$AdS_{(3)}$ within Kerr Geometry and the Entropy of the Near Horizon Extreme Kerr

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Abstract

Here we show that the Near Horizon Extremal Kerr geometry contains a warped $\mathrm{AdS}_{(3)}$ geometry. We start from the Kerr metric and by making appropriate approximations we take the extremal near horizon limit. Then calculating the Killing vectors we show that these form a $\mathrm{SL}(2)\times\mathrm{U}(1)$ algebra and as such carry the same symmetries as warped $\mathrm{AdS}_{(3)}$. This allows us to show that a dual conformal field theory exists and then to determine the entropy of the black hole by microscopic considerations. We find that this entropy is the same as the macroscopic Bekenstein-Hawking entropy.

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1 Introduction

In the 70's Bardeen, Carter and Hawking postulated that certain quantities, which appear in black hole expressions can be held analogous to classical thermodynamic parameters. The first of these is the area of the black hole event horizon. Like classical entropy the area of a black hole can never decrease [1]. The second is the surface gravity of the black hole that acts as an analogy to the temperature. It is important to note that the area and surface gravity are distinct from the entropy and temperature of a black hole but behave in a similar way. For a brief view of the arguments that support this theory see Appendix B: Black hole Thermodynamics.

Originally the thermodynamics of black holes posed a curious puzzle when held analogous to that of other systems. The problem we consider in this work relates to the entropy of black holes. The original work on the entropy of black holes by Bekenstein, Hawking and others [1] [2] [3], due to macroscopic considerations, gave the entropy as

$$S = \frac{A}{4\pi G} \tag{1.1}$$

where A is the event horizon area of the black hole.

But the more fundamental problem at the time was that there existed no microscopic accounting for this entropy. Compared to the work of Boltzmann in the 19th century on explaining the laws of thermodynamics of gasses and liquids by means of statistical reasoning of the fundamental constituents. It seemed important that a similar approach must be viable when considering black holes otherwise we would run into some serious contradictions.

This problem remained unsolved until the mid 90's when, using string theory, Strominger and Vafa [4] successfully reproduced the Bekenstein-Hawking result (1.1) by counting microstates. This breakthrough was considered as evidence that string theory is the correct description of the universe. But shortly after this it was discovered that any consistent quantum theory of gravity containing these black holes characterised by a near-horizon region that contains an AdS factor has to reproduce the entropy in a similar way [5]. The key ingredient of this theory followed from the work of Brown and Henneaux [6], that any consistent completion of quantum gravity on the background of $AdS_{(3)}$ could be described by a 2-dimensional conformal field theory by purely symmetry consideration. Our understanding of these systems has since then progressed to the point where we can reasonably understand the 4-dimensional Kerr black holes we observe in the sky.

As such the Kerr/CFT correspondence, although it is informed by string theory it is not dependent upon it. We rather approach the problem by considering the diffeomorphism group of the Kerr black hole and as such it's symmetries along with some basic consistency arguments. This is analogous to the work of Boltzmann in the sense that we do not require the an exact completion of the theory in the UV but rather assume that there has to be some natural cut off, and no assumptions of Planck scale physics has to be made. Therefore we can use the results obtained whether or not string theory is the correct description of the universe.

We continue here with a handling of the Kerr geometry as we focus on the near horizon extreme Kerr geometry (NHEK). Eventually showing that the above geometry contains a warped $AdS_{(3)}$ factor. We therefore know that the NHEK geometry has a dual 2 dimensional CFT.

2 Diffeomorphisms and Symmetry

Since we approach the analysis of the NHEK geometry and the corresponding CFT not from the point of string theory but using only symmetry considerations it is useful to first look into the diffeomorphisms of curved spaces.

A diffeomorphism is an isomorphism on smooth manifolds that maps one manifold to another, in such a way that both the function and its inverse are smooth. Given a diffeomorphism $\phi : M \to M$, where M is the manifold and an arbitrary tensor $T(x)$. We can define a differential operator on tensor fields which describes the rate of change of the tensor as it changes under the diffeomorphism. This can be done since we can form the difference between the tensor at point p and its value at point $\phi(p)$. But a single diffeomorphism is not sufficient and we therefore consider a one-parameter family of diffeomorphisms ϕ_t, such that it is a smooth map from $\mathbf{R} \times M \to M$ such that for every $t\epsilon\mathbf{R}$,ϕ_t is a diffeomorphism and $\phi_t \circ \phi_s = \phi_{t+s}$.

One-parameter families of diffeomorphisms are closely related to vector fields. Consider for example a point p under the entire family of ϕ_t, we see that it describes a curve in M, and these curves fill the entire manifold since it is true for all points p. An example of this on S^2 is $\phi_t(\theta, \phi) = (\theta, \phi+t)$. We can then define a vector field $V^\mu(x)$ to be the set of tangent vectors to each of these curves at every point. We then define the integral curves of the vector field to be those curves $x^\mu(t)$ that solve

$$\frac{dx^\mu}{dt} = V^\mu.$$

These are always guaranteed to exist as long as we don't run into the edge of the manifold, which in most cases can be avoided using an appropriate coordinate system. The diffemorphisms ϕ_t now represent the flow down the integral curves and the associated vector field is called the generator of the diffeomorphism.

We then define the Lie derivative as a way to evaluate the change of a tensor field along a vector field [7],

$$\mathcal{L}_V T = \begin{matrix} lim \\ t \to 0 \end{matrix} \left(\frac{\triangle_t T}{t} \right) = \begin{matrix} lim \\ t \to 0 \end{matrix} \left(\frac{\phi_t T(\phi(p)) - T(p)}{t} \right). \tag{2.1}$$

With some work, which we will not go into detail here, the Lie derivative can then be written in terms of the covariant derivative derived from the metric. We are interested in the Lie derivative of the metric which is given by [7],

$$\mathcal{L} g_{\mu\nu} = 2 \bigtriangledown_{(\mu} V_{\nu)} \tag{2.2}$$

We now get to describing the symmetries of the system mathematically. A diffeomorphism ϕ is a symmetry if some tensor T is invariant under it, i.e.

$$\phi T = T$$

Although this is useful, it is more common to have a one-parameter family of symmetries generated by a vector field $V^\mu(x)$ which satisfies

$$\mathcal{L}_V T = 0$$

We are particularly interested in the symmetries of the metric and by extension the the symmetries of the space-time. A diffeomorphism of this type is called an isometry and it satisfies

$$L_V g_{\mu\nu} = 0$$

or equivalently

$$\nabla_{(\mu} V_{\nu)} = 0 \tag{2.3}$$

The most important property of the Killing vectors for our purpose is that they describe the symmetries of a given manifold. And that all the distinct Killing vectors of a space together form the isometry group of that space. We can then use this group to identify a dual conformal field theory to describe the curved space system.

For the purpose of finding a dual conformal field theory we need to show that the Near Horizon Extremal Kerr metric has an embedded AdS symmetry. Now $AdS_{(3)}$ has the symmetry group SL(2)×SL(2) as a Hopf-fibration of S^1 over $AdS_{(2)}$ and is the analogue of S^3 with symmetry group SU(2)×SU(2). Like the squashing of the 3-sphere by deforming the S^1 fibre radius over the S^2, we can warp the $AdS_{(3)}$ by deforming the corresponding fibre radius. In the squashed sphere this breaks the one SU(2) symmetry down to U(1). In the warped $AdS_{(3)}$ case it breaks the one SL(2) symmetry down to U(1), and the warped $AdS_{(3)}$ now has the symmetry group SL(2)×U(1) [8]. For our purpose we need to find a way to describe the symmetry group of a space time. This is done by taking the commutation relations of the Killing vectors of a given space time, hence determining the corresponding algebra. For the most part the symmetry group we are interested in is the special linear group SL(2,R) which has the commutation relations ,

$$\begin{aligned} [T, R] &= R \\ [T, S] &= -S \\ [R, S] &= T \end{aligned} \tag{2.4}$$

for three vectors T, R and S.

3 The Kerr Solution

The Kerr Solution to Einstein's equations is a generalisation of the Schwarzschild solution in order to account for the angular momentum of the black hole. In Boyer-Lindquist coordinates the metric is given by [8]

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - asin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{sin^2\theta}{\rho^2}\left(\left(r^2+a^2\right)d\phi - adt\right)^2 + \rho^2 d\theta^2 \tag{3.1}$$

Where

$$\Delta = r^2 - 2Mr + a^2 \tag{3.2}$$

$$\rho^2 = r^2 + a^2 cos^2\theta \tag{3.3}$$

$$a = \frac{J}{M} \tag{3.4}$$

We are of course also using Natural Units, meaning that $G = c = 1$ which also means that mass (M) and angular momentum (J) have units of inverse length. This solution can be seen as having two horizon solutions, i.e. inner horizon (r_-) and outer horizon (r_+), for $\Delta = 0$. We then have that

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{3.5}$$

We then have two interesting cases we can consider. For $a = 0$ we have that $(r_+) = 2M$ which is just the standard Schwarzschild solution, in other words just a stationary black hole. The other case of interest is the extremal case in which if $a = M = \sqrt{J}$ we have that $(r_\pm) = M$. To understand this it makes sense to consider the angular velocity at the equator of the outer horizon (3.6). See Appendix A.

$$\begin{aligned} \Omega_H &= \frac{a}{2Mr_+} \\ &= \frac{1}{2r_+} \text{ , if } a = M \end{aligned} \tag{3.6}$$

So for the second case above we have that the equator on the horizon rotates at the speed of light, and as such is bounded. We can see this clearly by expressly taking $J > M^2$, we then see that the singularity at the horizon vanishes and the only singularity left is at $r = 0$. But this violates cosmic censorship since we now have a naked singularity. We can therefore say that the the angular momentum of the black hole is bounded by $J \leq M^2$. The above limit also helps in simplifying the system we work with, as will be discussed later.

Considering now the temperature of at the horizon of the black hole (B.12) (Appendix A), and taking $a = M$ we see that the Hawking-temperature is $T_H = 0$. This implies that the extremal rotating black hole acts like a ground state of the Kerr family. As such it is the extremal limit $a = M$ we are interested in and not the stationary case where $a = 0$, which initially might look like the simpler object.

4 Near Horizon Extreme Limit of Kerr

Generally we take limits to first look at a simplified version of a problem. As such we now exclusively deal with the simplest object in the Kerr solution family, the Near Horizon Extremal Kerr (NHEK) geometry. A heuristic explanation of why the problem simplifies in this limit is due to the frame dragging of the ergoshpere. This means that inside the ergoshpere particles have to move with the black hole. In the extremal limit, as discussed in the previous section, at the equator of the horizon the speed of the horizon approaches the speed of light. So particles in this region of space have to move with the same velocity and rotate in the same direction as the black hole. Hence only chiral degrees of freedom are considered, this is important for simplifying the CFT. Another justification for using the NHEK geometry is the extra symmetry gained. The generic Kerr metric (3.1) only produces two Killing vectors [9]

$$\begin{aligned} T &= \partial_t \\ R &= \partial_\phi \end{aligned} \tag{4.1}$$

We will see later that taking the NHEK limit we gain additional symmetries as shown later.

4.1 Extremal Limit

We originally obtained the extreme Kerr solution for $a = M$, and renaming the some of the parameters for use later we have that the metric (3.1) becomes (4.2), but the only real change is (4.3).

$$ds^2 = -\frac{\Delta}{\rho^2}\left(d\hat{t} - asin^2\theta d\hat{\phi}\right)^2 + \frac{\rho^2}{\Delta}d\hat{r}^2 + \frac{sin^2\theta}{\rho^2}\left(\left(\hat{r}^2 + a^2\right)d\hat{\phi} - ad\hat{t}\right)^2 + \rho^2 d\theta^2 \tag{4.2}$$

Where

$$\Delta = (\hat{r} + a)^2 \tag{4.3}$$

$$\rho^2 = \hat{r}^2 + a^2cos^2\theta \tag{4.4}$$

4.2 Near Horizon

We are now interested in the region close to the horizon. So we zoom in on the region $r = M$ by means of a scaling parameter $\lambda \to 0$, as was first used by Bardeen and Horowitz [10]

$$\begin{aligned} r &= \frac{\hat{r} - M}{\lambda M} \\ t &= \frac{\lambda \hat{t}}{2M} \\ \phi &= \hat{\phi} - \frac{\hat{t}}{2M} \end{aligned} \tag{4.5}$$

Where we keep r, t and ϕ fixed. Clearly for any finite r as $\lambda \to 0$ then $\hat{r} \to M$. So the parameter r is on the horizon of the black hole. Taking this limit we obtain the smooth near horizon extremal Kerr geometry:

$$ds^2 = 2\Omega^2 J \left[\frac{dr^2}{r^2} + d\theta^2 - r^2 dt^2 + \Lambda \left(d\phi + r dt \right)^2 \right] \tag{4.6}$$

with

$$\Omega = \frac{1 + cos^2\theta}{2} \ , \Lambda = \frac{2sin\theta}{1 + cos^2\theta} \tag{4.7}$$

Since the solution above (4.6) is the limit of a coordinate transform of a solution to Einstein's Equations it is still a solution. Now the Killing vectors of the NHEK metric are given as[1]

$$\begin{aligned} R &= \partial_\phi \\ K &= -t\partial_t + r\partial_r \\ T &= \partial_t \end{aligned} \tag{4.8}$$

This shows that the NHEK geometry carries enhanced symmetry with respect to the generic Kerr solution. We can further improve on this by taking a spacial slice of the metric, as done in the next section.

5 AdS$_{(3)}$ slice of Kerr

Now finally we come to analysing the underlying AdS$_{(3)}$ geometry of the NHEK geometry. To this end we take a slice of the Kerr space-time at fixed polar coordinate θ. This means that both Λ and Ω are constants and we have the metric below:

$$ds^2 = 2\Omega^2 J \left[\frac{dr^2}{r^2} - r^2 dt^2 + \Lambda \left(d\phi + r dt \right)^2 \right] \tag{5.1}$$

If we take the value of $\theta = \theta_0$ such that $\Lambda = 1$ we get the exact metric for AdS$_{(3)}$ in Boyer-Lindquist coordinates (5.4). We can check this by calculating the Killing vectors by solving (2.3). Anti-de Sitter space is defined as a maximally symmetric quadratic surface embedded in a flat space of signature $(+...+--)$, that is a solution to Einstein's equations [11]. Therefore 2+1 anti-de Sitter space, or $AdS_{(3)}$ is the hypersurface

$$-u^2 - v^2 + x^2 + y^2 = -l^2 \tag{5.2}$$

embedded in the flat space with metric

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \tag{5.3}$$

[1] For the calculations determining the Killing vectors see Appendix C.

The above will have an isometry group of SO(2,2) which is locally isomorphic to SO(2,1)×SO(2,1). But we also know that the special linear group SL(2,R) is a double covering of SO(2,1). So the isometries of $AdS_{(3)}$ are SL(2,R)×SL(2,R) [11].

$$ds^2 = 2\Omega^2 J\left[\frac{dr^2}{r^2} - r^2dt^2 + (d\phi + rdt)^2\right] \tag{5.4}$$

Solving (2.3) and by choice of appropriate solution constants we have the following Killing vectors[2]:

$$\begin{aligned} L_1 &= \partial_t \\ L_2 &= t\partial_t - r\partial_r \\ L_3 &= \frac{1-r^2t^2}{2r^2}\partial_t - rt\partial_r - \frac{1}{r}\partial_\phi \end{aligned} \tag{5.5}$$

$$\begin{aligned} K_1 &= \partial_\phi \\ K_2 &= \frac{e^\phi}{r}\partial_t + re^\phi\partial_r - e^\phi\partial_\phi \\ K_3 &= -\frac{e^\phi}{r}\partial_t + re^{-\phi}\partial_r + e^\phi\partial_\phi \end{aligned} \tag{5.6}$$

It is then easy to check that the commutation relations of these vectors are given by

$$\begin{gathered} [L_2, L_3] = L_3 \\ [L_2, L_1] = -L_1 \\ [L_3, L_1] = L_2 \\ \text{and} \\ [K_2, K_3] = K_3 \\ [K_2, K_1] = -K_1 \\ [K_3, K_1] = K_2 \end{gathered} \tag{5.7}$$

with the rest zero. We then see that we have a SL(2,R)×SL(2,R) isometry group as expected.

But for an arbitrary θ the isometry group found from solving (2.3) for (5.1) is that of a warped $AdS_{(3)}$. The Killing vectors we find are:

$$\begin{aligned} T &= \partial_t \\ K &= \partial_\phi \\ R &= t\partial_t - r\partial_r \\ S &= -t^2\left(1+\Lambda-\Lambda^2\right)\partial_t + \frac{1}{r^2}\partial_t + 2rt\left(1+\Lambda-\Lambda^2\right)\partial_r + \frac{2}{r}\partial_\phi \end{aligned} \tag{5.8}$$

[2]For the calculations determining the Killing vectors as well as their commutation relations see Appendix C.

Now it is clear that K commutes with all the other Killing vectors and therefore forms a U(1) group. The remaining three Killing vectors clearly do not commute with one another. Checking the commutation relations of T, K and S we see that they form a SL(2,R) like group.

$$\begin{aligned}
[T,R] &= T \\
[R,S] &= S \\
[S,T] &= 2\left(1+\Lambda-\Lambda^2\right)R
\end{aligned} \tag{5.9}$$

From this we can show that a corresponding 2-dimensional conformal field theory exists. And as such we can continue in our approach to determine the microscopic theory of the entropy of the black hole.

6 Conformal Field Theory

Now that we have established the diffeomorphism group of the NHEK geometry, we can continue to the conformal field theory to study the excitations around the space-time. This requires that we impose new boundary conditions since we lost the asymptotically flat region when we took the near horizon limit. For a consistent set of boundary conditions there exists an associated asymptotic symmetry group, defined as

$$ASG = \frac{Allowed\ Symmetry\ Transformations}{Trivial\ Symmmetry\ Transformations} \tag{6.1}$$

The allowed symmetry transformations refer to the diffeomorphisms consistent with the boundary conditions and the trivial to the generators of the symmetries that vanish after we implement the constraints [12]. Consistency of the theory depends on the generators of the ASG to be well defined and not diverge at the boundary. For example in asymptotically flat space we require the metric to fall off as $1/r$, then the ASG is simply the Poincare group. The boundary conditions used in this case are [12]

$$\begin{aligned}
h_{tt} = O(r^2), h_{t\phi} = h_{\phi\phi} = O(1), h_{t\theta} = h_{\phi\theta} = h_{\theta\theta} = h_{\phi r} = O(r^{-1}), \\
h_{tr} = h_{\theta r} = O(r^{-2}), h_{rr} = O(r^{-3})
\end{aligned} \tag{6.2}$$

where $h_{\mu\nu}$ is symmetric and the deviation from the full Kerr metric.

Now the most general diffeomorphism that preserves the boundary conditions (6.2) is [12]

$$\xi = \left[\varepsilon(\phi) + O(r^{-2})\right]\partial_\phi + \left[r\varepsilon'(\phi) + O(1)\right]\partial_r + \left[C + O(r^{-3})\right]\partial_t + \left[O(r^{-1})\right]\partial_\theta \tag{6.3}$$

where $\varepsilon(\phi)$ is an arbitrary smooth function dependant on the boundary coordinate ϕ and C is an arbitrary constant.

From calculating the generators of the theory we see that the sub-leading terms belong to the trivial symmetry group. So the ASG contains one copy of the conformal group of the circle generated by

$$\zeta_\varepsilon = \varepsilon(\phi)\partial_\phi - r\varepsilon'(\phi)\partial_r \tag{6.4}$$

.

It is then convenient to define a basis

$$\zeta_n = \zeta\left(-e^{-in\phi}\right) = -e^{-in\phi}\partial_\phi - ine^{-in\phi}r\partial_r \tag{6.5}$$

These vector fields then obey the Virasoro algebra with the Lie bracket solving [13]

$$[\zeta_n, \zeta_m]_{L.B.} = \mathcal{L}_{\zeta_n}\zeta_m - \mathcal{L}_{\zeta_m}\zeta_n \tag{6.6}$$
$$i\,[\zeta_n, \zeta_m]_{L.B.} = (m-n)\,\zeta_{n+m} \tag{6.7}$$

The allowed symmetry transformations also include t translations by ∂_t, and it's conjugate conserved quantity $E = M^2 - J$ measures the deviation from extremality [8]. Since we are only interested in the extremal Kerr black hole we require that $E = M^2 - J = 0$, which is compatible with (6.3) since ∂_t commutes with the Virasoro algebra.

We now need to construct the generators of the diffeomorphisms (6.3) via the Dirac bracket. Then for every diffeomorphism ζ there is and associated generator in the form of a charge Q_ζ,constructed to obey

$$\{Q_\zeta, \Phi\}_{D.B.} = \mathcal{L}_\zeta\Phi \tag{6.8}$$

where Φ is any field in the theory. The charges, under the Dirac brackets, obey the same algebra as the symmetries themselves up to a possible central charge. From (6.8), using the Jacobi identity, we have that

$$\{Q_\zeta, Q_\xi\}_{D.B.} = Q_{[\zeta,\xi]} + c_{\zeta\xi} \tag{6.9}$$

where $c_{\zeta\xi}$ is the central term or central charge. The central charge is an extensive measure of the degrees of freedom of the system [13], this is needed for determining the entropy of the system by using statistical considerations. Now the central term can be calculated using classical gravity from Q_ζ. The infinitesimal charge differences between neighbouring geometries $g_{\mu\nu}$ and $g_{\mu\nu} + h_{\mu\nu}$ is given by [12]

$$\delta Q_\zeta = \frac{1}{8\pi G}\int_{\partial\Sigma} K_\zeta(h,g) \tag{6.10}$$

and the integral is over the boundary of the slice of the space-time, with [12]

$$\begin{aligned} K_\zeta(h,g) = &-\frac{1}{4}\varepsilon_{\alpha\beta\mu\nu}dx^\alpha \wedge dx^\beta[\zeta^\nu D^\mu h - \zeta^\nu D_\sigma h^{\mu\sigma} \\ &+ \zeta_\sigma D^\nu h^{\mu\sigma} + \frac{1}{2}hD^\nu\zeta^\mu + \frac{1}{2}h^{\sigma\nu}\left(D^\mu\zeta_\sigma - D_\sigma\zeta^\mu\right)] \end{aligned} \tag{6.11}$$

So we have, when applying (6.10) to fluctuations $h = \mathcal{L}_\xi g$, that

$$c_{\xi\zeta} = \frac{1}{8\pi G}\int_{\partial\Sigma} K_\zeta(\mathcal{L}_\xi g, g). \tag{6.12}$$

Now in terms of (6.5) we have

$$\{Q_{\zeta_m}, Q_{\zeta_n}\}_{D.B.} = Q_{[\zeta_n,\zeta_m]} - iJ(m^2+2m)\delta_{m+n}. \tag{6.13}$$

These classical charges have units of action so we can use $\hbar$ to define a dimensionless quantity L_n so that we can apply the usual quantisation rule, $\{\cdot,\cdot\}_{D.B.} \to -\frac{i}{\hbar}[\cdot,\cdot]$.

$$\hbar L_n = Q_{\zeta_n} + \frac{3J}{2}\delta_n \tag{6.14}$$

The quantum charge algebra is then given by [8]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{J}{\hbar}m(m^2-1)\delta_{m+n} \tag{6.15}$$

This result shows that we have found a Virasoro algebra as the asymptotic symmetry algebra of the $AdS_{(3)}$-like geometry found in the fixed θ slice of the NHEK geometry. We also know that CFT's necessarily have a Virasoro symmetry and we can therefore associate a 2-dimensional CFT with the $AdS_{(3)}$ geometry. We can then read of the central charge of the NHEK geometry as [12]

$$c = \frac{12J}{\hbar} \tag{6.16}$$

It is important to note that the central charge is independent of the boundary conditions as long as the diffeomorphisms are allowed. We now have that for the NHEK geometry states with E=0 must form representations of the Virasoro algebra and hence comprise the chiral half of the a 2-dimensional CFT. We now have to tools to determine the entropy of a black hole from microscopic considerations.

7 Cardy Entropy

The Bekenstein-Hawking entropy of the extreme Kerr geometry (B.13) is $S_{B.H.} = 2\pi J$, note that we have now taken $\hbar$ to be one. We would now like to reproduce this in terms of the microstate degeneracy of the CFT at finite temperature. But we know the Hawking-Temperature of the extreme Kerr solution is zero. So the two relevant conserved quantities we use is the energy (conjugate to the Hawking-temperature) and the angular momentum (conjugate to the angular velocity). So counting microstates near the horizon moving at the speed of light produces a quantum number that is a linear combination of the energy and the angular momentum.

In general black holes represent a mixed state and the quantum fields around the black hole are in a thermal state. So the Boltzmann weighting is [8]

$$e^{-(\omega - m\Omega_H)/T_H} \tag{7.1}$$

with $\Omega_H = 1/2M$ for the NHEK.

Now as we take the extreme limit $T_H \to 0$, but this does not mean that (7.1) is necessarily trivial. Due to the existence of states where $\omega \to m\Omega_H$, which means that the particles are moving in the same direction as the black hole close to the speed of light, i.e. they are all chiral. To show this we take the extreme limit and in the near horizon coordinates (4.5) we then have

$$e^{i\omega t + im\phi} = e^{-\frac{i}{\lambda}(2M\omega - m)t + im\phi} = e^{-in_R t + in_L \phi} \tag{7.2}$$

Where $n_L = m$ and $n_R = \frac{1}{\lambda}(2M\omega - m)$ are the left and right charges associated with ∂_ϕ and ∂_t in the near horizon region. We can now write

$$e^{-(\omega - m\Omega_H)/T_H} = e^{-n_L/T_L - n_R/T_R} \tag{7.3}$$

with

$$T_L = \frac{r_+ - M}{2\pi(r_+ - a} \tag{7.4}$$

$$T_R = \frac{r_+ - M}{2\pi\lambda r_+} \tag{7.5}$$

and if $M^2 \to J$ as in the extreme limit

$$T_L = \frac{1}{2\pi} \tag{7.6}$$

$$T_R = 0 \tag{7.7}$$

So the field is populated with left movers with the Boltzmann distribution $e^{-2\pi n_L}$ and there are no right movers, hence the chiral part of the CFT.

Now according to the Cardy entropy for a unitary CFT, and with c_L the central charge calculated previously since there are only left movers in the extreme case, we have the entropy,

$$S = \frac{\pi^2}{3} c_L T_L \tag{7.8}$$

$$= 2\pi J \tag{7.9}$$

Which is the Bekenstein-Hawking entropy of the NHEK black hole derived from quantum and statistical considerations.

8 Concluding Statements

We have therefore shown that there exists a conformal field theory dual to the Kerr space-time that we can use to describe dynamics of the black hole.

The importance of this is that we now have a both a microscopic as well as a macroscopic explanation for the entropy of a black hole and as such we have a theory that is consistent with the other systems we see in nature. This has only been shown here for the near horizon extremal Kerr geometry, but to apply this to more general black holes new ideas and techniques may be required.

As such we have treated the extremal Kerr black hole as a sort of ground state to the black hole family, since the system is simplified by the additional isometries of the NHEK case. Future research can now be done on generalising the concepts in this paper to those of the generic Kerr geometry or the other special case mentioned previously, the Schwarzschild black hole.

Before other black holes are explored there are some other interesting questions still left unanswered. Among these, the fact that there are boundary conditions found that give either a left or right-moving Virasoro asymptotic symmetry group, but are there boundary conditions that give both? Another puzzle is how we understand that the near horizon extremal Kerr region disappears for finite $M^2 - J$. Where the hidden conformal symmetry comes from and why does the Cardy formula for entropy work [8]. Currently the favoured approach to answering these questions is the embedding of the Kerr/CFT into string theory. Some progress has been made with this approach on charged, spinning black holes in five dimensions as in [14].

Then of course there is the big question, constructing the dual CFT. This means that the spectrum of the field theory has to be found and constructing the correlation functions in addition to the number of states, as done here when finding the central charge.

Appendices

A Properties of Black holes

A.1 Area

From the Kerr metric (3.1) we find the angular part of the metric as

$$d\omega^2 = \rho^2 d\theta^2 + \sin^2\theta \left(r^2 + a^2 + \frac{2Mr}{\rho^2} a^2 \sin^2\theta \right) d\phi^2 \tag{A.1}$$

$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{A.2}$$

Then we can write

$$d\omega^2 = \gamma_{AB} dx^A dx^B \tag{A.3}$$

The area of the event horizon is then given by

$$A = \int \sqrt{det\gamma_{AB}} dx^A dx^B \tag{A.4}$$

integrated over $0 < \theta < \pi$ and $0 \leq \phi \leq 2\pi$. Since we are interested in the NHEK geometry, and it simplifies the integral, we take the extreme limit of the metric, by $M^2 \to J$. The area is then calculated as

$$A = 2M^2 \int \sin\theta d\theta d\phi \tag{A.5}$$

$$= 8\pi M^2 \tag{A.6}$$

A.2 Angular Velocity

The angular velocity of the Kerr black hole can be given by [9]

$$\Omega = \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \tag{A.7}$$

After some simplification we have the angular velocity as (with Δ as in (3.2));

$$\Omega = \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2 - \lambda a^2 \sin^2\theta} \tag{A.8}$$

We then take the velocity at the horizon, i.e. $r \to r_+$ and therefore $\Delta = 0$ to get

$$\Omega_H = \frac{a}{(r_+^2 + a^2)} = \frac{a}{2Mr_+} \tag{A.9}$$

A.3 Surface Gravity

The surface gravity, κ, of a black hole is defined as the coefficient relating the covariant directional derivative of the horizon normal vector l^μ along itself to l^μ [15] , i.e.

$$l^\nu \bigtriangledown_\nu l^\mu = \kappa l^\mu \tag{A.10}$$

It can also be thought of as the red-shifted acceleration of a particle staying still on the horizon. Although the acceleration of a particle at the horizon is infinite the infinite red-shift at the horizon between time to proper time compensates for this. The surface gravity of the Kerr black hole is then given by [15],

$$\kappa = \frac{1}{2}\frac{r_+ - r_-}{r_+^2 + a^2} = \frac{\sqrt{M^2 - a^2}}{r_+^2 + a^2} \tag{A.11}$$

B Black hole Thermodynamics

To expand the analogies of black hole mechanics to thermodynamics, to a more complete system of thermodynamic laws we need to expand the relations to the first and second laws of thermodynamics. Let us consider the equation relating the mass energy of a black hole with the irreducible mass M_{irr} (related to the area of a black hole since the area cannot decrease) and the angular momentum J. [15]

$$M^2 = M_{irr}^2 + \frac{J^2}{4M_{irr}} \tag{B.1}$$

$$= \frac{A}{16\pi} + \frac{4\pi J^2}{A} \tag{B.2}$$

Taking the total derivative of the mass energy we have

$$dM = \frac{\partial M}{\partial J} dJ + \frac{\partial M}{\partial A} dA \tag{B.3}$$

and from (B.2) we have

$$\frac{\partial M}{\partial J} = \frac{1}{2M}\frac{\partial M^2}{\partial J} \tag{B.4}$$

$$= \frac{4\pi J}{MA} = \Omega_H \tag{B.5}$$

and

$$\frac{\partial M}{\partial A} = \frac{1}{2M}\frac{\partial M^2}{\partial A} \tag{B.6}$$

$$= \frac{1}{2M}\left(\frac{1}{16\pi} - \frac{4\pi J^2}{A^2}\right) \tag{B.7}$$

$$= \frac{\kappa}{8\pi} \tag{B.8}$$

So we now have the equation

$$dM = \Omega_H dJ + \frac{\kappa}{8\pi} dA \tag{B.9}$$

This closely resembles the first law of thermodynamics where the first term on the right hand side is the work term as in a rotating fluid. To complete this we then relate the area to the entropy, by $S_{BH} = \alpha A$. Where α is a constant of dimension inverse length squared. We then have form (B.9) that

$$T_H dS_{BH} = \alpha \frac{\kappa}{8\pi} dA \tag{B.10}$$

so the Hawking temperature is then given by

$$T_H = \frac{\kappa}{8\pi\alpha} \tag{B.11}$$

We then have that the Hawking Temperature as given below [8]

$$T_H = \frac{r_+ - M}{4\pi M r_+} \tag{B.12}$$

and the Bekenstein-Hawking entropy of the NHEK from (1.1) and (A.6) as [8]

$$S_{BH} = 2mMr_+ \tag{B.13}$$

C Calculations

The calculations in determining the Killing vectors as well as their commutation relations were done with Maple 18. The calculations done are given below in order of appearance.

NHEK Killing vectors

> $restart:$

> $with(Physics):$

> $Setup(mathematicalnotation = true):$

> $Setup(dimension = [4, `+`]):$

> $Coordinates(A = (t, r, \text{theta}, \text{phi})):$

Default differentiation variables for d_, D_ and dAlembertian are: $\{A = (t, r, \theta, \phi)\}$

Systems of spacetime Coordinates are: $\{A = (t, r, \theta, \phi)\}$ **(1)**

> $Define(V):$

Defined objects with tensor properties **(2)**

$$> Setup\left(g_{\mu,\nu} = \left[\left[-2\left(\frac{1}{2}+\frac{1}{2}\cos(\theta)^2\right)^2 J r^2\left(1-\frac{2\sin(\theta)}{1+\cos(\theta)^2}\right), 0, 0, \frac{8\left(\frac{1}{2}+\frac{1}{2}\cos(\theta)^2\right)^2 J r\sin(\theta)^2}{\left(1+\cos(\theta)^2\right)^2}\right], \left[0, \frac{2\left(\frac{1}{2}+\frac{1}{2}\cos(\theta)^2\right)^2 J}{r^2}, 0, 0\right], \left[0, 0, 2\left(\frac{1}{2}+\frac{1}{2}\cos(\theta)^2\right)^2 J, 0\right], \left[\frac{8\left(\frac{1}{2}+\frac{1}{2}\cos(\theta)^2\right)^2 J r\sin(\theta)^2}{\left(1+\cos(\theta)^2\right)^2}, 0, 0, \frac{8\left(\frac{1}{2}+\frac{1}{2}\cos(\theta)^2\right)^2 J\sin(\theta)^2}{\left(1+\cos(\theta)^2\right)^2}\right]\right]\right)$$

$$\left[metric = \left\{(1,1) = -2\left(\frac{1}{2}+\frac{\cos(\theta)^2}{2}\right)^2 J r^2\left(1-\frac{2\sin(\theta)}{1+\cos(\theta)^2}\right), (1,4) = \frac{8\left(\frac{1}{2}+\frac{\cos(\theta)^2}{2}\right)^2 J r\sin(\theta)^2}{\left(1+\cos(\theta)^2\right)^2}, (2,2) = \frac{2\left(\frac{1}{2}+\frac{\cos(\theta)^2}{2}\right)^2 J}{r^2}, (3,3) = 2\left(\frac{1}{2}+\frac{\cos(\theta)^2}{2}\right)^2 J, (4,4) = \frac{8\left(\frac{1}{2}+\frac{\cos(\theta)^2}{2}\right)^2 J\sin(\theta)^2}{\left(1+\cos(\theta)^2\right)^2}\right\}\right] \quad \textbf{(3)}$$

> $PDEtools{:-}declare(V(A)):$

$V(t, r, \theta, \phi)$ *will now be displayed as* V **(4)**

> $sys2 := KillingVectors(V_{\sim alpha}, output = equations)$

$$sys2 := \left[\left(V^1\right)_t = -\frac{V^2}{r}, \left(V^1\right)_r = 0, \left(V^1\right)_\theta = 0, \left(V^1\right)_\phi = 0, \left(V^2\right)_t = 0, \left(V^2\right)_r = \frac{V^2}{r}, \left(V^2\right)_\theta = 0, \left(V^2\right)_\phi = 0, \left(V^4\right)_t = 0, \left(V^4\right)_r = 0, \left(V^4\right)_\theta = 0, \left(V^4\right)_\phi = 0, V^3 = 0 \right] \tag{5}$$

```
> with(PDEtools, casesplit, declare) :
> with(DEtools, gensys) :
> sol := pdsolve(sys2)
```

$$sol := \left\{ V^1 = -\_C2\, t + \_C3, V^2 = \_C2\, r, V^3 = 0, V^4 = \_C1 \right\} \tag{6}$$

```
> _C1 := 1 : _C2 := 0 : _C3 := 0 :
  for i from 1 to 4 do:
   E[1, i] := simplify(rhs(sol[i]));
  end do:

> _C1 := 0 : _C2 := 1 : _C3 := 0 :
  for i from 1 to 4 do:
  E[2, i] := simplify(rhs(sol[i]));
  end do:

> _C1 := 0 : _C2 := 0 : _C3 := 1 :
  for i from 1 to 4 do:
   E[3, i] := simplify(rhs(sol[i]));
  end do:

> with(PDEtools, SymmetryCommutator, InfinitesimalGenerator)
```

$$[SymmetryCommutator, InfinitesimalGenerator] \tag{7}$$

```
> for h from 1 to 3 do;
  S[h] := [_ξ_t = E[h, 1], _ξ_r = E[h, 2], _ξ_theta = E[h, 3], _ξ_phi = E[h, 4]];
  end do:

> for k from 1 to 3 do
  K[k] := InfinitesimalGenerator(S[k], u(t, r, phi, theta));
  end do;
```

$$K_1 := f \mapsto \frac{\partial}{\partial \phi} f$$

$$K_2 := f \mapsto -t \left(\frac{\partial}{\partial t} f \right) + r \left(\frac{\partial}{\partial r} f \right)$$

$$K_3 := f \mapsto \frac{\partial}{\partial t} f \tag{8}$$

Killing vectors and their commutation relations for AdS_(3) from the NHEK metric

> restart :

> with(Physics) :

> Setup(mathematicalnotation = true) :

> Setup(dimension = [3, `+`]) :

The dimension and signature of the tensor space are set to: [3, +] **(1)**

> Coordinates(A = (t, r, phi)) :

Default differentiation variables for d_, D_ and dAlembertian are: $\{A = (t, r, \phi)\}$

Systems of spacetime Coordinates are: $\{A = (t, r, \phi)\}$ **(2)**

> Define(V) :

Defined objects with tensor properties **(3)**

$$> Setup\left(g_{\mu,\nu} = \begin{bmatrix} 0 & 0 & r \\ 0 & \frac{1}{r^2} & 0 \\ r & 0 & 1 \end{bmatrix}\right)$$

$$\left[metric = \left\{(1,3) = r,\ (2,2) = \frac{1}{r^2},\ (3,3) = 1\right\}\right] \quad \textbf{(4)}$$

> PDEtools:−declare(V(A)) :

$V(t, r, \phi)$ *will now be displayed as* V **(5)**

> sys2 := KillingVectors($V_{\sim alpha}$, output = equations)

$$sys2 := \left[\left(V^1\right)_t = \frac{-\left(V^3\right)_\phi r - V^2}{r},\ \left(V^1\right)_r = \frac{-\left(V^3\right)_r r^2 - \left(V^2\right)_\phi}{r^3},\ \left(V^1\right)_\phi = -\frac{\left(V^3\right)_\phi}{r},\right. \quad \textbf{(6)}$$

$$\left(V^2\right)_{\phi,\phi} = -\left(V^3\right)_\phi r,\ \left(V^2\right)_t = -\left(V^3\right)_r r^3,\ \left(V^2\right)_r = \frac{V^2}{r},\ \left(V^3\right)_{r,r} = -\frac{2\left(V^3\right)_r}{r},\ \left(V^3\right)_{\phi,r}$$

$$\left. = 0,\ \left(V^3\right)_{\phi,\phi} = -\frac{\left(V^2\right)_\phi}{r},\ \left(V^3\right)_t = 0\right]$$

> KillingVectors($V_{\sim alpha}$)

$$\left\{V^1 = -\frac{\_C1\, t^2}{2} - \_C5\, t - \frac{\_C3\, e^{\phi}}{r} - \frac{\_C4\, e^{-\phi}}{r} - \frac{\_C1}{2 r^2} + \_C6,\ V^2 = r\left(\_C1\, t - \_C3\, e^{\phi}\right.\right. \quad \textbf{(7)}$$

$$\left.\left. + \_C4\, e^{-\phi} + \_C5\right),\ V^3 = \_C2 + \_C3\, e^{\phi} + \_C4\, e^{-\phi} + \frac{\_C1}{r}\right\}$$

> with(PDEtools, casesplit, declare) :

> with(DEtools, gensys) :

> sol := pdsolve(sys2)

$$sol := \left\{V^1 = -\frac{\_C1\, t^2}{2} - \_C5\, t - \frac{\_C3\, e^{\phi}}{r} - \frac{\_C4\, e^{-\phi}}{r} - \frac{\_C1}{2 r^2} + \_C6,\ V^2 = r\left(\_C1\, t\right.\right. \quad \textbf{(8)}$$

$$-\_C3\, e^{\phi} + \_C4\, e^{-\phi} + \_C5\big),\ V^3 = \_C2 + \_C3\, e^{\phi} + \_C4\, e^{-\phi} + \frac{\_C1}{r}\Big\}$$

```
> _C1 := 1 : _C2 := 0 : _C3 := 0 : _C4 := 0 : _C5 := 0 : _C6 := 0 :
  for i from 1 to 3 do:
    E[1, i] := simplify(rhs(sol[i]));
  end do:
```

```
> _C1 := 0 : _C2 := 1 : _C3 := 0 : _C4 := 0 : _C5 := 0 : _C6 := 0 :
  for i from 1 to 3 do:
   E[2, i] := simplify(rhs(sol[i]));
  end do:
```

```
> _C1 := 0 : _C2 := 0 : _C3 := 1 : _C4 := 0 : _C5 := 0 : _C6 := 0 :
  for i from 1 to 3 do:
    E[3, i] := simplify(rhs(sol[i]));
  end do:
```

```
> _C1 := 0 : _C2 := 0 : _C3 := 0 : _C4 := 1 : _C5 := 0 : _C6 := 0 :
  for i from 1 to 3 do:
    E[4, i] := simplify(rhs(sol[i]));
  end do:
```

```
> _C1 := 0 : _C2 := 0 : _C3 := 0 : _C4 := 0 : _C5 := 1 : _C6 := 0 :
  for i from 1 to 3 do:
    E[5, i] := simplify(rhs(sol[i]));
  end do:
```

```
> _C1 := 0 : _C2 := 0 : _C3 := 0 : _C4 := 0 : _C5 := 0 : _C6 := 1 :
  for i from 1 to 3 do:
    E[6, i] := simplify(rhs(sol[i]));
  end do:
```

```
> with(PDEtools, SymmetryCommutator, InfinitesimalGenerator)
```

$$[SymmetryCommutator, InfinitesimalGenerator] \tag{9}$$

```
> for h from 1 to 6 do;
   S[h] := [_ξ_t = E[h, 1], _ξ_r = E[h, 2], _ξ_phi = E[h, 3]];
  end do
```

$$S_1 := \left[\_\xi_t = -\frac{r^2 t^2 + 1}{2\, r^2},\ \_\xi_r = r\, t,\ \_\xi_\phi = \frac{1}{r}\right]$$

$$S_2 := \left[\_\xi_t = 0,\ \_\xi_r = 0,\ \_\xi_\phi = 1\right]$$

$$S_3 := \left[\_\xi_t = -\frac{e^{\phi}}{r},\ \_\xi_r = -r\, e^{\phi},\ \_\xi_\phi = e^{\phi}\right]$$

$$S_4 := \left[\_\xi_t = -\frac{e^{-\phi}}{r},\ \_\xi_r = r\, e^{-\phi},\ \_\xi_\phi = e^{-\phi}\right]$$

$$S_5 := \left[\_\xi_t = -t,\ \_\xi_r = r,\ \_\xi_\phi = 0\right]$$

$$S_6 := [\_\xi_t = 1, \_\xi_r = 0, \_\xi_\phi = 0] \tag{10}$$

```
> for k from 1 to 6 do
  K[k] := InfinitesimalGenerator(S[k], u(t, r, phi));
  end do;
```

$$K_1 := f \mapsto -\frac{(r^2 t^2 + 1)\left(\frac{\partial}{\partial t} f\right)}{2 r^2} + r t \left(\frac{\partial}{\partial r} f\right) + \frac{\frac{\partial}{\partial \phi} f}{r}$$

$$K_2 := f \mapsto \frac{\partial}{\partial \phi} f$$

$$K_3 := f \mapsto -\frac{e^{\phi}\left(\frac{\partial}{\partial t} f\right)}{r} - r e^{\phi}\left(\frac{\partial}{\partial r} f\right) + e^{\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$K_4 := f \mapsto -\frac{e^{-\phi}\left(\frac{\partial}{\partial t} f\right)}{r} + r e^{-\phi}\left(\frac{\partial}{\partial r} f\right) + e^{-\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$K_5 := f \mapsto -t\left(\frac{\partial}{\partial t} f\right) + r\left(\frac{\partial}{\partial r} f\right)$$

$$K_6 := f \mapsto \frac{\partial}{\partial t} f \tag{11}$$

```
> for p from 1 to 6 do;
  for l from p to 6 do;
   if SymmetryCommutator(K[l], K[p], u(t, r, phi)) ≠ (f ↦ 0) then
     print([T[p], T[l]] = SymmetryCommutator(K[l], K[p], u(t, r, phi)))
   end if;
  end do;
  end do;
```

$$[T_1, T_5] = f \mapsto \frac{(r^2 t^2 + 1)\left(\frac{\partial}{\partial t} f\right)}{2 r^2} - r t\left(\frac{\partial}{\partial r} f\right) - \frac{\frac{\partial}{\partial \phi} f}{r}$$

$$[T_1, T_6] = f \mapsto -t\left(\frac{\partial}{\partial t} f\right) + r\left(\frac{\partial}{\partial r} f\right)$$

$$[T_2, T_3] = f \mapsto \frac{e^{\phi}\left(\frac{\partial}{\partial t} f\right)}{r} + r e^{\phi}\left(\frac{\partial}{\partial r} f\right) - e^{\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$[T_2, T_4] = f \mapsto -\frac{e^{-\phi}\left(\frac{\partial}{\partial t} f\right)}{r} + r e^{-\phi}\left(\frac{\partial}{\partial r} f\right) + e^{-\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$[T_3, T_4] = f \mapsto 2 e^{-\phi} e^{\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$[T_5, T_6] = f \mapsto -\left(\frac{\partial}{\partial t} f\right) \tag{12}$$

Killing vectors and their commutation relations for warped AdS_(3) from the NHEK metric

```
> restart
> with(Physics) :
> Setup(mathematicalnotation = true) :
> Setup(dimension = [3, `+`]) :
```

The dimension and signature of the tensor space are set to: [3, +] **(1)**

```
> Coordinates(A = (t, r, phi)) :
```

Default differentiation variables for d_, D_ and dAlembertian are: $\{A = (t, r, \phi)\}$

Systems of spacetime Coordinates are: $\{A = (t, r, \phi)\}$ **(2)**

```
> Define(V) :
```

Defined objects with tensor properties **(3)**

$$> Setup\left(g_{\mu,\nu} = \begin{bmatrix} 0 & 0 & r \\ 0 & \frac{1}{r^2} & 0 \\ r & 0 & 1 \end{bmatrix}\right)$$

$$\left[metric = \left\{(1,3) = r, (2,2) = \frac{1}{r^2}, (3,3) = 1\right\}\right] \quad \textbf{(4)}$$

```
> PDEtools:-declare(V(A)) :
```

$V(t, r, \phi)$ *will now be displayed as* V **(5)**

$$> sys2 := KillingVectors\left(V_{\sim alpha}, output = equations\right)$$

$$sys2 := \left[\left(V^1\right)_t = \frac{-\left(V^3\right)_\phi r - V^2}{r}, \left(V^1\right)_r = \frac{-\left(V^3\right)_r r^2 - \left(V^2\right)_\phi}{r^3}, \left(V^1\right)_\phi = -\frac{\left(V^3\right)_\phi}{r}, \right. \quad \textbf{(6)}$$
$$\left(V^2\right)_{\phi,\phi} = -\left(V^3\right)_\phi r, \left(V^2\right)_t = -\left(V^3\right)_r r^3, \left(V^2\right)_r = \frac{V^2}{r}, \left(V^3\right)_{r,r} = -\frac{2\left(V^3\right)_r}{r}, \left(V^3\right)_{\phi,r}$$
$$\left. = 0, \left(V^3\right)_{\phi,\phi} = -\frac{\left(V^2\right)_\phi}{r}, \left(V^3\right)_t = 0\right]$$

```
> with(PDEtools, casesplit, declare) :
> with(DEtools, gensys) :
> sol := pdsolve(sys2)
```

$$sol := \left\{V^1 = -\frac{\_C1\, t^2}{2} - \_C5\, t - \frac{\_C3\, e^\phi}{r} - \frac{\_C4\, e^{-\phi}}{r} - \frac{\_C1}{2 r^2} + \_C6, V^2 = r\left(\_C1\, t \right.\right. \quad \textbf{(7)}$$
$$\left.\left. - \_C3\, e^\phi + \_C4\, e^{-\phi} + \_C5\right), V^3 = \_C2 + \_C3\, e^\phi + \_C4\, e^{-\phi} + \frac{\_C1}{r}\right\}$$

```
> _C1 := 1 : _C2 := 0 : _C3 := 0 : _C4 := 0 : _C5 := 0 : _C6 := 0 :
  for i from 1 to 3 do:
    E[1, i] := simplify(rhs(sol[i]));
  end do:

> _C1 := 0 : _C2 := 1 : _C3 := 0 : _C4 := 0 : _C5 := 0 : _C6 := 0 :
```

```
for i from 1 to 3 do:
 E[2, i] := simplify(rhs(sol[i]));
 end do:
```

```
> _C1 := 0 : _C2 := 0 : _C3 := 1 : _C4 := 0 : _C5 := 0 : _C6 := 0 :
for i from 1 to 3 do:
  E[3, i] := simplify(rhs(sol[i]));
 end do:
```

```
> _C1 := 0 : _C2 := 0 : _C3 := 0 : _C4 := 1 : _C5 := 0 : _C6 := 0 :
for i from 1 to 3 do:
  E[4, i] := simplify(rhs(sol[i]));
 end do:
```

```
> _C1 := 0 : _C2 := 0 : _C3 := 0 : _C4 := 0 : _C5 := 1 : _C6 := 0 :
for i from 1 to 3 do:
  E[5, i] := simplify(rhs(sol[i]));
 end do:
```

```
> _C1 := 0 : _C2 := 0 : _C3 := 0 : _C4 := 0 : _C5 := 0 : _C6 := 1 :
for i from 1 to 3 do:
  E[6, i] := simplify(rhs(sol[i]));
 end do:
```

```
> with(PDEtools, SymmetryCommutator, InfinitesimalGenerator) :
```

```
> for h from 1 to 6 do;
 S[h] := [_ξ_t = E[h, 1], _ξ_r = E[h, 2], _ξ_phi = E[h, 3]];
 end do:
```

```
> for k  from 1 to 6 do
 K[k] := InfinitesimalGenerator(S[k], u(t, r, phi));
 end do;
```

$$K_1 := f \mapsto -\frac{(r^2 t^2 + 1)\left(\frac{\partial}{\partial t} f\right)}{2 r^2} + r\, t \left(\frac{\partial}{\partial r} f\right) + \frac{\frac{\partial}{\partial \phi} f}{r}$$

$$K_2 := f \mapsto \frac{\partial}{\partial \phi} f$$

$$K_3 := f \mapsto -\frac{e^{\phi}\left(\frac{\partial}{\partial t} f\right)}{r} - r\, e^{\phi}\left(\frac{\partial}{\partial r} f\right) + e^{\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$K_4 := f \mapsto -\frac{e^{-\phi}\left(\frac{\partial}{\partial t} f\right)}{r} + r\, e^{-\phi}\left(\frac{\partial}{\partial r} f\right) + e^{-\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$K_5 := f \mapsto -t\left(\frac{\partial}{\partial t} f\right) + r\left(\frac{\partial}{\partial r} f\right)$$

$$K_6 := f \mapsto \frac{\partial}{\partial t} f \tag{8}$$

```
> for p from 1 to 6 do;
   for l from p to 6 do;
    if SymmetryCommutator(K[l], K[p], u(t, r, phi)) ≠ (f ↦ 0) then
      print([T[p], T[l]] = SymmetryCommutator(K[l], K[p], u(t, r, phi)))
    end if;
   end do;
   end do;
```

$$[T_1, T_5] = f \mapsto \frac{(r^2 t^2 + 1)\left(\frac{\partial}{\partial t} f\right)}{2 r^2} - r t \left(\frac{\partial}{\partial r} f\right) - \frac{\frac{\partial}{\partial \phi} f}{r}$$

$$[T_1, T_6] = f \mapsto -t \left(\frac{\partial}{\partial t} f\right) + r \left(\frac{\partial}{\partial r} f\right)$$

$$[T_2, T_3] = f \mapsto \frac{e^{\phi}\left(\frac{\partial}{\partial t} f\right)}{r} + r\, e^{\phi}\left(\frac{\partial}{\partial r} f\right) - e^{\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$[T_2, T_4] = f \mapsto -\frac{e^{-\phi}\left(\frac{\partial}{\partial t} f\right)}{r} + r\, e^{-\phi}\left(\frac{\partial}{\partial r} f\right) + e^{-\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$[T_3, T_4] = f \mapsto 2\, e^{-\phi} e^{\phi}\left(\frac{\partial}{\partial \phi} f\right)$$

$$[T_5, T_6] = f \mapsto -\left(\frac{\partial}{\partial t} f\right) \tag{9}$$

References

[1] J. M. Bardeen, B. Carter, and S. W. Hawking. The four laws of black hole mechanics. *Comm. Math. Phys.*, 31(2):161–170, 1973.

[2] Jacob Bekenstein. Black holes and entropy. *Phys. Rev. D*, 7:2333–2346, Apr 1973.

[3] Jacob Bekenstein. Generalized second law of thermodynamics in black-hole physics. *Phys. Rev. D*, 9:3292–3300, Jun 1974.

[4] Andrew Strominger and Cumrun Vafa. Microscopic origin of the Bekenstein-Hawking entropy. *Phys.Lett.*, B379:99–104, 1996.

[5] Andrew Strominger. Black hole entropy from near horizon microstates. *JHEP*, 9802:009, 1998.

[6] J. D. Brown and Marc Henneaux. Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity. *Comm. Math. Phys.*, 104(2):207–226, 1986.

[7] Sean M. Carroll. Lecture notes on general relativity. 1997.

[8] Irene Bredberg, Cynthia Keeler, Vyacheslav Lysov, and Andrew Strominger. Cargese Lectures on the Kerr/CFT Correspondence. *Nucl.Phys.Proc.Suppl.*, 216:194–210, 2011.

[9] Robert M. Wald. *General Relativity*. The University od Chicago Press, 1984.

[10] James M. Bardeen and Gary T. Horowitz. The Extreme Kerr throat geometry: A Vacuum analog of AdS(2) x S**2. *Phys.Rev.*, D60:104030, 1999.

[11] Ingemar Bengtsson and Patrik Sandin. Anti de Sitter space, squashed and stretched. *Class.Quant.Grav.*, 23:971–986, 2006.

[12] Monica Guica, Thomas Hartman, Wei Song, and Andrew Strominger. The Kerr/CFT Correspondence. *Phys.Rev.*, D80:124008, 2009.

[13] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Graduate texts in contemporary physics. Springer, New York, 1997.

[14] Monica Guica and Andrew Strominger. Microscopic Realization of the Kerr/CFT Correspondence. *JHEP*, 1102:010, 2011.

[15] Thibault Damour. The Entropy of black holes: A Primer. 2004.