans Bethe discovered an elegant solution to this problem in 1931, which will be discussed in this text [4]. It allows one to solve for only the the lowest energy states, instead of diagonalising the entire Hamiltonian. Bethe considered an isotropic spin chain, called the XXX spin chain. A additional parameter can be added to the Hamiltonian describing this chain, influencing only the z-component of this Hamiltonian. This is called the XXZ spin chain. The XXZ spin chain does not have the simple symmetry of the XXX spin chain, but it remains integrable, and Bethe's solution is still applicable.

A recent development is that spin chains were found to be related to conformal field theories. Specifically, in $\mathcal{N} = 4$ Super Yang-Mills (SYM), certain operators (those in the $SU(2)$ sector) can be mapped to the XXX spin chain [5]. More generalised operators are mapped onto more advanced spin chains. In the light of AdS/CFT correspondence, the symmetries of these operators in the conformal field theory could tell us about symmetries in the anti-de Sitter background, where they are much harder to see. One motivation to consider the XXZ spin chain is that it is related to conformal deformations of the $\mathcal{N} = 4$ SYM theory and thus plays an important part in AdS/CFT correspondence [6].

In this project, we will first give some background on group theory, the mathematical language of symmetries. We will then consider the XXX spin chain and look at its symmetries. We will then move on to the more general case of the deformed XXZ spin chain, where we will solve for the low lying eigenvalues of its Hamiltonian. We will also find the symmetry of a open XXZ spin chain, which is the quantum group $U_q(sl(2))$.

# 2 Mathematical Background

## 2.1 Lie Groups, Lie Algebras and Representations

In this section we will give some basic definitions of group theory used in this project[1].

A *group*, $(G, \cdot)$ is a set, $G$ together with a binary operation, $\mu$

$$\mu : G \otimes G \to G \tag{2.1}$$

$$(g_1, g_2) \to g_1 \cdot g_2 \tag{2.2}$$

usually called multiplication, that satisfies the following axioms:

**Associativity:** for all $g_1, g_2, g_3 \in G$

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \tag{2.3}$$

**Identity:** There exists an element $e \in G$, such that for all $g \in G$

$$e \cdot g = g \cdot e = g \tag{2.4}$$

[1]These definitions are based on those given by [7].

**Inverse:** for each $g \in G$ there exists an element $g^{-1}$, such that

$$g \cdot g^{-1} = g^{-1} \cdot g = e \tag{2.5}$$

It can easily be proved that the identity element is unique. It should also be noted that, in general $a \cdot b \neq b \cdot a$, i.e. multiplication is not commutative. If this property holds the group is said to be Abelian.

Groups are very useful to describe symmetries. A symmetry can be seen as a transformation of a configuration space that leaves a specific property invariant. The combination of two symmetry transformations is also a symmetry transformation. These transformations are associative and reversible. Thus these transformations form a group.

A *Lie group* $G$ is a group which is also a differentiable manifold, such that the multiplication and inverse maps

$$\mu : G \otimes G \to G \tag{2.6}$$

$$(g_1, g_2) \to g_1 \cdot g_2 \tag{2.7}$$

and

$$\nu : G \to G \tag{2.8}$$

$$g \to g^{-1} \tag{2.9}$$

are smooth. We define a differentiable manifold intuitively as a topological space that can be described by a family of open charts, or maps to linear vector spaces, which are connected in a differentiable manner where they overlap. In simple terms, Lie groups have an infinite number of elements, that are labelled by continuous parameters, and the multiplication and inverse maps depend smoothly on these parameters.

We will also define a *homomorphism* to be a map from one group to another that conserves the multiplication rules of the groups. A homomorphism that is bijective is called an *isomorphism*.

A *Lie Algebra*, $\mathcal{A}$, is a vector space over a field, and a binary operation, called a *Lie bracket*

$$[\cdot, \cdot] : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \tag{2.10}$$

which satisfies the following properties (if $x, y, z$ are elements of the algebra and $a, b$ of the field):

**Bilinearity**

$$[ax + by, z] = a[x, z] + b[y, z] \tag{2.11}$$

**Antisymmetry**

$$[x, y] = -[y, x] \tag{2.12}$$

**Jacobi identity**

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \tag{2.13}$$

We usually consider matrices as algebra elements, where the Lie bracket becomes the commutator known from quantum mechanics ($[A,B] = AB - BA$). This satisfies all the properties above.

Each Lie group has a corresponding Lie algebra. The elements of the group can be found by exponentiating the corresponding algebra element,

$$g(\alpha) = e^{i\alpha T}. \tag{2.14}$$

The generators, or a basis for the algebra can be found by taking a Taylor expansion of a group element, and considering an infinitesimal displacement from the identity (i.e. $\alpha \ll 1$). Then

$$g(\alpha) = 1 + i\alpha T. \tag{2.15}$$

The number of linearly independent generators is called the dimension of the algebra. These generators satisfy some commutation relations. These commutation relations can be used to define the algebra. However, in general, we can find other matrices, of different dimensions, which satisfy the same commutation relation. We call these different *representations* of the algebra. The representations act naturally on some vector space, called the *representation module*, although this is sometimes informally also called a representation. The dimension of the representation module is called the dimension of the representation, which should not be confused with the dimension of the algebra.

## 2.2 A closer look at SU(2)

The group $SU(N)$ is defined to be the set of unitary $N \times N$ matrices with determinant 1, i.e. that if $A \in SU(N)$ then

$$A^\dagger A = \mathbb{I}_{N\times N} \text{ and } \det A = 1. \tag{2.16}$$

If we consider $SU(2)$, and we find generators of the corresponding algebra, $su(2)$, using (2.15), it is easy to see that the properties (2.16) imply that the generators must be hermitian (or anti-hermitian) and traceless. Thus the general form for a generator, $T$, of $SU(2)$ is

$$T = \begin{pmatrix} x & y+iz \\ y-iz & -x \end{pmatrix}. \tag{2.17}$$

Thus it is easy to see that the dimension of $su(2)$ is 3. As a basis for $su(2)$ we can choose the Pauli matrices, $\sigma^i$, $i \in \{x, y, z\}$ defined as

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.18}$$

These matrices satisfy the commutation relations

$$\left[\sigma^i, \sigma^j\right] = 2i\epsilon^{ijk}\sigma^k \tag{2.19}$$

where $\epsilon^{ijk}$ is the Levi-Civita symbol (and we denote $x, y, z$ by $1, 2, 3$ respectively). The Pauli matrices form a 2-dimensional representation of SU(2). We can form a $2^N$-dimensional representation of SU(2) by defining what we call the $N$-site *total spin operators*[2], defined as

$$S^i = \sum_{l=1}^{N} S^i_l \tag{2.20}$$

$$= \frac{1}{2}\sum_{l=1}^{N} \sigma^i_l. \tag{2.21}$$

We can easily check that the total spin operators also satisfy the su(2) commutation relations as defined in eq. (2.19).

$$\left[S^i, S^j\right] = \frac{1}{4}\sum_{m=1}^{N}\sum_{n=1}^{N}\left[\sigma^i_m, \sigma^j_n\right] \tag{2.22}$$

$$= \frac{1}{4}\sum_{m=1}^{N}\sum_{n=1}^{N}\delta_{mn}\left[\sigma^i_m, \sigma^j_n\right] \tag{2.23}$$

$$= \frac{1}{4}\sum_{m=1}^{N}\left[\sigma^i_m, \sigma^j_m\right] \tag{2.24}$$

$$= \frac{i\epsilon^{ijk}}{2}\sum_{m=1}^{N}\sigma^k_m \tag{2.25}$$

$$= i\epsilon^{ijk}S^k \tag{2.26}$$

So apart from normalisation, we see that the total spin operators satisfy the su(2) commutation relations, and therefore define an $2^N$-dimensional representation of SU(2).

[2]The $N$-site total spin operators are defined as the co-product of the 1-particle spin operators. This is discussed in Section 6 of this text.

## 3 The XXX Spin Chain

### 3.1 The Model

This model is a 1-dimensional chain of $N$ sites, with a spin particle at each site. We will only consider the case where this is a spin-$\frac{1}{2}$ particle, but higher spins can also be treated. The Hamiltonian for this model can be written in many ways, we will define it as

$$\mathcal{H} = J \sum_{l=1}^{N} \left( \sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \sigma_l^z \otimes \sigma_{l+1}^z \right), \tag{3.1}$$

where we define

$$\sigma_l^i = \overbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}^{j-1 \text{ times}} \otimes \sigma^i \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \tag{3.2}$$

and where $\sigma^i$ is the $i$'th Pauli spin matrix, and $\mathbb{I}$ is the $2 \times 2$ identity matrix. We will be treating the case with periodic boundary conditions, where $\sigma_{N+1}^i = \sigma_1^i$. Since the Pauli matrices act on the spin half complex Hilbert space, $\mathbb{C}^2$, the Hamiltonian acts on the $2^N$-dimensional Hilbert space, comprised of the direct product of N spin-$\frac{1}{2}$ spaces, which we will denote as

$$\mathbb{C}^2{}^{\otimes N} = \overbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{N \text{ times}}. \tag{3.3}$$

We can rewrite this Hamiltonian in terms of raising and lowering operators.

The raising and lowering operators ($\sigma^+$ and $\sigma^-$ respectively) are defined as

$$\sigma^+ = \frac{1}{2} \left( \sigma^x + i\sigma^y \right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.4}$$

$$\sigma^- = \frac{1}{2} \left( \sigma^x - i\sigma^y \right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.5}$$

We see that

$$\sigma_l^+ \otimes \sigma_{l+1}^- = \frac{1}{4} \left( \sigma_l^x + i\sigma_l^y \right) \otimes \left( \sigma_{l+1}^x - i\sigma_{l+1}^y \right) \tag{3.6}$$

$$= \frac{1}{4} \left( \sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + i\sigma_l^y \otimes \sigma_{l+1}^x - i\sigma_l^x \otimes \sigma_{l+1}^y \right) \tag{3.7}$$

and

$$\sigma_l^- \otimes \sigma_{l+1}^+ = \frac{1}{4} \left( \sigma_l^x - i\sigma_l^y \right) \otimes \left( \sigma_{l+1}^x + i\sigma_{l+1}^y \right) \tag{3.8}$$

$$= \frac{1}{4} \left( \sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y - i\sigma_l^y \otimes \sigma_{l+1}^x + i\sigma_l^x \otimes \sigma_{l+1}^y \right). \tag{3.9}$$

Thus we see that

$$\sigma_l^+ \otimes \sigma_{l+1}^- + \sigma_l^- \otimes \sigma_{l+1}^+ = \frac{1}{2}\left(\sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y\right), \tag{3.10}$$

and therefore we can write $\mathcal{H}$ as

$$\mathcal{H} = J \sum_{l=1}^{N} \left(2\left(\sigma_l^+ \otimes \sigma_{l+1}^- + \sigma_l^- \otimes \sigma_{l+1}^+\right) + \sigma_l^z \otimes \sigma_{l+1}^z\right). \tag{3.11}$$

We can also rewrite the Hamiltonian in terms of the two site permutation matrix, $\mathcal{P}$, defined as

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.12}$$

This matrix acts on a two site Hilbert space as

$$\mathcal{P}(x \otimes y) = \mathcal{P}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \tag{3.13}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{pmatrix} \tag{3.14}$$

$$= \begin{pmatrix} x_1 y_1 \\ x_2 y_1 \\ x_1 y_2 \\ x_2 y_2 \end{pmatrix} \tag{3.15}$$

$$= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = y \otimes x. \tag{3.16}$$

It is easy to confirm that

$$\sigma^+ \otimes \sigma^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma^- \otimes \sigma^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \sigma^z \otimes \sigma^z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.17}$$

thus we see that

$$2\left(\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+\right) + \sigma^z \otimes \sigma^z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.18}$$

$$= 2\mathcal{P} - \mathbb{I}_{4\times 4} \tag{3.19}$$

and therefore if we denote by $\mathcal{P}_{l,l+1}$ the operator that acts as $\mathcal{P}$ on the $l$ and $l+1$'th Hilbert spaces, and trivially on the rest, we can write the Hamiltonian as

$$\mathcal{H} = 2J \sum_{l=1}^{N} \mathcal{P}_{l,l+1} - NJ\mathbb{I}_{N\times N}. \tag{3.20}$$

### 3.2 Symmetry

We will first show that the XXX Hamiltonian is translationally invariant. To do this, we define the shift operator, T, which shifts the states on the chain one lattice position to the right. (The state at the $N$'th position gets shifted to the first position on the chain). Since the Hamiltonian only consists of nearest neighbour interactions, we get the same result if we act first with $\mathcal{H}$ and then T, or the other way around. Thus we can conclude that

$$[\mathcal{H}, \mathrm{T}] = 0. \tag{3.21}$$

We now show the XXX Hamiltonian, with periodic boundary conditions, has an su(2) symmetry algebra, by showing that it commutes with the total spin operators defined in (2.20). We now first check that $S^z$ commutes with the Hamiltonian.

$$[\mathcal{H}, S^z] = \left[ J \sum_{l=1}^{N} \left( \sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \sigma_l^z \otimes \sigma_{l+1}^z \right), \sum_{m=1}^{N} \frac{1}{2} \sigma_m^z \right] \tag{3.22}$$

$$= \frac{J}{2} \sum_{l=1}^{N} \sum_{m=1}^{N} \left[ \sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \sigma_l^z \otimes \sigma_{l+1}^z, \sigma_m^z \right] \tag{3.23}$$

$$= \frac{J}{2} \sum_{l=1}^{N} \sum_{m=1}^{N} \left( \left[ \sigma_l^x \otimes \sigma_{l+1}^x, \sigma_m^z \right] + \left[ \sigma_l^y \otimes \sigma_{l+1}^y, \sigma_m^z \right] + \left[ \sigma_l^z \otimes \sigma_{l+1}^z, \sigma_m^z \right] \right) \tag{3.24}$$

All the terms where $k < j$, or where $k > j + 1$ commute trivially, so this reduces to

$$[\mathcal{H}, S^z] = \frac{J}{2}\sum_{l=1}^{N}\left(\left[\sigma_l^x \otimes \sigma_{l+1}^x, \sigma_l^z\right] + \left[\sigma_l^y \otimes \sigma_{l+1}^y, \sigma_l^z\right] + \left[\sigma_l^z \otimes \sigma_{l+1}^z, \sigma_l^z\right] + \right.$$
$$\left. \left[\sigma_l^x \otimes \sigma_{l+1}^x, \sigma_{l+1}^z\right] + \left[\sigma_l^y \otimes \sigma_{l+1}^y, \sigma_{l+1}^z\right] + \left[\sigma_l^z \otimes \sigma_{l+1}^z, \sigma_{l+1}^z\right]\right). \quad (3.25)$$

We first just consider the first term

$$\left[\sigma_l^x \otimes \sigma_{l+1}^x, \sigma_l^z\right] = \left(\cdots \otimes \mathbf{1} \otimes \sigma_l^x \sigma_l^z \otimes \sigma_{l+1}^x \otimes \mathbf{1} \otimes \cdots\right) - \left(\cdots \otimes \mathbf{1} \otimes \sigma_l^z \sigma_l^x \otimes \sigma_{l+1}^x \otimes \mathbf{1} \otimes \cdots\right) \quad (3.26)$$
$$= \left(\cdots \otimes \mathbf{1} \otimes [\sigma_l^x, \sigma_l^z] \otimes \sigma_{l+1}^y \otimes \mathbf{1} \otimes \cdots\right) \quad (3.27)$$
$$= [\sigma_l^x, \sigma_l^z] \otimes \sigma_{l+1}^y \quad (3.28)$$
$$= \left(-2i\sigma_l^y\right) \otimes \sigma_{l+1}^y. \quad (3.29)$$

We now see that

$$[\mathcal{H}, S^z] = \frac{J}{2}\sum_{l=1}^{N}\left(\left[\sigma_l^x \otimes \sigma_{l+1}^x, \sigma_l^z\right] + \left[\sigma_l^y \otimes \sigma_{l+1}^y, \sigma_l^z\right] + \left[\sigma_l^z \otimes \sigma_{l+1}^z, \sigma_l^z\right] + \right.$$
$$\left. \left[\sigma_l^x \otimes \sigma_{l+1}^x, \sigma_{l+1}^z\right] + \left[\sigma_l^y \otimes \sigma_{l+1}^y, \sigma_{l+1}^z\right] + \left[\sigma_l^z \otimes \sigma_{l+1}^z, \sigma_{l+1}^z\right]\right) \quad (3.30)$$
$$= \frac{J}{2}\sum_{l=1}^{N}\left([\sigma_l^x, \sigma_l^z] \otimes \sigma_{l+1}^x + \left[\sigma_l^y, \sigma_l^z\right] \otimes \sigma_{l+1}^y + [\sigma_l^z, \sigma_l^z] \otimes \sigma_{l+1}^z + \right.$$
$$\left. \sigma_l^x \otimes \left[\sigma_{l+1}^x, \sigma_{l+1}^z\right] + \sigma_l^y \otimes \left[\sigma_{l+1}^y, \sigma_{l+1}^z\right] + \sigma_l^z \otimes \left[\sigma_{l+1}^z, \sigma_{l+1}^z\right]\right) \quad (3.31)$$
$$= iJ\sum_{l=1}^{N}\left(-\sigma_l^y \otimes \sigma_{l+1}^x + \sigma_l^x \otimes \sigma_l^y + 0 - \sigma_l^x \otimes \sigma_{l+1}^y + \sigma_l^y \otimes \sigma_{l+1}^x + 0\right) \quad (3.32)$$
$$= 0. \quad (3.33)$$

Recall that for the case when $j = N$, $\sigma_{N+1}^i = \sigma_1^i$, and the expression above is still valid. Since the Hamiltonian commutes with the total z spin, this is a conserved quantity, and we can divide the Hilbert space of possible states into different sectors, each labelled by the z component of the spin, as discussed in [8] on p. 10.

A similar result applies for $S^x$ and $S^y$. We reproduce the key steps below:

$$[\mathcal{H}, S^x] = \frac{J}{2}\sum_{l=1}^{N}\sum_{m=1}^{N}\left[\sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \sigma_l^z \otimes \sigma_{l+1}^z, \sigma_m^x\right] \tag{3.34}$$

$$= \frac{J}{2}\sum_{l=1}^{N}\Big([\sigma_l^x, \sigma_l^x] \otimes \sigma_{l+1}^x + \left[\sigma_l^y, \sigma_l^x\right] \otimes \sigma_{l+1}^y + [\sigma_l^z, \sigma_l^x] \otimes \sigma_{l+1}^z +$$

$$\sigma_l^x \otimes \left[\sigma_{l+1}^x, \sigma_{l+1}^x\right] + \sigma_l^y \otimes \left[\sigma_{l+1}^y, \sigma_{l+1}^x\right] + \sigma_l^z \otimes \left[\sigma_{l+1}^z, \sigma_{l+1}^x\right]\Big) \tag{3.35}$$

$$= iJ\sum_{l=1}^{N}\left(0 + \sigma_l^y \otimes \sigma_{l+1}^z - \sigma_l^y \otimes \sigma_{l+1}^z + 0 + \sigma_l^z \otimes \sigma_{l+1}^y - \sigma_l^z \otimes \sigma_{l+1}^y\right) \tag{3.36}$$

$$= 0 \tag{3.37}$$

and also

$$[\mathcal{H}, S^y] = \frac{J}{2}\sum_{l=1}^{N}\sum_{m=1}^{N}\left[\sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \sigma_l^z \otimes \sigma_{l+1}^z, \sigma_m^y\right] \tag{3.38}$$

$$= \frac{J}{2}\sum_{l=1}^{N}\Big(\left[\sigma_l^x, \sigma_l^y\right] \otimes \sigma_{l+1}^x + \left[\sigma_l^y, \sigma_l^y\right] \otimes \sigma_{l+1}^y + \left[\sigma_l^z, \sigma_l^y\right] \otimes \sigma_{l+1}^z +$$

$$\sigma_l^x \otimes \left[\sigma_{l+1}^x, \sigma_{l+1}^y\right] + \sigma_l^y \otimes \left[\sigma_{l+1}^y, \sigma_{l+1}^y\right] + \sigma_l^z \otimes \left[\sigma_{l+1}^z, \sigma_{l+1}^y\right]\Big) \tag{3.39}$$

$$= iJ\sum_{l=1}^{N}\left(\sigma_l^z \otimes \sigma_{l+1}^x + 0 - \sigma_l^x \otimes \sigma_l^z + \sigma_l^x \otimes \sigma_{l+1}^z + 0 - \sigma_l^z \otimes \sigma_{l+1}^x\right) \tag{3.40}$$

$$= 0. \tag{3.41}$$

Thus we can conclude the Hamiltonian has SU(2) symmetry, and we can expect the eigenvalues to be arranged in SU(2) multiplets[3].

## 4 The XXZ Spin Chain

We can break the su(2) symmetry of the XXX spin chain, and define a Hamiltonian $\mathcal{H}_\Delta$ as

$$\mathcal{H}_\Delta = J\sum_{l=1}^{N}\left(\sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \Delta\sigma_l^z \otimes \sigma_{l+1}^z\right) \tag{4.1}$$

where $\Delta$ is called the anisotropy parameter. This Hamiltonian acts on the same spin chain as the XXX spin chain, is translationally invariant and still commutes with the total z-projection

[3]For example, the $N = 2$ chain, with 4 states, contains a singlet with one energy, and a triplet of states with a different energy.

of the spin (as can be seen by looking at the proof for the undeformed case (3.23)), but no longer commutes with $S^x$ or $S^y$. This Hamiltonian describes an integrable model, that can be solved using the Bethe ansatz. In the next section we will look at how this eigenvalue problem can be solved.

# 5 The Coordinate Bethe Ansatz

We will now show how to solve for the energy eigenvalues of the XXZ spin chain Hamiltonian[4], using the technique developed by Hans Bethe in 1931. This approach leads to a good physical understanding of the spin chain.

We will start by dividing the Hilbert space into different sectors, each with a different total number of spins down, which we will call $M$. We can do this since the Hamiltonian commutes with $S^z$. For simplicity, we will only solve for the low lying states on a reasonably long chain. Thus, for all the cases we consider, we will assume that $M << \frac{N}{2}$.

## 5.1 The reference state

The $M = 0$ sector, with all the spins pointing up, contains only one state, which we will choose to be our reference state[5], and denote by $|0\rangle$. We can create other states by acting on this state with the lowering operator, $S_l^- = \sigma_l^-$, which lowers a spin at the $l$'th position.

We observe that $\sigma_l^+$, $\sigma_l^-$ and $\sigma_l^z$ act on a general state with an up or down spin at the $l$'th position as indicated in Table 1:

**Table 1**. Action of spin operators on a general spin chain state

| | $\vert\ldots\uparrow\ldots\rangle$ | $\vert\ldots\downarrow\ldots\rangle$ |
|---|---|---|
| $\sigma_l^+$ | 0 | $\vert\ldots\uparrow\ldots\rangle$ |
| $\sigma_l^-$ | $\vert\ldots\downarrow\ldots\rangle$ | 0 |
| $\sigma_l^z$ | $\vert\ldots\uparrow\ldots\rangle$ | $-\vert\ldots\downarrow\ldots\rangle$ |

We see that $|0\rangle$ is an eigenfunction of $\mathcal{H}_\Delta$, with eigenvalue $E_0 = JN\Delta$:

[4]This is also applicable to the XXX spin chain, just set $\Delta = 1$.

[5]We could equally well have chosen the state with all spins down as our reference state.

$$\mathcal{H}_\Delta |0\rangle = J \sum_{l=1}^{N} \left(2\left(\sigma_l^+ \otimes \sigma_{l+1}^- + \sigma_l^- \otimes \sigma_{l+1}^+\right) + \Delta\sigma_l^z \otimes \sigma_{l+1}^z\right) |\uparrow\uparrow \dots \uparrow\rangle \tag{5.1}$$

$$= J\Delta \sum_{l=1}^{N} \left(\sigma_l^z \otimes \sigma_{l+1}^z\right) |\uparrow\uparrow \dots \uparrow\rangle \tag{5.2}$$

$$= JN\Delta |\uparrow\uparrow \dots \uparrow\rangle . \tag{5.3}$$

This state is the ground state of the Hamiltonian in the ferromagnetic case, where the spins tend to align. In this case, the excitations of the reference state, called *spin waves* are particle-like. For a Hamiltonian describing an anti-ferromagnetic system, the ground state is a much more complicated structure, and the spin waves we will consider, although they are still solutions, are not excitations of the ground state.

### 5.2 The single spin wave states

For the $M = 1$ sector, we can choose as a basis all the states where one spin is down at a position $n$, for $n$ from 1 to $N$. We can create these states using the lowering operator,

$$|n\rangle = \sigma_n^- |0\rangle . \tag{5.4}$$

These states are not eigenvalues of the Hamiltonian, since the Hamiltonian is translationally invariant, but these states do not respect that symmetry. We want some linear combination of these states that has this symmetry. A wavelike state will have this translational symmetry, so we define our states as

$$|\psi_k\rangle = \sum_{n=1}^{N} f(n) |n\rangle = \sum_{n=1}^{N} e^{ikn} |n\rangle , \tag{5.5}$$

and impose the boundary condition that $f(n + N) = f(n)$. This implies that $k = \frac{2\pi m}{N}$ with $m \in \{0, 1, 2, \dots, N - 1\}$. Thus there are N such states, matching the dimension of the sector. These states are eigenstates of the shift operator

$$\mathrm{T}\,|\psi_k\rangle = \sum_{n=1}^{N} e^{ikn}\,\mathrm{T}\,|n\rangle \tag{5.6}$$

$$= \sum_{n=1}^{N-1} e^{ikn}\,|n+1\rangle + e^{ikN}\,|1\rangle \tag{5.7}$$

$$= \sum_{n=1}^{N-1} e^{ikn}\,|n+1\rangle + e^{ik(1+N-1)}\,|1\rangle \tag{5.8}$$

$$= \sum_{n=1}^{N} e^{ik(n-1)}\,|n\rangle \tag{5.9}$$

$$= e^{-ik}\sum_{n=1}^{N} e^{ikn}\,|n\rangle = e^{-ik}\,|\psi_k\rangle \tag{5.10}$$

and of the Hamiltonian. To see this we first act with $\mathcal{H}_\Delta$ on $|n\rangle$ (we assume the down-spin is away from the boundary, the case with the spin at positions 1 or $N$ is similar).

$$\mathcal{H}_\Delta\,|n\rangle = J\sum_{l=1}^{N}\left(2\left(\sigma_l^+\otimes\sigma_{l+1}^- + \sigma_l^-\otimes\sigma_{l+1}^+\right) + \Delta\sigma_l^z\otimes\sigma_{l+1}^z\right)|\uparrow\ldots\uparrow\downarrow\uparrow\ldots\uparrow\rangle \tag{5.11}$$

$$= 2J\left(|\uparrow\ldots\uparrow\uparrow\downarrow\ldots\uparrow\rangle + |\uparrow\ldots\downarrow\uparrow\uparrow\ldots\uparrow\rangle\right) + J\Delta(N-4)\,|\uparrow\ldots\uparrow\downarrow\uparrow\ldots\uparrow\rangle \tag{5.12}$$

$$= 2J\left(|n+1\rangle + |n-1\rangle\right) + J\Delta(N-4)\,|n\rangle \tag{5.13}$$

The $\sigma_l^+\otimes\sigma_{l+1}^-$ term gives zero, unless it acts on a down-spin followed by an up-spin, so only acts non-trivially when $l = n$, where it moves the down-spin to the right. Similarly, the $\sigma_l^-\otimes\sigma_{l+1}^+$ only acts non-trivially when $l = n-1$, and moves the down-spin to the right. The $\sigma_l^z\otimes\sigma_{l+1}^z$ term gives a contribution of $|n\rangle$ if the neighbouring spins are aligned, and $-|n\rangle$ if they are not. Thus, since we have two anti-aligned pairs, we have

$$\sum_{l=1}^{N}\left(\sigma_l^z\otimes\sigma_{l+1}^z\right)|n\rangle = (N-2)\,|n\rangle + 2\left(-|n\rangle\right) = (N-4)\,|n\rangle\,. \tag{5.14}$$

We now can see that (remembering that the spin chains wrap around so that $|N+1\rangle = |1\rangle$ and $|1-1\rangle = |N\rangle$ )

$$
\begin{aligned}
\mathcal{H}_\Delta |\psi_k\rangle &= \sum_{n=1}^{N} e^{ikn} \mathcal{H}_\Delta |n\rangle && (5.15)\\
&= J \sum_{n=1}^{N} e^{ikn} \left(2\left(|n+1\rangle + |n-1\rangle\right) + \Delta(N-4)|n\rangle\right) && (5.16)\\
&= 2J \left(\sum_{n=1}^{N} e^{ikn} |n+1\rangle + \sum_{n=1}^{N} e^{ikn} |n-1\rangle\right) + J\Delta(N-4)|\psi_k\rangle && (5.17)\\
&= 2J \left(e^{-ik} \sum_{n=1}^{N} e^{ikn} |n\rangle + e^{ik} \sum_{n=1}^{N} e^{ikn} |n\rangle\right) + J\Delta(N-4)|\psi_k\rangle && (5.18)\\
&= 2J \left(e^{ik} + e^{-ik}\right) |\psi_k\rangle + J\Delta(N-4)|\psi_k\rangle && (5.19)\\
&= J\left(4\cos(k) + \Delta(N-4)\right)|\psi_k\rangle . && (5.20)
\end{aligned}
$$

Thus we can conclude that $|\psi_k\rangle$ is an eigenstate of $\mathcal{H}$, with eigenvalue

$$
\begin{aligned}
E_k &= J\left(4\cos(k) + \Delta(N-4)\right) && (5.21)\\
&= JN\Delta + 4J(\cos(k) - \Delta). && (5.22)
\end{aligned}
$$

This state contains a single spin wave, with wave number $k$.

### 5.3 States with two spin waves

In the $M = 2$ sector we can choose a basis

$$
|n_1, n_2\rangle = \sigma^-_{n_1}\sigma^-_{n_2}|0\rangle , \tag{5.23}
$$

with the condition that $n_1 < n_2$, to avoid over counting states, and to take into account that lowering a down-spin annihilates the state. The eigenvectors of $\mathcal{H}_\Delta$ will be a linear combination of these basis vectors:

$$
|\psi_2\rangle = \sum_{1\le n_1<n_2\le N} f(n_1,n_2)\,|n_1,n_2\rangle . \tag{5.24}
$$

Bethe made the ansatz that the form of $f(n_1, n_2)$ is

$$
f(n_1,n_2) = A_{12} e^{ik_1n_1 + ik_2n_2} + A_{21} e^{ik_2n_1 + ik_1n_2}, \tag{5.25}
$$

which satisfies the periodicity condition $f(n_2, n_1 + N) = f(n_1, n_2)$, if the following equations hold

$$e^{ik_1 N} = \frac{A_{12}}{A_{21}} \text{ and } e^{ik_2 N} = \frac{A_{21}}{A_{12}}. \tag{5.26}$$

We define the *scattering amplitudes for spin waves* as

$$\hat{S}_{12} = \frac{A_{21}}{A_{12}}, \qquad \hat{S}_{21} = \frac{A_{12}}{A_{21}}, \tag{5.27}$$

in terms of which equations (5.26) are

$$e^{ik_1 N}\hat{S}_{12}(k_1, k_2) = 1, \qquad e^{ik_2 N}\hat{S}_{21}(k_1, k_2) = 1. \tag{5.28}$$

These equations imply that if a spin wave travels all the way around the chain, it experiences a total phase shift of 1, made up of two contributions, one kinematic part ($e^{ik1}N$ or $e^{ik_2 N}$), and a phase shift produced in the interaction with the other spin wave. We can rewrite the Bethe ansatz in terms of the scattering amplitude as

$$f(n_1, n_2) = A_{12}\left(e^{ik_1 n_1 + ik_2 n_2} + \hat{S}_{12} e^{ik_2 n_1 + ik_1 n_2}\right). \tag{5.29}$$

We solve for the scattering amplitude $\hat{S}_{12}$, by considering the eigenvalue equation

$$E_{k_1,k_2} |\psi_2\rangle = \mathcal{H}_\Delta |\psi_2\rangle \tag{5.30}$$

$$E_{k_1,k_2} \sum_{1 \le n_1 < n_2 \le N} f(n_1, n_2) |n_1, n_2\rangle = \mathcal{H}_\Delta \sum_{1 \le n_1 < n_2 \le N} f(n_1, n_2) |n_1, n_2\rangle. \tag{5.31}$$

Since we assume our states are orthonormal we have, for any $m_1 < m_2$

$$E_{k_1,k_2} f(m_1, m_2) = \sum_{1 \le n_1 < n_2 \le N} \langle m_1, m_2| \mathcal{H}_\Delta |n_1, n_2\rangle f(n_1, n_2). \tag{5.32}$$

To analyse this equation, we first consider the action of $\mathcal{H}_\Delta$ on a general basis state, $|n_1, n_2\rangle$. Consider the case where the two down spins are not adjacent.[6] Thus we have $n_1 + 1 < n_2$. Then the operators in $\mathcal{H}_\Delta$ act as in (5.11) and we have

[6]This is applicable even when the down-spins are separated by only one up-spin.

$$\mathcal{H}_\Delta |n_1, n_2\rangle = J\sum_{l=1}^{N} \left(2\left(\sigma_l^+ \otimes \sigma_{l+1}^- + \sigma_l^- \otimes \sigma_{l+1}^+\right) + \Delta\sigma_l^z \otimes \sigma_{l+1}^z\right)|n_1, n_2\rangle \tag{5.33}$$

$$= 2J\left(|n_1+1, n_2\rangle + |n_1, n_2+1\rangle + |n_1-1, n_2\rangle + |n_1, n_2-1\rangle\right) + J\Delta(N-8)\,|n_1, n_2\rangle\,. \tag{5.34}$$

If the spins are adjacent $n_1 + 1 = n_2$, we see that

$$\mathcal{H}_\Delta |n_1, n_2\rangle = J\sum_{l=1}^{N} \left(2\left(\sigma_l^+ \otimes \sigma_{l+1}^- + \sigma_l^- \otimes \sigma_{l+1}^+\right) + \Delta\sigma_l^z \otimes \sigma_{l+1}^z\right)|n_1, n_2\rangle \tag{5.35}$$

$$= 2J\left(|n_1, n_2+1\rangle + |n_1-1, n_2\rangle\right) + J\Delta(N-4)\,|n_1, n_2\rangle\,. \tag{5.36}$$

Using (5.34) and (5.36), we now see that, if $m_1 + 1 < m_2$

$$\sum_{1\le n_1<n_2\le N} \langle m_1, m_2|\,\mathcal{H}\,|n_1, n_2\rangle\, f(n_1, n_2) = 2J\left(f(m_1+1, m_2) + f(m_1, m_2+1) + f(m_1-1, m_2) + f(m_1, m_2-1)\right) + J\Delta(N-8)f(m_1, m_2) \tag{5.37}$$

and if $m_1 + 1 = m_2$

$$\sum_{1\le n_1<n_2\le N} \langle m_1, m_2|\,\mathcal{H}\,|n_1, n_2\rangle\, f(n_1, n_2) = 2J\left(f(m_1, m_2+1) + f(m_1-1, m_2))\right) + J\Delta(N-4)f(m_1, m_2). \tag{5.38}$$

To clean up notation a little, we will now define

$$\alpha = k_1 n_1 + k_2 n_2 \text{ and } \beta = k_2 n_1 + k_1 n_2. \tag{5.39}$$

From (5.32), we now see that, for the case of non-adjacent spins

$$
\begin{aligned}
E_{k_1,k_2} f(n_1,n_2) &= 2J\left(f(n_1+1,n_2) + f(n_1,n_2+1) + f(n_1-1,n_2) + f(n_1,n_2-1)\right) \\
&\qquad + J\Delta(N-8) f(n_1,n_2) && (5.40) \\
&= 2JA_{12}\left(\left(e^{i\alpha}e^{ik_1} + \hat{S}_{12}e^{i\beta}e^{ik_2}\right) + \left(e^{i\alpha}e^{ik_2} + \hat{S}_{12}e^{i\beta}e^{ik_1}\right) + \left(e^{i\alpha}e^{-ik_1} + \hat{S}_{12}e^{i\beta}e^{-ik_2}\right)\right. \\
&\qquad \left. + \left(e^{i\alpha}e^{-ik_2} + \hat{S}_{12}e^{i\beta}e^{-ik_1}\right)\right) + J\Delta(N-8) f(n_1,n_2) && (5.41) \\
&= J\left[2\left(e^{ik_1} + e^{-ik_1} + e^{ik_2} + e^{-ik_2}\right) + \Delta(N-8)\right] f(n_1,n_2) && (5.42) \\
&= \left[JN\Delta + 4J(\cos(k_1) - \Delta) + 4J(\cos(k_2) - \Delta)\right] f(n_1,n_2). && (5.43)
\end{aligned}
$$

Thus we can conclude that $E_{k_1,k_2} = JN\Delta + 4J(\cos(k_1) - \Delta) + 4J(\cos(k_2) - \Delta)$. For the case with adjacent spins, we see that

$$
E_{k_1,k_2} f(n_1,n_1+1) = 2J\left(f(n_1,n_1+2) + f(n_1-1,n_1+1)\right) + J\Delta(N-4) f(n_1,n_1+1). \quad (5.44)
$$

This should have the same eigenvalue of $\mathcal{H}_\Delta$, since the total z-spin commutes with $\mathcal{H}_\Delta$. If we equate (5.40) (with $n_1 + 1 = n_2$) and (5.44) we find

$$
\begin{aligned}
& 0 = f(n_1,n_1) + f(n_1+1,n_1+1) - 2\Delta f(n_1,n_1+1) && (5.45) \\
\Rightarrow \quad & 0 = e^{i(k_1n_1+k_2n_1)} + \hat{S}_{12}e^{i(k_2n_1+k_1n_1)} + e^{i(k_1n_1+k_2n_1)}e^{i(k_1+k_2)} + \\
& \qquad 2\hat{S}_{12}e^{i(k_2n_1+k_1n_1)}e^{i(k_1+k_2)} - \Delta e^{i(k_1n_1+k_2n_1)}e^{ik_2} - 2\Delta\hat{S}_{12}e^{i(k_2n_1+k_1n_1)e^{ik_1}} && (5.46) \\
\Rightarrow \quad & 0 = 1 + \hat{S}_{12} + e^{i(k_1+k_2)} + \hat{S}_{12}e^{i(k_1+k_2)} - 2\Delta e^{ik_2} - 2\Delta\hat{S}_{12}e^{ik_1}. && (5.47)
\end{aligned}
$$

Thus we see that

$$
\begin{aligned}
\hat{S}_{12}\left(1 - 2\Delta e^{ik_1} + e^{i(k_1+k_2)}\right) &= \left(1 - 2\Delta e^{ik_2} + e^{i(k_1+k_2)}\right) && (5.48) \\
\Rightarrow \qquad \hat{S}_{12} &= \frac{1 - 2\Delta e^{ik_1} + e^{i(k_1+k_2)}}{1 - 2\Delta e^{ik_2} + e^{i(k_1+k_2)}}. && (5.49)
\end{aligned}
$$

Thus, to solve for our eigenstates, we must find $k_1$ and $k_2$ that are simultaneous solutions to the equations:

$$
\begin{aligned}
e^{ik_1N} &= \frac{1 - 2\Delta e^{ik_2} + e^{i(k_1+k_2)}}{1 - 2\Delta e^{ik_1} + e^{i(k_1+k_2)}} && (5.50) \\
e^{ik_2N} &= \frac{1 - 2\Delta e^{ik_1} + e^{i(k_1+k_2)}}{1 - 2\Delta e^{ik_2} + e^{i(k_1+k_2)}}. && (5.51)
\end{aligned}
$$

### 5.4 States with three spin waves

In the $M = 3$ sector, the basis states in are

$$|n_1, n_2, n_3\rangle = \sigma_1^- \sigma_2^- \sigma_3^- |0\rangle \,, \tag{5.52}$$

with $n_1 < n_2 < n_3$. A general linear combination of all the basis vectors is

$$|\psi_3\rangle = \sum_{1 \leq n_1 < n_2 < n_3 \leq N} f(n_1, n_2, n_3) \, |n_1, n_2, n_3\rangle \,. \tag{5.53}$$

The Bethe ansatz is

$$\begin{aligned} f(n_1, n_2, n_3) = A_{123} e^{i(k_1 n_1 + k_2 n_2 + k_3 n_3)} + A_{132} e^{i(k_1 n_1 + k_3 n_2 + k_2 n_3)} + A_{213} e^{i(k_2 n_1 + k_1 n_2 + k_3 n_3)} + \\ A_{231} e^{i(k_2 n_1 + k_3 n_2 + k_1 n_3)} + A_{312} e^{i(k_3 n_1 + k_1 n_2 + k_2 n_3)} + A_{321} e^{i(k_3 n_1 + k_2 n_2 + k_1 n_3)} . \end{aligned} \tag{5.54}$$

As before, the periodicity condition is

$$f(n_2, n_3, n_1 + N) = f(n_1, n_2, n_3) \tag{5.55}$$

which, together with (5.54), implies that

$$e^{ik_1 N} = \frac{A_{123}}{A_{231}} = \frac{A_{132}}{A_{321}} \tag{5.56}$$

$$e^{ik_2 N} = \frac{A_{231}}{A_{312}} = \frac{A_{213}}{A_{132}} \tag{5.57}$$

$$e^{ik_3 N} = \frac{A_{312}}{A_{123}} = \frac{A_{321}}{A_{213}} \tag{5.58}$$

From these equations ((5.58), (5.57) and (5.56) respectively) we see that

$$\frac{A_{213}}{A_{123}} = \frac{A_{321}}{A_{312}}, \quad \frac{A_{312}}{A_{132}} = \frac{A_{231}}{A_{213}}, \quad \frac{A_{321}}{A_{231}} = \frac{A_{132}}{A_{123}}. \tag{5.59}$$

We can also see that

$$
\begin{aligned}
e^{ik_1N}\frac{A_{231}}{A_{123}} &= e^{ik_1N}\frac{A_{213}}{A_{123}}\frac{A_{231}}{A_{213}} = 1, && (5.60)\\
e^{ik_2N}\frac{A_{312}}{A_{231}} &= e^{ik_2N}\frac{A_{312}}{A_{321}}\frac{A_{321}}{A_{231}} = 1, && (5.61)\\
e^{ik_3N}\frac{A_{123}}{A_{312}} &= e^{ik_3N}\frac{A_{132}}{A_{312}}\frac{A_{123}}{A_{132}} = 1. && (5.62)
\end{aligned}
$$

If we define

$$
\hat{S}_{12} = \frac{A_{213}}{A_{123}} = \frac{A_{321}}{A_{312}}, \quad \hat{S}_{13} = \frac{A_{312}}{A_{132}} = \frac{A_{231}}{A_{213}}, \quad \hat{S}_{23} = \frac{A_{321}}{A_{231}} = \frac{A_{132}}{A_{123}}, \quad (\hat{S}_{ij})^{-1} = \hat{S}_{ji}, \quad (5.63)
$$

then, similar to equations (5.28), we see that

$$
\begin{aligned}
e^{ik_1N}\hat{S}_{12}\hat{S}_{13} &= 1, && (5.64)\\
e^{ik_2N}\hat{S}_{21}\hat{S}_{23} &= 1, && (5.65)\\
e^{ik_3N}\hat{S}_{31}\hat{S}_{32} &= 1. && (5.66)
\end{aligned}
$$

We also see that the scattering of any two particles is independent of the position of the third particle. We can rewrite the Bethe ansatz as

$$
\begin{aligned}
f(n_1,n_2,n_3) = A_{123}&\left(e^{i(k_1n_1+k_2n_2+k_3n_3)} + \hat{S}_{12}e^{i(k_2n_1+k_1n_2+k_3n_3)}\right) + \\
A_{132}&\left(e^{i(k_1n_1+k_3n_2+k_2n_3)} + \hat{S}_{13}e^{i(k_3n_1+k_1n_2+k_2n_3)}\right) + \\
A_{231}&\left(e^{i(k_2n_1+k_3n_2+k_1n_3)} + \hat{S}_{23}e^{i(k_3n_1+k_2n_2+k_1n_3)}\right). \qquad (5.67)
\end{aligned}
$$

Considering the equation

$$
E_{k_1,k_2}f(m_1,m_2,m_3) = \sum_{1\le n_1<n_2<n_3\le N} \langle m_1,m_2,m_3|\,\mathcal{H}_\Delta\,|n_1,n_2,n_3\rangle\, f(n_1,n_2,n_3) \qquad (5.68)
$$

and considering the case where $m_1, m_2$ and $m_3$ are well separated, and following a similar method to the $M = 2$ case, we can conclude that in this sector, the energy eigenvalues of the Hamiltonian are

$$E_{k_1,k_2,k_3} = JN\Delta + 4J(\cos(k_1) - \Delta) + 4J(\cos(k_2) - \Delta) + 4J(\cos(k_3) - \Delta). \tag{5.69}$$

If we consider a region where $m_2 = m_1 + 1$, and $m_3 > m_1 + 2$, we find the equation

$$f(m_1, m_1, m_3) + f(m_1 + 1, m_1 + 1, m_3) - 2\Delta f(m_1, m_1 + 1, m_3). \tag{5.70}$$

Which implies that the scattering matrices must satisfy

$$\hat{S}_{ij} = \frac{1 - 2\Delta e^{ik_i} + e^{i(k_i+k_j)}}{1 - 2\Delta e^{ik_j} + e^{i(k_i+k_j)}}. \tag{5.71}$$

### 5.5 A general state

If we now consider the general case when $M \geq 2$, our basis vectors are

$$|n_1, n_2, \ldots, n_M\rangle = \sigma_1^- \sigma_2^- \ldots \sigma_M^- |0\rangle, \tag{5.72}$$

with $n_1 < n_2 < \cdots < n_M$. A general linear combination of all the basis vectors is

$$|\psi_M\rangle = \sum_{1 \leq n_1 < n_2 < \cdots < n_M \leq N} f(n_1, \ldots, n_M) |n_1, \ldots, n_M\rangle. \tag{5.73}$$

The Bethe ansatz for this sector is

$$f(n_1, \ldots, n_M) = \sum_{p\ \in \mathscr{P}_M} A_p e^{i(k_{p(1)}n_1 + \cdots + k_{p(M)}n_M)}, \tag{5.74}$$

where $\mathscr{P}_M$ denotes the set of all $M!$ permutations of the labels $1, 2, \ldots, M$, and we sum over all these permutations. The periodicity condition becomes $f(n_2, \ldots, n_M, n_1 + N) = f(n_1, \ldots, n_M)$. The solutions to the eigenvalue problem are the Bethe ansatz states, with the spin wave momenta $k_i$ satisfying, for all $i = 1, \ldots, M$ the equations

$$e^{ik_iN} = \prod_{\substack{j=1 \\ j\neq i}}^{M} \hat{S}_{ji}(k_j, k_i) \tag{5.75}$$

with $\hat{S}_{ji}(k_j, k_i)$ given by (5.71). The energy eigenvalues are given by

$$E_{k_1,k_2,\ldots,k_M} = JN\Delta + \sum_{i=1}^{M} 4J(\cos(k_i) - \Delta). \tag{5.76}$$

Thus we have reduced the problem of finding the eigenvalues of a $2^N \times 2^N$ matrix to the problem of finding solutions of a much smaller set of equations. These equations are called the *Bethe ansatz equations*. We will not attempt to simplify or solve these equations any further. A good introduction is given by [9].

# 6 Hopf Algebras

## 6.1 Definition of a Hopf algebra

An algebra over a field is defined to be a vector space, together with a binary operation, generally called multiplication, $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ , which is bilinear (2.11). We denote the multiplication $m(a \otimes b)$ by $a \cdot b$. Assume $\mathcal{A}$ is an unital associative algebra over $\mathbb{C}$ with unit $\mathbf{1}$. We define the unit map

$$\imath : \mathbb{C} \to \mathcal{A} \tag{6.1}$$

$$\lambda \to \lambda \mathbf{1} \tag{6.2}$$

This allows us to formally multiply an algebra element $a$ with a complex scalar $\lambda$. We have

$$\imath(\lambda) \cdot a = \lambda a = a\lambda = a \cdot \imath(\lambda). \tag{6.3}$$

We now want to define a co-algebra. Suppose $\mathcal{C}$ is a vector space over $\mathbb{C}$, and suppose now there is an operation, $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, called the co-multiplication, or co-product, which, for every $c \in \mathcal{C}$ satisfies

$$(\Delta \otimes \mathbf{1})(\Delta(c)) = (\mathbf{1} \otimes \Delta)(\Delta(c)). \tag{6.4}$$

If this property holds we say $\Delta$ is *co-associative*. Also suppose there exists a map, $\epsilon : \mathcal{C} \to \mathbb{C}$, called the *co-unit*, such that, for every $a \in \mathcal{A}$

$$(\epsilon \otimes \mathbf{1})(\Delta(a)) = (\mathbf{1} \otimes \epsilon)(\Delta(a)) = a. \tag{6.5}$$

The structure $(\mathcal{C}, \Delta, \epsilon)$ is called a *co-algebra*. If $\mathcal{A}$ is simultaneously an algebra and a co-algebra, it is called a *bi-algebra* if the co-unit and co-multiplication are homomorphisms, i.e. we have, for $\forall\ a, b \in \mathcal{A}$

$$\epsilon(a \cdot b) = \epsilon(a)\epsilon(b), \tag{6.6}$$

$$\Delta(a \cdot b) = \Delta(a)\Delta(b). \tag{6.7}$$

A *Hopf algebra* is a bi-algebra that has an additional map $\gamma : \mathcal{A} \to \mathcal{A}$, called the antipode, which is an antihomomorphism

$$\gamma(a \cdot b) = \gamma(a) \cdot \gamma(b), \tag{6.8}$$

and which satisfies,

$$m\left((\mathbf{1} \otimes \gamma)(\Delta(a))\right) = m\left((\gamma \otimes \mathbf{1})(\Delta(a))\right) = \epsilon(a)\mathbf{1}. \tag{6.9}$$

The antipode is the analogue of the inversion map for groups, that sends $g \to g^{-1}$. Just like the multiplication need not be commutative, the co-product may also not be be co-commutative. The co-product is co-commutative if the transpose of the co-product of an element is equal to the co-product of that element. We will be considering non-commutative, non-co-commutative Hopf algebras. A Hopf Algebra is called *quasi-triangular* if there exists an universal $\mathscr{R}$ matrix, $\mathscr{R} \in \mathcal{A} \otimes \mathcal{A}$ satisfying specific properties. These properties imply that the $\mathscr{R}$ matrix satisfies the Yang-Baxter equation. Quasi-triangular Hopf algebras are also called *quantum groups*.

Quasi-triangular Hopf algebras can be constructed from two Hopf algebras, $\mathcal{A}$ and $\mathcal{A}^\star$ that are isomorphic , and dual to each other in the sense that the product of $\mathcal{A}$ coincides with the co-product of $\mathcal{A}^\star$, and the co-product of $\mathcal{A}$ coincides with the product of $\mathcal{A}^\star$. This is called *Drinfeld's quantum double construction*, and is done by imposing a normal ordering on basis vectors from $\mathcal{A}$ and $\mathcal{A}^\star$. Details on quasi-triangular Hopf algebras and the Drinfeld double can be found in chapter 6 of [8].

### 6.2 The quantum group $\mathbf{U_q(sl(2))}$

As defined by [10] in Section 3.2, the quantum group $U_q(sl(2))$ is given by all linear combinations and formal powers of the generators $S^+, S^-, q^{\pm S^z}$, where q is a complex number, and the generators satisfy the relations

$$q^{S^z} S^{\pm} q^{-S^z} = q^{\pm 1} S^{\pm}, \tag{6.10}$$

$$\left[S^+, S^-\right] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}. \tag{6.11}$$

These relations reduce to the normal SU(2) relations if $q = 1$. It can be given a Hopf algebra structure by defining the co-product, antipode[7] and co-unit as

$$\Delta(q^{\pm S^z}) = q^{\pm S^z} \otimes q^{\pm S^z}, \tag{6.12}$$

$$\Delta(S^{\pm}) = q^{S^z} \otimes S^{\pm} + S^{\pm} \otimes q^{-S^z}, \tag{6.13}$$

$$\gamma(q^{\pm S^z}) = q^{\mp S^z}, \tag{6.14}$$

$$\gamma(S^{\pm}) = -q^{\mp 1} S^{\pm}, \tag{6.15}$$

$$\epsilon(q^{\pm S^z}) = 1, \tag{6.16}$$

$$\epsilon(S^{\pm}) = 0. \tag{6.17}$$

Since we consider all powers of the elements, the multiplication is normal multiplication, and not the Lie bracket as in Lie algebras. If we use a matrix representation this is matrix multiplication. In Appendix A we show how the co-product, co-unit and antipode satisfy the properties of a Hopf algebra. We are interested in spin-$\frac{1}{2}$ representations of $U_q(sl(2))$. If we consider the representation module $\mathbb{C}^2$, it can easily be checked that, where $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ and the $\sigma^i$ are the usual Pauli matrices,

$$S^z = \frac{\sigma^z}{2}, \qquad S^{\pm} = \sigma^{\pm} \tag{6.18}$$

satisfy the relations (6.10) and (6.11). We can find representations on representation modules of the form $\mathbb{C}^2 \otimes^N$ by using the co-product. We start by defining $\Delta^{(2)}(a) = \Delta(a)$ , and then define $\Delta^{(n)}$ recursively as

$$\Delta^{(n)} = (\Delta \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})(\Delta^{(n-1)}(a)), \qquad \text{for } n \geq 3, \tag{6.19}$$

which we do since $\Delta$ is co-associative [8]. We see that

[7]We are using the transpose of the co-product that is usually defined. This means that the antipode we should use is the inverse antipode, $\gamma^{'}(a) = [\gamma(a)]^{-1}$. For details on this, see [8], p. 186

[8]Any Lie algebra can trivially be given a co-commutative Hopf algebra structure by defining the co-product $\Delta(x) = x \otimes 1 + 1 \otimes x$. This is useful in normal quantum mechanics, for example the total spin of a two-particle system is the spin of the first plus that of the second. In fact, this is how we define the $N$-site total spin operators with the co-product, as mentioned before. We see $S^i = \Delta^N(\frac{\sigma^i}{2}) = \frac{1}{2}\sum_{l=1}^{N} \sigma_l^i$.

$$\Delta^{(3)}(q^{\pm S^z}) = (\Delta \otimes \mathbf{1})(q^{\pm S^z} \otimes q^{\pm S^z}) \tag{6.20}$$
$$= \Delta(q^{\pm S^z}) \otimes q^{\pm S^z} \tag{6.21}$$
$$= q^{\pm S^z} \otimes q^{\pm S^z} \otimes q^{\pm S^z}. \tag{6.22}$$

We generalise this to $\Delta^{(n)}(q^{\pm S^z}) = q^{\pm S^z} \otimes q^{\pm S^z} \otimes \cdots \otimes q^{\pm S^z}$ where $q^{\pm S^z}$ occurs $n$ times. We also see that

$$\Delta^{(3)}(S^{\pm}) = (\Delta \otimes \mathbf{1})(q^{S^z} \otimes S^{\pm} + S^{\pm} \otimes q^{-S^z}) \tag{6.23}$$
$$= \Delta(q^{S^z}) \otimes S^{\pm} + \Delta(S^{\pm}) \otimes q^{-S^z} \tag{6.24}$$
$$= q^{S^z} \otimes q^{S^z} \otimes S^{\pm} + q^{S^z} \otimes S^{\pm} \otimes q^{-S^z} + S^{\pm} \otimes q^{-S^z} \otimes q^{-S^z}. \tag{6.25}$$

In general, we see that

$$\Delta^{(n)}(S^{\pm}) = \sum_{i=1}^{n} q^{S^z} \otimes \cdots \otimes q^{S^z} \otimes S_i^{\pm} \otimes q^{-S^z} \otimes \cdots \otimes q^{-S^z}, \tag{6.26}$$

where by $S_i^{\pm}$ we indicate that the matrix $S^{\pm}$ is occupying the $i$'th position in the tensor product. On a space $\mathbb{C}^2 \otimes^N$ a representation of $U_q(sl(2))$ is given by the generators $\Delta^{(N)}(S^+), \Delta^{(N)}(S^-), \Delta^{(N)}(q^{S^z}), \Delta^{(N)}(q^{-S^z})$ [11].

# 7 The quantum group symmetry of the XXZ Hamiltonian

We consider a spin-$\frac{1}{2}$ chain of length $N$, and the open spin chain Hamiltonian

$$\mathcal{H} = \sum_{l=1}^{N-1} \left(\sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \Delta \sigma_l^z \otimes \sigma_{l+1}^z\right) + \Gamma\left(\sigma_1^z - \sigma_N^z\right) \tag{7.1}$$

where

$$\Delta = \frac{q + q^{-1}}{2}, \qquad \Gamma = \frac{q - q^{-1}}{2}. \tag{7.2}$$

The last term is a boundary term. The specific choice made leaves the Bethe equations unchanged from those for the closed chain, and thus this chain has the same eigenvalues as the closed chain (This is discussed in chapter 3 of [8]). We see that

$$\sum_{l=1}^{N-1} (\sigma_l^z - \sigma_{l+1}^z) = (\sigma_1^z - \sigma_2^z) + (\sigma_2^z - \sigma_3^z) + \cdots + (\sigma_{N-1}^z - \sigma_N^z) = \sigma_1^z - \sigma_N^z \tag{7.3}$$

and thus we can write the Hamiltonian as

$$\mathcal{H} = \sum_{l=1}^{N-1} H_{l,l+1} \tag{7.4}$$

where

$$H_{l,l+1} = \sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \Delta \sigma_l^z \otimes \sigma_{l+1}^z + \Gamma \left(\sigma_l^z - \sigma_{l+1}^z\right). \tag{7.5}$$

We want to show that this Hamiltonian commutes with the generators of $U_q(sl(2))$, as defined above. We will use a similar method to that used by [12]. We first consider the case where $N = 2$. We see that

$$\mathcal{H} = H_{1,2} = \sigma_1^x \otimes \sigma_2^x + \sigma_1^y \otimes \sigma_2^y + \Delta \sigma_1^z \otimes \sigma_2^z + \Gamma \left(\sigma_1^z - \sigma_2^z\right) \tag{7.6}$$

$$= \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & 2\Gamma - \Delta & 2 & 0 \\ 0 & 2 & -2\Gamma - \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix}. \tag{7.7}$$

For notational simplicity, we define $A = 2\Gamma - \Delta = \frac{q - 3q^{-1}}{2}$ and $B = -2\Gamma - \Delta = \frac{-3q + q^{-1}}{2}$. The generators of $U_q(sl(2))$ are

$$\Delta(q^{S^z}) = q^{S^z} \otimes q^{S^z} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \tag{7.8}$$

$$\Delta(q^{-S^z}) = q^{-S^z} \otimes q^{-S^z} = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{7.9}$$

$$\Delta(S^+) = q^{S^z} \otimes S^+ + S^+ \otimes q^{-S^z} = \begin{pmatrix} 0 & q^{\frac{1}{2}} & q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} \\ 0 & 0 & 0 & q^{-\frac{1}{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.10}$$

$$\Delta(S^-) = q^{S^z} \otimes S^- + S^- \otimes q^{-S^z} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{\frac{1}{2}} & 0 & 0 & 0 \\ q^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & q^{-\frac{1}{2}} & 0 \end{pmatrix}. \tag{7.11}$$

We can evaluate the commutators of the Hamiltonian with these generators. We see that

$$\left[H_{1,2},\Delta(q^{\pm S^z})\right]=\begin{pmatrix}\Delta&0&0&0\\0&A&2&0\\0&2&B&0\\0&0&0&\Delta\end{pmatrix}\begin{pmatrix}q^{\pm}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&q^{\mp}\end{pmatrix}-\begin{pmatrix}q^{\pm}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&q^{\mp}\end{pmatrix}\begin{pmatrix}\Delta&0&0&0\\0&A&2&0\\0&2&B&0\\0&0&0&\Delta\end{pmatrix} \tag{7.12}$$

$$=\begin{pmatrix}q^{\pm}\Delta&0&0&0\\0&A&2&0\\0&2&B&0\\0&0&0&q^{\mp}\Delta\end{pmatrix}-\begin{pmatrix}q^{\pm}\Delta&0&0&0\\0&A&2&0\\0&2&B&0\\0&0&0&q^{\mp}\Delta\end{pmatrix} \tag{7.13}$$

$$=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=0. \tag{7.14}$$

and

$$\left[H_{1,2},\Delta(S^{+})\right]=\begin{pmatrix}\Delta&0&0&0\\0&A&2&0\\0&2&B&0\\0&0&0&\Delta\end{pmatrix}\begin{pmatrix}0&q^{\frac{1}{2}}&q^{-\frac{1}{2}}&0\\0&0&0&q^{\frac{1}{2}}\\0&0&0&q^{-\frac{1}{2}}\\0&0&0&0\end{pmatrix}-\begin{pmatrix}0&q^{\frac{1}{2}}&q^{-\frac{1}{2}}&0\\0&0&0&q^{\frac{1}{2}}\\0&0&0&q^{-\frac{1}{2}}\\0&0&0&0\end{pmatrix}\begin{pmatrix}\Delta&0&0&0\\0&A&2&0\\0&2&B&0\\0&0&0&\Delta\end{pmatrix} \tag{7.15}$$

$$=\begin{pmatrix}0&q^{\frac{1}{2}}\Delta&q^{-\frac{1}{2}}\Delta&0\\0&0&0&q^{\frac{1}{2}}A+2q^{-\frac{1}{2}}\\0&0&0&q^{-\frac{1}{2}}B+2q^{\frac{1}{2}}\\0&0&0&0\end{pmatrix}-\begin{pmatrix}0&q^{\frac{1}{2}}A+2q^{-\frac{1}{2}}&q^{-\frac{1}{2}}B+2q^{\frac{1}{2}}&0\\0&0&0&q^{\frac{1}{2}}\Delta\\0&0&0&q^{-\frac{1}{2}}\Delta\\0&0&0&0\end{pmatrix} \tag{7.16}$$

$$=\begin{pmatrix}0&\left(q^{\frac{1}{2}}\Delta-q^{\frac{1}{2}}A-2q^{-\frac{1}{2}}\right)&\left(q^{-\frac{1}{2}}\Delta-q^{-\frac{1}{2}}B-2q^{\frac{1}{2}}\right)&0\\0&0&0&\left(q^{\frac{1}{2}}A+2q^{-\frac{1}{2}}-q^{\frac{1}{2}}\Delta\right)\\0&0&0&\left(q^{-\frac{1}{2}}B+2q^{\frac{1}{2}}-q^{-\frac{1}{2}}\Delta\right)\\0&0&0&0\end{pmatrix}. \tag{7.17}$$

All the entries of this matrix are zero due to the cancellations

$$q^{\frac{1}{2}}\Delta - q^{\frac{1}{2}}A - 2q^{-\frac{1}{2}} = q^{\frac{1}{2}}\left(\Delta - A - 2q^{-1}\right) \tag{7.18}$$

$$= \frac{q^{\frac{1}{2}}}{2}\left((q+q^{-1}) - (q-3q^{-1}) - 4q^{-1}\right) = 0 \tag{7.19}$$

and

$$q^{-\frac{1}{2}}\Delta - q^{-\frac{1}{2}}B - 2q^{\frac{1}{2}} = q^{-\frac{1}{2}}\left(\Delta - B - 2q\right) \tag{7.20}$$

$$= \frac{q^{-\frac{1}{2}}}{2}\left((q+q^{-1}) - (-3q+q^{-1}) - 4q\right) = 0. \tag{7.21}$$

Thus we conclude

$$\left[H_{1,2}, \Delta(S^+)\right] = 0. \tag{7.22}$$

Similarly, we see that

$$\left[H_{1,2}, \Delta(S^-)\right] = \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & A & 2 & 0 \\ 0 & 2 & B & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{\frac{1}{2}} & 0 & 0 & 0 \\ q^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & q^{-\frac{1}{2}} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{\frac{1}{2}} & 0 & 0 & 0 \\ q^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & q^{-\frac{1}{2}} & 0 \end{pmatrix}\begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & A & 2 & 0 \\ 0 & 2 & B & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix} \tag{7.23}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{\frac{1}{2}}A + 2q^{-\frac{1}{2}} & 0 & 0 & 0 \\ q^{-\frac{1}{2}}B + 2q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}}\Delta & q^{-\frac{1}{2}}\Delta & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{\frac{1}{2}}\Delta & 0 & 0 & 0 \\ q^{-\frac{1}{2}}\Delta & 0 & 0 & \\ 0 & q^{\frac{1}{2}}A + 2q^{-\frac{1}{2}} & q^{-\frac{1}{2}}B + 2q^{\frac{1}{2}} & 0 \end{pmatrix} \tag{7.24}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \left(q^{\frac{1}{2}}A + 2q^{-\frac{1}{2}} - q^{\frac{1}{2}}\Delta\right) & 0 & 0 & 0 \\ \left(q^{-\frac{1}{2}}B + 2q^{\frac{1}{2}} - q^{-\frac{1}{2}}\Delta\right) & 0 & 0 & 0 \\ 0 & \left(q^{\frac{1}{2}}\Delta - q^{\frac{1}{2}}A - 2q^{-\frac{1}{2}}\right) & \left(q^{-\frac{1}{2}}\Delta - q^{-\frac{1}{2}}B - 2q^{\frac{1}{2}}\right) & 0 \end{pmatrix} \tag{7.25}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0. \tag{7.26}$$

We can conclude that a chain of length 2, has $U_q(sl(2))$ symmetry. We can now consider the general case, for $N \geq 3$. Firstly

$$\begin{aligned}\left[\mathcal{H}, \Delta^{(N)}(q^{\pm S^z})\right] &= \sum_{l=1}^{N-1} \left[H_{l,l+1}, \Delta^{(N)}(q^{\pm S^z})\right] &(7.27)\\ &= q^{\pm S^z} \otimes \cdots \otimes H_{l,l+1}(q^{\pm S^z} \otimes q^{\pm S^z}) \otimes \cdots \otimes q^{\pm S^z} - \\ &\quad q^{\pm S^z} \otimes \cdots \otimes (q^{\pm S^z} \otimes q^{\pm S^z})H_{l,l+1} \otimes \cdots \otimes q^{\pm S^z} &(7.28)\\ &= q^{\pm S^z} \otimes \cdots \otimes \left[H_{l,l+1}, q^{\pm S^z} \otimes q^{\pm S^z}\right] \otimes \cdots \otimes q^{\pm S^z} &(7.29)\\ &= 0. &(7.30)\end{aligned}$$

We have used the result (7.12), that the two-site Hamiltonian commutes with $q^{\pm S^z} \otimes q^{\pm S^z}$. Next we define

$$\Phi_i^{\pm} = q^{S^z} \otimes \cdots \otimes q^{S^z} \otimes S_i^{\pm} \otimes q^{-S^z} \otimes \cdots \otimes q^{-S^z}, \tag{7.31}$$

where $S_i^{\pm}$ occupies the i'th position in the tensor product $N$ terms long. This allows us to write (6.26) as

$$\Delta^{(N)}(S^{\pm}) = \sum_{i=1}^{N} \Phi_i^{\pm}. \tag{7.32}$$

Using this definition we see

$$\left[\mathcal{H}, \Delta^{(N)}(S^{\pm})\right] = \sum_{l=1}^{N-1} \sum_{i=1}^{N} \left[H_{l,l+1}, \Phi_i^{\pm}\right]. \tag{7.33}$$

For each $l \in \{1, 2, \ldots, N-1\}$ we split the sum $\sum_{i=1}^{N} \left[H_{l,l+1}, \Phi_i^{\pm}\right]$ into four parts:

$$\sum_{i=1}^{N} \left[H_{l,l+1}, \Phi_i^{\pm}\right] = \sum_{i=1}^{l-1} \left[H_{l,l+1}, \Phi_i^{\pm}\right] + \left[H_{l,l+1}, \Phi_l^{\pm}\right] + \left[H_{l,l+1}, \Phi_{l+1}^{\pm}\right] + \sum_{i=l+2}^{N} \left[H_{l,l+1}, \Phi_i^{\pm}\right]. \tag{7.34}$$

For the first part with $i < l$

$$\begin{aligned}\left[H_{l,l+1},\Phi_i^{\pm}\right] &= q^{S^z}\otimes\cdots\otimes H_{l,l+1}(q^{S^z}\otimes q^{S^z})\otimes\cdots\otimes q^{S^z}\otimes S_i^{\pm}\otimes q^{-S^z}\otimes\cdots\otimes q^{-S^z}-\\ &\quad q^{S^z}\otimes\cdots\otimes(q^{S^z}\otimes q^{S^z})H_{l,l+1}\otimes\cdots\otimes q^{S^z}\otimes S_i^{\pm}\otimes q^{-S^z}\otimes\cdots\otimes q^{-S^z} \qquad (7.35)\\ &= q^{S^z}\otimes\cdots\otimes\left[H_{l,l+1},q^{S^z}\otimes q^{S^z}\right]\otimes\cdots\otimes q^{S^z}\otimes S_i^{\pm}\otimes q^{-S^z}\otimes\cdots\otimes q^{-S^z} \qquad (7.36)\\ &= 0, \qquad (7.37)\end{aligned}$$

and for the last part, where $i > l+1$

$$\begin{aligned}\left[H_{l,l+1},\Phi_i^{\pm}\right] &= q^{S^z}\otimes\cdots\otimes S_i^{\pm}\otimes q^{-S^z}\otimes\cdots\otimes H_{l,l+1}(q^{-S^z}\otimes q^{-S^z})\otimes\cdots\otimes q^{-S^z}-\\ &\quad q^{S^z}\otimes\cdots\otimes S_i^{\pm}\otimes q^{-S^z}\otimes\cdots\otimes(q^{-S^z}\otimes q^{-S^z})H_{l,l+1}\otimes\cdots\otimes q^{-S^z} \qquad (7.38)\\ &= q^{S^z}\otimes\cdots\otimes S_i^{\pm}\otimes q^{-S^z}\otimes\cdots\otimes\left[H_{l,l+1},q^{-S^z}\otimes q^{-S^z}\right]\otimes\cdots\otimes q^{-S^z} \qquad (7.39)\\ &= 0. \qquad (7.40)\end{aligned}$$

In both the above results we use (7.12). For the two middle terms where $i = l, l+1$

$$\begin{aligned}&\left[H_{l,l+1},\Phi_l^{\pm}\right]+\left[H_{l,l+1},\Phi_{l+1}^{\pm}\right]\\ &\quad= \big(q^{S^z}\otimes\cdots\otimes q^{S^z}\otimes H_{l,l+1}(S_l^{\pm}\otimes q^{-S^z})\otimes q^{-S^z}\otimes\cdots\otimes q^{-S^z}\\ &\qquad - q^{S^z}\otimes\cdots\otimes q^{S^z}\otimes(S_l^{\pm}\otimes q^{-S^z})H_{l,l+1}\otimes q^{-S^z}\otimes\cdots\otimes q^{-S^z}\big)+\\ &\qquad \big(q^{S^z}\otimes\cdots\otimes q^{S^z}\otimes H_{l,l+1}(q^{S^z}\otimes S_{l+1}^{\pm})\otimes q^{-S^z}\otimes\cdots\otimes q^{-S^z}\\ &\qquad - q^{S^z}\otimes\cdots\otimes q^{S^z}\otimes(q^{S^z}\otimes S_{l+1}^{\pm})H_{l,l+1}\otimes q^{-S^z}\otimes\cdots\otimes q^{-S^z}\big) \qquad (7.41)\\ &\quad= q^{S^z}\otimes\cdots\otimes q^{S^z}\otimes\left[H_{l,l+1},S_l^{\pm}\otimes q^{-S^z}+q^{S^z}\otimes S_{l+1}^{\pm}\right]\otimes\cdots\otimes q^{-S^z} \qquad (7.42)\\ &\quad= 0. \qquad (7.43)\end{aligned}$$

We used the results (7.22) and (7.23) in the last line. Thus, we see the open chain of length $N$ with these specific boundary conditions has $U_q(sl(2))$ as a symmetry group [9]. The boundary conditions may seem restrictive, but we note that

$$\lim_{\substack{N_+\to\infty\\ N_-\to-\infty}}\sum_{i=N_-}^{N_+}(\sigma_i^z-\sigma_{i+1}^z)=0. \qquad (7.44)$$

Thus Hamiltonian

[9]The finite periodic chain considered before does not have this symmetry. The symmetries of finite closed chains are subtle, and will not be discussed here. See for example [13].

$$\mathcal{H}_\infty = \sum_{\mathbb{Z}} \left( \sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \Delta \sigma_l^z \otimes \sigma_{l+1}^z \right) \tag{7.45}$$

is equivalent to the Hamiltonian

$$\mathcal{H}_\infty^{'} = \sum_{\mathbb{Z}} \left( \sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \Delta \sigma_l^z \otimes \sigma_{l+1}^z + \Gamma \left( \sigma_l^z - \sigma_{l+1}^z \right) \right), \tag{7.46}$$

and we see that the XXZ spin chain without boundary terms has $U_q(sl(2))$ symmetry in the limit as $N \to \infty$.

## 8 Conclusion

The symmetries of a system are a very useful property to investigate, as they can help solve a large number of problems and understand those solutions better. We saw a small hint of this in the project, the eigenstates we were trying to find had to be translationally symmetric. This helped Bethe make his ansatz for the form of the solutions. We found that the XXX spin chain has SU(2) symmetry. We also found that the deformed XXZ chain has a more complicated symmetry, described by the quantum group $U_q(sl(2))$. We can see from the calculation showing this that this symmetry is not trivial, but rather a unexpected result of the fortuitous cancelling of a number of terms.

We also considered the coordinate Bethe ansatz, and saw it greatly simplifies the eigenvalue problem, reducing the problem of finding the eigenvalues of a large matrix to solving a few equations. This can easily be done computationally, and can be done separately for every sector, so if only the first few excitations above the ground state are needed, only theses need to be calculated.

This project only touches on a small part of all the knowledge gained on integrable spin chains. A lot more can be said about the integrability of the model and different techniques like the Algebraic Bethe ansatz can be considered. The techniques we have applied can be generalised to open spin chains, higher spin chains, and different models, like the XYZ model, which deforms the chain even further, yet still leaves the model integrable. Spin chains emerge from seemingly unconnected fields, like the statistical mechanics of 2-dimensional lattices, and the operators in conformal field theories. This all shows why integrable models are worth studying, and have been studied so intensively in the past. They serve as a test bed for new techniques and methods, and give a greater understanding of the "workings" of a theory than a model which can only be solved numerically or by perturbation. These better techniques and understanding can then be applied to more realistic models.

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## A Appendix: $U_q(sl(2))$ is a Hopf algebra

The goal of this appendix is to show that the quantum group, $U_q(sl(2))$, consisting of all the linear combinations and powers of the generators $S^+, S^-, q^{\pm S^z}$, under the relations

$$q^{S^z} S^{\pm} q^{-S^z} = q^{\pm 1} S^{\pm}, \tag{A.1}$$

$$\left[S^+, S^-\right] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}, \tag{A.2}$$

and with co-product, co-unit and antipode defined as

$$\Delta(q^{\pm S^z}) = q^{\pm S^z} \otimes q^{\pm S^z}, \tag{A.3}$$

$$\Delta(S^{\pm}) = q^{S^z} \otimes S^{\pm} + S^{\pm} \otimes q^{-S^z}, \tag{A.4}$$

$$\gamma(q^{\pm S^z}) = q^{\mp S^z}, \tag{A.5}$$

$$\gamma(S^{\pm}) = -q^{\mp 1} S^{\pm}, \tag{A.6}$$

$$\epsilon(q^{\pm S^z}) = 1, \tag{A.7}$$

$$\epsilon(S^{\pm}) = 0, \tag{A.8}$$

satisfies the properties defining a Hopf algebra, as stated in Section 6. Hopefully this will make the meaning of these definitions clear. The definitions above assume the co-unit and co-product are homomorphisms, and the antipode is an anti-homomorphism. That is, we assume (6.6), (6.7) and (6.8) hold. We will show that the generators satisfy the required properties. A general algebra element, which is a linear combination of all the powers of these generators will then also satisfy these properties. We can show that the assumptions that co-unit and co-product are homomorphisms are compatible with the defining relations, (A.1) and (A.2). Indeed, we see that if we apply $\epsilon$ to the LHS of (A.1), we have

$$\epsilon\left(q^{S^z} S^{\pm} q^{-S^z}\right) = \epsilon\left(q^{S^z}\right) \epsilon\left(S^{\pm}\right) \epsilon\left(q^{-S^z}\right) = 1 \cdot 0 \cdot 1 = 0, \tag{A.9}$$

and if we apply it to the RHS we also get

$$\epsilon\left(q^{\pm 1} S^{\pm}\right) = q^{\pm 1} \epsilon\left(S^{\pm}\right) = 0. \tag{A.10}$$

Similarly, if we consider (A.2) we see

$$\epsilon\left(\left[S^+,S^-\right]\right)=\epsilon\left(\frac{q^{2S^z}-q^{-2S^z}}{q-q^{-1}}\right) \tag{A.11}$$

$$=\frac{1}{q-q^{-1}}\left(\epsilon\left((q^{S^z})^2\right)-\epsilon\left((q^{-S^z})^2\right)\right) \tag{A.12}$$

$$=\frac{1}{q-q^{-1}}\left(1\cdot 1-1\cdot 1\right)=0, \tag{A.13}$$

and

$$\left[\epsilon(S^+),\epsilon(S^-)\right]=[0,0]=0. \tag{A.14}$$

We can do the same with $\Delta$. From (A.1) we have

$$\Delta\left(q^{S^z}\right)\Delta\left(S^\pm\right)\Delta\left(q^{-S^z}\right)=\left(q^{S^z}\otimes q^{S^z}\right)\left(q^{S^z}\otimes S^\pm+S^\pm\otimes q^{-S^z}\right)\left(q^{-S^z}\otimes q^{-S^z}\right) \tag{A.15}$$

$$=q^{S^z}q^{S^z}q^{-S^z}\otimes q^{S^z}S^\pm q^{-S^z}+q^{S^z}S^\pm q^{-S^z}\otimes q^{S^z}q^{-S^z}q^{-S^z} \tag{A.16}$$

$$=q^{S^z}\otimes q^{S^z}S^\pm q^{-S^z}+q^{S^z}S^\pm q^{-S^z}\otimes q^{-S^z} \tag{A.17}$$

$$=q^{S^z}q^{S^z}q^{-S^z}\otimes q^{S^z}S^\pm q^{-S^z}+q^{S^z}S^\pm q^{-S^z}\otimes q^{S^z}q^{-S^z}q^{-S^z} \tag{A.18}$$

$$=q^{S^z}\otimes q^{\pm 1}S^\pm+q^{\pm 1}S^\pm\otimes q^{-S^z} \tag{A.19}$$

$$=q^{\pm 1}\Delta(S^\pm). \tag{A.20}$$

By applying $\Delta$ to (A.2) we have

$$\Delta\left(\left[S^+,S^-\right]\right)=\frac{1}{q-q^{-1}}\left(\Delta(q^{2S^z})-(q^{-2S^z})\right) \tag{A.21}$$

$$=\frac{1}{q-q^{-1}}\left(q^{2S^z}\otimes q^{2S^z}-q^{-2S^z}\otimes q^{-2S^z}\right). \tag{A.22}$$

We also see that

$$\left[\Delta(S^+),\Delta(S^-)\right]=\left[q^{S^z}\otimes S^++S^+\otimes q^{-S^z},q^{S^z}\otimes S^-+S^-\otimes q^{-S^z}\right] \tag{A.23}$$

$$=\left(q^{S^z}\otimes S^++S^+\otimes q^{-S^z}\right)\left(q^{S^z}\otimes S^-+S^-\otimes q^{-S^z}\right)-\left(S^+\leftrightarrow S^-\right) \tag{A.24}$$

$$=\left(q^{S^z}q^{S^z}\otimes S^+S^-+q^{S^z}S^-\otimes S^+q^{-S^z}+S^+q^{S^z}\otimes q^{-S^z}S^-+ S^+S^-\otimes q^{-S^z}q^{-S^z}\right)-\left(S^+\leftrightarrow S^-\right), \tag{A.25}$$

where by $(S^+ \leftrightarrow S^-)$ we denote a term exactly like the preceding term, with $S^+$ and $S^-$ interchanged. Note that (A.1) implies $q^{S^z} S^\pm = q^{\pm 1} S^\pm q^{S^z}$ and $q^{-S^z} S^\pm = q^{\mp 1} S^\pm q^{-S^z}$. This allows us to cancel the cross terms above:

$$\begin{aligned}
&(q^{S^z} S^- \otimes S^+ q^{-S^z} + S^+ q^{S^z} \otimes q^{-S^z} S^-) - (q^{S^z} S^+ \otimes S^- q^{-S^z} + S^- q^{S^z} \otimes q^{-S^z} S^+) \\
&\quad = (q^{S^z} S^- \otimes S^+ q^{-S^z} - S^- q^{S^z} \otimes q^{-S^z} S^+) + (S^+ q^{S^z} \otimes q^{-S^z} S^- - q^{S^z} S^+ \otimes S^- q^{-S^z}) \qquad \text{(A.26)} \\
&\quad = (q^{S^z} S^- \otimes S^+ q^{-S^z} - (q^{+1} q^{S^z} S^-) \otimes (q^{-1} S^+ q^{-S^z})) \\
&\qquad + (S^+ q^{S^z} \otimes q^{-S^z} S^- - (q^{+1} S^+ q^{S^z}) \otimes (q^{-1} q^{-S^z} S^-)) \qquad \text{(A.27)} \\
&\quad = 0. \qquad \text{(A.28)}
\end{aligned}$$

This means

$$\begin{aligned}
\left[\Delta(S^+), \Delta(S^-)\right] &= \left(q^{S^z} q^{S^z} \otimes S^+ S^- + S^+ S^- \otimes q^{-S^z} q^{-S^z}\right) - \left(S^+ \leftrightarrow S^-\right) && \text{(A.29)} \\
&= q^{2S^z} \otimes \left[S^+, S^-\right] + \left[S^+, S^-\right] \otimes q^{-2S^z} && \text{(A.30)} \\
&= \frac{1}{q - q^{-1}} \left(q^{2S^z} \otimes (q^{2S^z} - q^{-2S^z}) + (q^{2S^z} - q^{-2S^z}) \otimes q^{-2S^z}\right) && \text{(A.31)} \\
&= \frac{1}{q - q^{-1}} \left(q^{2S^z} \otimes q^{2S^z} - q^{-2S^z} \otimes q^{-2S^z}\right) && \text{(A.32)} \\
&= \Delta\left(\left[S^+, S^-\right]\right) && \text{(A.33)}
\end{aligned}$$

Since we assumed $\epsilon$ and $\Delta$ were homomorphisms, and this was found to be compatible with the defining relations of the quantum group, we see that they are defined in a way that respects the group structure, and are valid homomorphisms. It is easy to see that the co-product is co-associative (6.4), indeed

$$(\Delta \otimes \mathbf{1})(\Delta(q^{\pm S^z})) = q^{\pm S^z} \otimes q^{\pm S^z} \otimes q^{\pm S^z} = (\mathbf{1} \otimes \Delta)(\Delta(q^{\pm S^z})) \qquad \text{(A.34)}$$

and

$$(\Delta \otimes \mathbf{1})(\Delta(S^\pm)) = q^{S^z} \otimes q^{S^z} \otimes S^\pm + q^{S^z} \otimes S^\pm \otimes q^{-S^z} + S^\pm \otimes q^{-S^z} \otimes q^{-S^z} = (\mathbf{1} \otimes \Delta)(\Delta(S^\pm)). \qquad \text{(A.35)}$$

The co-unit must satisfy

$$(\epsilon \otimes \mathbf{1})(\Delta(a)) = (\mathbf{1} \otimes \epsilon)(\Delta(a)) = a. \qquad \text{(A.36)}$$

We will show only half of this relation. The other half follows in exactly the same way. Firstly, for $q^{\pm S^z}$

$$(\epsilon \otimes \mathbf{1})(\Delta(q^{\pm S^z})) = \epsilon(q^{\pm S^z}) \otimes q^{\pm S^z} = 1 \otimes q^{\pm S^z} \tag{A.37}$$

and for $S^{\pm}$

$$(\epsilon \otimes \mathbf{1})(\Delta(S^{\pm})) = \epsilon(q^{S^z}) \otimes S^{\pm} + \epsilon(S^{\pm}) \otimes q^{-S^z} \tag{A.38}$$

$$= 1 \otimes S^{\pm} + 0 \otimes q^{-S^z} = S^{\pm}. \tag{A.39}$$

The antipode must satisfy

$$m\left((\mathbf{1} \otimes \gamma)(\Delta(a))\right) = m\left((\gamma \otimes \mathbf{1})(\Delta(a))\right) = \epsilon(a)\mathbf{1}. \tag{A.40}$$

Again, we only show one half of this relation. For $q^{\pm S^z}$

$$m\left((\mathbf{1} \otimes \gamma)(\Delta(q^{\pm S^z}))\right) = m\left(q^{\pm S^z} \otimes \gamma(q^{\pm S^z})\right) \tag{A.41}$$

$$= m\left(q^{\pm S^z} \otimes q^{\mp S^z}\right) \tag{A.42}$$

$$= q^{\pm S^z} \cdot q^{\mp S^z} = \mathbf{1} = \epsilon(q^{\pm S^z})\mathbf{1}. \tag{A.43}$$

For $S^{\pm}$ we see

$$m\left((\mathbf{1} \otimes \gamma)(\Delta(S^{\pm}))\right) = m\left(q^{S^z} \otimes \gamma(S^{\pm}) + S^{\pm} \otimes \gamma(q^{-S^z})\right) \tag{A.44}$$

$$= m\left(q^{S^z} \otimes -q^{\mp 1}S^{\pm}\right) + m\left(S^{\pm} \otimes q^{S^z}\right) \tag{A.45}$$

$$= -q^{\mp 1}q^{S^z}S^{\pm} + S^{\pm}q^{S^z} \tag{A.46}$$

using (A.1) we see

$$m\left((\mathbf{1} \otimes \gamma)(\Delta(S^{\pm}))\right) = S^{\pm}q^{S^z} - q^{\mp 1}q^{S^z}S^{\pm} \tag{A.47}$$

$$= S^{\pm}q^{S^z} - q^{\mp 1}(q^{\pm 1}S^{\pm}q^{S^z}) \tag{A.48}$$

$$= 0 = \epsilon(\Delta(S^{\pm})\mathbf{1}, \tag{A.49}$$

thus confirming (A.40)