General Relativity and the Black Ring Coordinate System

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1 Introduction

1.1 Background Knowledge

General Relativity was developed by Albert Einstein between 1905 and 1915 [16], it was developed to unify special relativity and Newton's gravitational law as Newton's law alone could not describe certain observed phenomena - such as the small anomalies in the orbit of Mercury.

This lead to the notion of large objects distorting spacetime structure, essentially incorporating gravity as a geometric property of spacetime. The theory of General Relativity explained all observed gravitational phenomena that were previously unexplained, however, it predicted more unobserved phenomena such as gravitational lensing and gravitational waves [18] (both of which have now been observed). Apart from this it also predicted black holes, objects so dense that near them spacetime is distorted to the point where light cannot escape.

1.2 Tensor Calculus and the Einstein Equation

We require a way to describe the geometry of the space we are dealing with hence we define a metric, labelled $g_{\mu\nu}$, on this space, which is a mapping between vectors and one-forms (also known as covariant vectors) at every point[17]. Defining the notion of a metric on a space allows us to measure distances between points, this is important, we know how to measure distances on locally flat spaces, but in curved spaces measuring distance is not quite as simple.Thus we define a 'connection' on the manifold called the Christoffel symbols, ${\Gamma^\alpha}_{\mu\nu}$, which gives us a way to measure distances on a manifold. The definition of the Christoffel symbols in terms of the metric tensor is,

$$ {\Gamma^\alpha}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial\rho g_{\mu\nu}) \tag{1.1} $$

These Christoffel symbols lead the way to very important equations in general relativity, namely the Riemann tensor:

$$ {R^\rho}_{\sigma\mu\nu} = \partial_\mu {\Gamma^\rho}_{\nu\sigma} - \partial_\nu {\Gamma^\rho}_{\mu\sigma} + {\Gamma^\rho}_{\mu\lambda}{\Gamma^\lambda}_{\nu\sigma} - {\Gamma^\rho}_{\nu\lambda}{\Gamma^\lambda}_{\mu\sigma} $$

which has some interesting properties when it's indices are lowered, such as it is antisymmteric and it is invariant under a change in its first two indices. We can also contract the first and third indices to define the Ricci tensor,

$$ R_{\mu\nu} = {R^\lambda}_{\mu\lambda\nu} $$

which is symmetric in its lower indices. Furthermore we can define the Ricci scalar:

$$R = g^{\mu\nu}R_{\mu\nu} = R^{\mu}{}_{\mu}$$

using the Bianchi identities[1] we can now write the Einstein tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \tag{1.2}$$

Einstein then wanted to find an equation that superceded Poisson's equation in curved spacetime for the Newtonian potential,

$$\nabla^2\Phi = 4\pi G\rho \tag{1.3}$$

where ρ is the mass density, the corresponding tensor generalisation of the mass density is the stress energy tensor $T^{\mu\nu}$. In trying to generalize Poisson's equation, it was necessary to consider the physics of the universe, this static cosmology lead to the introduction of a **cosmological constant**, Λ, which can be thought of as the vacuum energy density. The derivation of Einstein's equation is given in Sean Carroll [7], the final result is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu} \tag{1.4}$$

this is the fundamental equation of general relativity. However, for the purposes of this study, we will be looking at Ricci flat, vaccuum solutions ($\Lambda = 0$), this would imply that:

$$R^{\mu\nu} = 0 \Rightarrow G^{\mu\nu} = 0 \tag{1.5}$$

This is a set of second order, non-linear partial differential equations that are very difficult to solve analytically. To solve these equations it is much easier to guess the form of a function (write an ansatz) and hope that by substituting it into equation 1.5 leads to simpler equations that are easier to solve. To write this ansatz one would need to start with a coordinate system adapted to the physics one wants to describe.

1.3 Thermodynamics and AdS/CFT Correspondence

Classically, black holes are perfect absorbers and thus do not emit radiation hence they have a physical temperature of absolute zero [21]. This perfect absorption led to Wheeler's cup of tea [6] - the question of if you dropped a cup of tea into a passing black hole, the initial state is a cup of tea and a black hole, while the final state is a slightly larger black hole with no tea, so where did the information (entropy) go? Bekenstein answered this question using thought experiments [3] and found that

[1] as done in Schutz[17]

the black hole entropy must be proportional to the horizon area. Thus, information is stored along the event horizon. Shortly after this discovery, Hawking showed the black holes do radiate due to quantum effects[12], namely the creation of a particle-antiparticle pair just outside the event horizon, where one of which crosses the event horizon and the other does not in a very short amount of time such that the uncertainty principle is not broken. The radiation as a result of this process was thus called Hawking Radiation.

However this would imply that some quantum information may be destroyed during this process. There is however, a way around this and that is the AdS/CFT correspondence, theorised by Juan Maldacena[13] which provides a relationship between a gravitational theory that is approximately AdS (Anti-de Sitter Space)[2] in d+1 dimension and a non-gravitational quantum field theory in d dimensions. This is an incredibly useful tool in solving difficult quantum gravity problems by reducing them to easier conformal field theory problems and is an active research topic in string theory.

2 General Relativity in 4 Dimensions

2.1 Schwarzschild Solution

Karl Schwarzschild was the first physicist to find a solution to Einstein's equations. As stated in the previous section, it is not easy to find a solution to equation 1.5, however Schwazschild did this by assuming a static, stationary point mass with spherical symmetry. The 4 dimensional Schwarzschild metric is given by[7]:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{2.1}$$

This is a unique solution to Einsteins equations in vacuum as proved by Birkhoff's theorem. There are two singular points, $r = 0$ which is a true singularity and the metric blows up there, and $r = 2GM$ which is only a coordinate singularity and corresponds to the black hole **event horizon**, the surface beyond which a particle cannot escape. The properties of black holes can be graphically illustrated using Penrose diagrams which provide an easy way to understand the structure of the space around a black hole and the causal relation between different points in spacetime. Figure 2.1 shows the Penrose diaram for Schwarszchild, time goes upward in these diagrams. Light rays travel along 45°lines, hence any two points that are not in

[2]As it is shown in the next sections, the near horizon limit of popular black holes is indeed approximately AdS and hence we can study the thermodynamics of a system as this is where information is stored.

this 45°lightcone are called spacelike seperated, for example: if points A and B are spacelike separated and an event occurs at B, an observer at A will see the event at B at a later time as opposed to seeing it almost instantaneously.

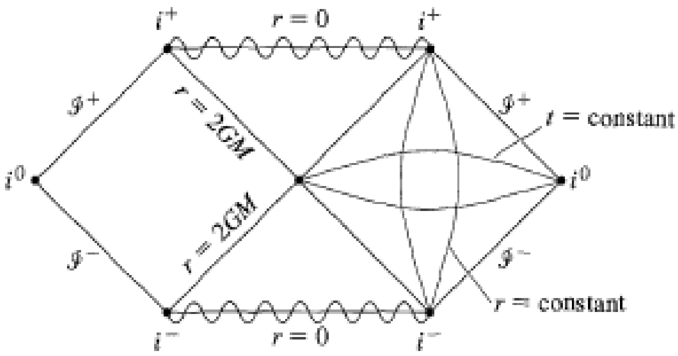

Figure 1: Penrose Diagram for a Schwarzschild Black Hole [17]

The horizontal lines labelled $r = 0$ correspond to the location of the black hole singularity, the lower corresponds to the past and the upper corresponds to the future, the lines labelled $r = 2GM$ correspond to the coordinate singularity that is the event horizon. By analysis of the diagram you can see that any particle outside the event horizon will orbit the black hole but as soon as at passes the event horizon, even if it is a photon, will eventually reach the singularity.

3 Other Black Hole Solutions and Generalisations to D Dimensions

3.1 Schwarzschild in d-Dimensions

D-dimensions refers to $(1, d-1)$ Lorentzian space where we have 1 temporal dimension and $d-1$ spatial dimensions. Referring back to the "Schwarzschild metric": to generalise to higher dimensions it was suggested by Tangherlini[19] to take the radius r to the area radius:

$$r \to r - \frac{16\pi G}{(d-2)(d-3)\omega_{d-2}} \frac{M}{r^{d-3}} \tag{3.1}$$

where ω_{d-2} is the volume of a unit $(d-2)$-dimension sphere. Introducing the 'mass parameter', μ,

$$\mu = \frac{16\pi GM}{(d-2)\omega_{d-2}} \tag{3.2}$$

we can rewrite the Schwarzschild solution generalised to higher dimensions as:

$$ds^2 = -(1 - \frac{\mu}{r^{d-3}})dt^2 + (1 - \frac{\mu}{r^{d-3}})^{-1}dr^2 + r^2 d\Omega^2_{d-2} \tag{3.3}$$

This metric is indeed Ricci flat and hence solves the Einstein equation. There is not much that has been done to the original 4-dimensional solution other than a rescaling of the mass term, M (to μ), and the falloff radius, $\frac{1}{r}$. However, this is the simplest generalisation one can do.

3.2 Kerr

The Kerr solution to Einstein's equation is a generalisation of Schwarszchild for a rotating black hole. It is most simple to look at the metric with natural units, in Boyer-Lindquist coordinates [5]:

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\sin^2\theta}{\rho^2}((r^2 + a^2)d\phi - adt)^2 + \rho^2 d\theta^2 \tag{3.4}$$

where

$$\Delta = r^2 - 2Mr + a^2 \tag{3.5}$$

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{3.6}$$

$$a = \frac{J}{M} \tag{3.7}$$

There is an interesting property of the Kerr event horizon, the even horizon in these coordinates is at $g^{rr} = 0$ which would imply

$$r^2 - 2Mr + a^2 = 0 \Rightarrow r_\pm = M \pm \sqrt{M^2 - a^2} \tag{3.8}$$

hence there are two radii at which the metric blows up which indicates two event horizons. From [5] we see that interesting things happen at $g_{tt} = 0$ (the location at which this is true are known as the static limits). this corresponds to

$$r_{E\pm} = M \pm \sqrt{M^2 - a^2\cos^2\theta} \tag{3.9}$$

Where r_E are the locations of the inner and outer ergospheres, The area inside which, a particle cannot remain stationary but must corotate with the black hole [5]. The outer ergosphere lies outside of the event horizon, as shown in figure 3.2, this region, has an interesting property as it is not inescapable (as it is not beyond the event horizon) but any particle within it, whether it enters the ergosphere in or opposite to the direction of rotation of the black hole, must corotate. Thus it is possible for a particle to extract energy from a rotating black hole by 'skipping' off the ergosphere like a stone would skip on water. This is known as the Penrose Process.

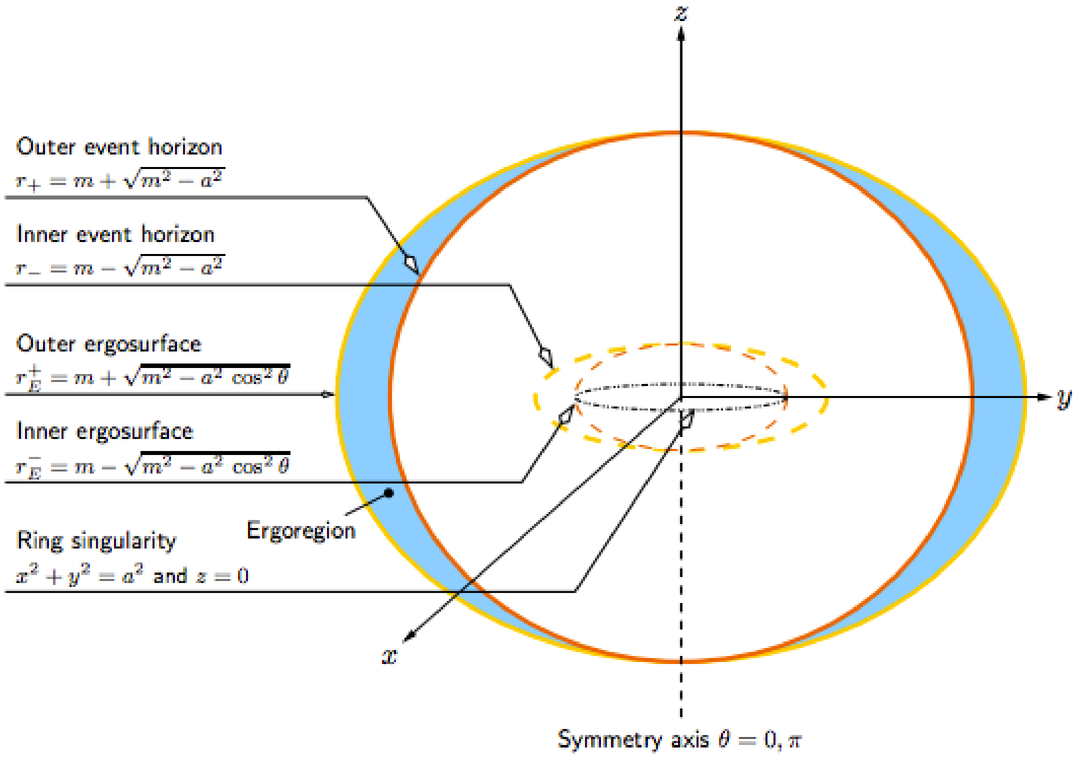

Figure 2: Locations of horizons and the ergosphere for Kerr [20]

3.3 5D Myers-Perry

It can be shown that the generalisation of Kerr to d dimensions differs for even and odd dimensions [15]. However, we will only concern ourselves with the 5D solution with metric:

$$\begin{aligned} ds^2 = -dt^2 + \frac{\mu}{\Sigma}(dt + a\sin^2\theta d\phi_1 + b\cos^2\theta d\phi_2)^2 + \frac{r^2\Sigma}{\Pi - \mu r^2}dr^2 \\ +\Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi_1^2 + (r^2 + b^2)\cos^2\theta d\phi_2^2 \end{aligned} \tag{3.10}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta + b^2\sin^2\theta \tag{3.11}$$

$$\Pi = (r^2 + a^2)(r^2 + b^2) \tag{3.12}$$

This is known as the 5d Myers-Perry black hole. By inspection it can be seen that the metric is singular at the points $\Sigma = 0$ and at $\Pi - \mu r^2 = 0$. Following [15], if $b^2 = a^2$ then the metric is entirely singular. However, if $b^2 \neq a^2$ then the metric is only singular at $\theta = 0$.

the horizon corresponds to the points where $\Pi - \mu r^2 = 0$ which yield the equations:

$$2r_{\pm}^2 = \mu - a^2 - b^2 \pm \sqrt{(\mu - a^2 - b^2)^2 - 4a^2b^2} \tag{3.13}$$

which implies that

$$\mu \geq a^2 + b^2 + 2|ab| \tag{3.14}$$

then by using equation 3.2 and the angular momentum around the x_i, y_i axis, given by $J^{y_i x_i} = \frac{2}{3} M a_i$, we can rewrite 3.14 as

$$M^3 \geq \frac{27\pi}{32}(J_1^2 + J_2^2 + 2|J_1 J_2|) \tag{3.15}$$

where $J_1 \equiv J^{y_1 x_1}$ and similar for J_2.

This is an interesting result, there is a limit placed on the angular momentum of Myers-Perry black holes in 5D, if the angular momenta excede the mass in equation 3.15 then there is no event horizon resulting in a naked singularity.

4 Black Rings

4.1 Non-Uniqueness in 5D

The no hair theorem postulates that a black hole can be completely characterised by only its mass, charge and angular momentum. We know that both Schwarzschild and Kerr are the unique solutions to Einstein's equations in 4D. However, it is possible to construct a 5D solution to Einstein's equations with the same mass and angular momentum of Myers-Perry, that is not Myers-Perry. In 2001, R.Emparan and H.S. Reall published a paper [10] on such a solution. This solution is fascinating, we have discussed how difficult it is to solve Einstein's equations and how it becomes easier when one assumes spherical symmetry, thus when writing an ansatz for a non-spherically symmetric black hole; one can expect a strange coordinate system.

4.2 How a Black Ring is Formed

The black ring solution is not spherically symmetric, so how do we start finding a solution? Let's consider a heuristic arguement: Consider a neutral black string in 5D constructed as the direct product of the Schwarzschild solution and a line. Hence the horizon topology is $\mathbb{R} \times S^2$ then we bend the ends of the string to form a circle with topology $S^1 \times S^2$. However, as one could imagine, this would not be a stable solution as the string's self attraction and tension would cause it to contract. To counter these forces by allowing the string to rotate along this S^1 topology. We have thus created a stable black ring with a horizon topology of $S^1 \times S^2$. This process was done by Emparan and Reall and proves to be a solution of Einstein's equations in 5D.

Hence, we have two solutions to Einstein's equation in 5D. Myers-Perry with horizon topology S^3 [1], and the black ring with horizon topology $S^1 \times S^2$. Which, as stated above, violates the no hair theorem.

4.3 Coordinate System

Before delving into black rings, it would be wise to understand the ring coordinate system and how it is constructed.$SO(4)$, the spatial rotation group in 4+1 dimensions contains two mutually commuting $U(1)$ groups, this means we can have rotations in two seperate planes. Writing the equations for these planes in polar coordinates,

$$x^1 = r_1 \cos\phi, \quad x^2 = r_1 \sin\phi, \quad x^3 = r_2 \cos\psi, \quad x^4 = r_2 \sin\psi \tag{4.1}$$

where ϕ and ψ are our angles of rotation such that we have independent angular rotation about these angles labeled by J_ϕ and J_ψ. As a convention we will describe rings along the (x^3, x^4)-plane with rotations along ψ.

The flat space metric for the coordinates in 4.1 is:

$$d\mathbf{x_4^2} = dr_1^2 + r_1^2 d\phi^2 + dr_2^2 + r_2^2 d\psi^2 \tag{4.2}$$

however, it would be beneficial to rewrite this metric in coordinates that resemble equipotential surfaces of the field created by a black ring. Instead of looking for a scalar field, it turns out that looking at a two-form potential $B_{\mu\nu}$ is much simpler. Hence we consider the black ring as a circular string that acts as an electric force of a 3-form field strength $H = dB$ which satisfies the equation,

$$\partial_\mu(\sqrt{-g} H^{\mu\nu\rho}) = 0 \tag{4.3}$$

the solution of this field $B_{t\psi}$ is given in [11] as

$$B_{t\psi} = -\frac{1}{2}\left(1 - \frac{R^2 + r_1^2 + r_2^2}{\Sigma}\right) \tag{4.4}$$

where

$$\Sigma = \sqrt{(R^2 + r_1^2 + r_2^2)^2 - 4R^2 r_2^2} \tag{4.5}$$

With the help of some computer programming (Shown in Appendix A) it can be shown that this equation for $B_{t\psi}$ satisfies equation 4.3 by simply taking the exterior derivative of $B_{t\psi}$, to calculate H, and substituting. We can then take the dual of the field H to calculate a one-form potential A. The dual of an electric string is a magnetic monopole. Hence, we want to show that $*H = F = dA$, again, using A_ϕ given by Emparan and Reall as,

$$A_\phi = -\frac{1}{2}\left(1 + \frac{R^2 - r_1^2 - r_2^2}{\Sigma}\right) \tag{4.6}$$

and taking the Hodge dual calculated above, it can be shown that this choice of A_ϕ does indeed satisfy this equation. Then a convenient choice of coordinates is

$$y = -\frac{R^2 + r_1^2 + r_2^2}{\Sigma}, \qquad x = \frac{R^2 - r_1^2 - r_2^2}{\Sigma} \tag{4.7}$$

with inverse,

$$r_1 = R\frac{\sqrt{1-x^2}}{x-y}, \qquad r_2 = R\frac{\sqrt{y^2-1}}{x-y} \tag{4.8}$$

To calculate the range of x and y we need to substitute our ring conditions into equation 4.8. We wanted our source at $r_1 = 0$ and $r_2 = R$ which implies,

$$-1 \leq x \leq 1 \tag{4.9}$$

and

$$-\infty \leq y \leq -1 \tag{4.10}$$

it can be seen by simply substituting these ranges into equation 4.8 that as $y \to -\infty$ we have the location of the ring source and as $x \to y \to -1$ we recover asymptotic infinity. $r_2 = 0$ occurs at $y = -1$ and corresponds to the plane of rotation around ψ. Due to the limited range of x values we have two separate rotations around ϕ, $x = 1$ which implies that $r_2 \leq R$ which is then the axis of rotation on the inside of the ring. $x = -1$ is then rotation around the outside of the ring ($r_2 \geq R$). Using these new coordinates and doing a very simple coordinate trasnformation to the "flat space metric" we obtain,

$$d\mathbf{x}_4^2 = \frac{R^2}{(x-y)^2}\left[(y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right] \tag{4.11}$$

which is symmetric under the transformation $(x, y) \to (y, x)$ To study the area near the ring horizon, it is beneficial to change coordinates to r and θ defined as:

$$r = -\frac{R}{y}, \qquad \cos\theta = x \tag{4.12}$$

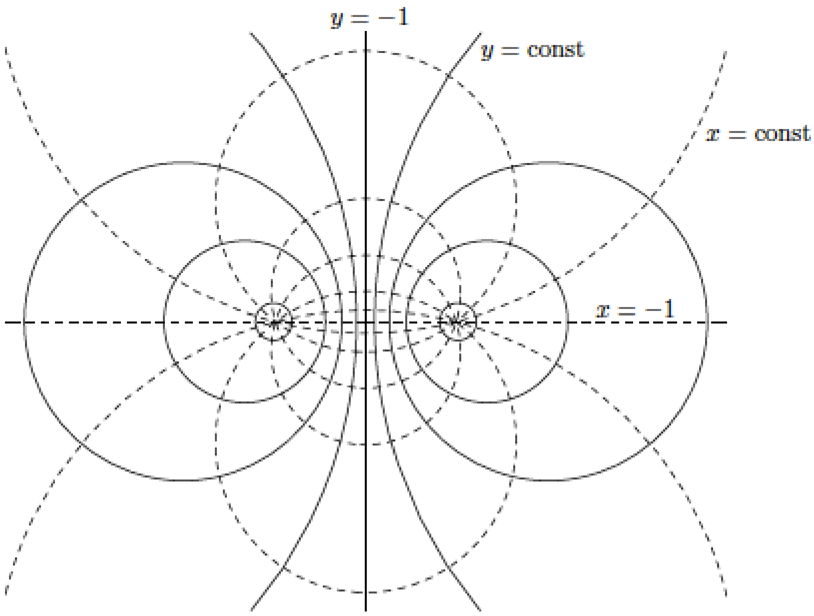

Figure 3: Ring coordinates for four dimensional flat space at constant ϕ, ψ and $\phi+\pi$, $\psi + \pi$ dashed circles corresponds to spheres at constant $|x| \in [0, 1]$, solid circles correspond to spheres at constant $y \in [-\infty, 1]$ spheres at constant y collapse to zero size at the location of the ring of radius R, $y = -\infty$. [11]

substituting in the ranges for x and y above we find the ranges as,

$$0 \leq r \leq R, \qquad 0 \leq \theta \leq \pi \tag{4.13}$$

which then changes equation 4.11 to,

$$d\mathbf{x_4^2} = \left(1 + \frac{r\cos\theta}{R}\right)^{-2} \left[\left(1 - \frac{r^2}{R^2}\right) R^2 d\psi^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.14}$$

From here it is easy to see that

$$\left(1 - \frac{r^2}{R^2}\right) R^2 d\psi^2 \tag{4.15}$$

looks like S^1 topology and

$$r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.16}$$

looks like S^2 topology. Hence in these coordinates we have ring-like topology, $S^1 \times S^2$. If we take equation 4.14 and look at the limit $r \to \infty$, we have that the metric is

$$d\mathbf{x_4^{2\prime}} \approx \left(1 + \frac{r^2}{R^2}\right) dr^2 + r^2 d\Omega^2 \tag{4.17}$$

up to a constant. This is the Ads metric [22], thus we call the black ring topology asymptotically AdS. The metric has a singularity at $r = R$ which corresponds to the rotation around ψ, in AdS this singular point corresponds to the AdS horizon.

4.4 Neutral Black Ring

The metric for the neutral black ring looks similar to equation 4.11 with the exception that now we place a black ring into the coordinate system and we no longer have flat space but curvature. The most convenient form for the metric can be written as:

$$ds^2 = -\frac{F(y)}{F(x)}\left(dt - CR\frac{1+y}{F(y)}d\psi\right)^2 + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right] \tag{4.18}$$

where,

$$F(\zeta) = 1 + \lambda\zeta, \qquad G(\zeta) = (1-\zeta^2)(1+\nu\zeta) \tag{4.19}$$

and

$$C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}} \tag{4.20}$$

λ and ν are dimensionless parameters which have range

$$0 < \nu \leq \lambda < 1 \tag{4.21}$$

substituting this into Einstein's equations with no cosmological constant we find that $G^{\mu\nu} = 0$ and hence (4.18) is indeed a solution (See Appendix B).

In (4.18), both x and y vary the same as in equations 4.9 and 4.10 respectively. Hence there are certains values of x and y such that the black ring metric blows up, specifically around the orbits of $\partial/\partial\phi$ and $\partial/\partial\psi$ at $x = y = -1$. This is most likely a coordinate singularity, to examine this further let us set $dy = d\psi = dt = 0$ and expand around $x = -1 + \epsilon$.

The relevant part of the metric is then

$$ds^2 = \frac{1}{(x-y)^2}\left[\frac{F(x)}{G(x)}dx^2 + G(x)d\phi^2\right] \tag{4.22}$$

Taylor expanding around ϵ and keeping only the first order terms,

$$\begin{aligned}\frac{F(x)}{G(x)} &= \frac{1+\lambda x}{(1-x^2)(1+\nu x)} \\ &= \frac{1-\lambda+\lambda\epsilon}{(2\epsilon-\epsilon^2)(1-\nu+\nu\epsilon)} \\ &\approx \frac{1}{2}\frac{1-\lambda}{1-\nu}\frac{1}{\epsilon}+\mathcal{O}(1)\end{aligned} \tag{4.23}$$

similarly,

$$\begin{aligned}G(x) &= (2\epsilon-\epsilon^2)(1-\nu+\nu\epsilon) \\ &\approx 2\epsilon(1-\nu)+\mathcal{O}(\epsilon^2)\end{aligned} \tag{4.24}$$

the prefactor term in 4.22 is $1/\epsilon^2$. Rewriting this,

$$ds^2 = \frac{1}{2}\frac{1-\lambda}{1-\nu}\frac{d\epsilon^2}{\epsilon^3}+\frac{2}{\epsilon}(1-\nu)d\phi^2 \tag{4.25}$$

we want to write this in the form

$$ds^2 = dr^2 + A^2r^2d\phi^2 \tag{4.26}$$

because then we have rescaled our ϕ parameter and hence we will avoid the singularity. The rescaling of angular variables is done to avoid a **conincal singularity** which is a singularity of the coordinate system and not a true singularity of the metric.

the periodicity of ϕ will be

$$\Delta\phi = \frac{2\pi}{A} \tag{4.27}$$

thus,

$$\begin{aligned} dr^2 &= \frac{1}{2}\frac{1-\lambda}{1-\nu}\frac{d\epsilon^2}{\epsilon^3} \\ \Rightarrow \qquad dr &= \pm\sqrt{\frac{1}{2}\frac{1-\lambda}{1-\nu}}\frac{d\epsilon}{\epsilon^{3/2}}\end{aligned} \tag{4.28}$$

choosing the minus sign and integrating,

$$\begin{aligned} \epsilon &= \frac{2}{r^2}\frac{1-\lambda}{1-\nu} \\ \Rightarrow \qquad ds^2 &= dr^2+\frac{(1-\nu)^2}{1-\lambda}r^2d\phi^2 \end{aligned} \tag{4.29}$$

$$\Rightarrow \qquad \Delta\phi = 2\pi\frac{\sqrt{1-\lambda}}{1-\nu} \tag{4.30}$$

due to the symmetric interchanging of x and y, $\Delta\phi = \Delta\psi$. at $x = +1$[3] all we have is a change in sign (as we are now Taylor expanding around $x = 1 + \epsilon$) and hence we have

$$\Delta\phi = 2\pi\frac{\sqrt{1+\lambda}}{1+\nu} \tag{4.31}$$

equating (4.30) and (4.31),

$$\lambda = \frac{2\nu}{1+\nu^2} \tag{4.32}$$

This relationship between λ and ν imposes that there are no conical singularities of the metric: not only that, it also leaves only two independent variables, R which can be interpreted as the ring radius and ν which could be interpreted as the mass. This makes sense as if we have the angular momentum $L = R \times p$ (where p is the momentum dependent on ν) then L must be such that the centrifugal force must balance the self attraction of the ring, hence there can only be two free parameters, thus what we have derived here in equation 4.32 is that the system is balanced under no external forces.

At $G(y) = 0$ the metric becomes singular, this is the point $y = -1/\nu$. However, doing the transformation:

$$dt = dv - CR\frac{1+y}{G(y)\sqrt{-F(y)}}dy, \qquad d\psi = d\psi' + \frac{\sqrt{-F(y)}}{G(y)}dy \tag{4.33}$$

we obtain the metric

$$ds^2 = -\frac{F(y)}{F(x)}\left(dv - CR\frac{1+y}{F(y)}d\psi'\right) + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi'^2 + 2\frac{dy\,d\psi'}{\sqrt{-F(y)}} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right] \tag{4.34}$$

which no longer has any singularity at $y = -1/\nu$. If we define,

$$V = \frac{\partial}{\partial v} + \Omega\left(\frac{\Delta\psi}{2\pi}\right)\frac{\partial}{\partial\psi'} \tag{4.35}$$

where

$$\Omega = \frac{1}{R}\sqrt{\frac{\lambda-\nu}{\lambda(1+\lambda)}} \tag{4.36}$$

and calculate its inner product along the constant surface $y = -1/\nu$, we find (as in Appendix C[4])

[3] $y = +1$ is not in the range of our coordinates so we do not consider it.

[4] The expression in the appendix is exactly zero, Maple is not sure how to simplify it, however it is easily solvable by hand.

$$V_\mu V^\mu = 0 \tag{4.37}$$

Thus V is a null vector. Also, if we consider V before the transformation where,

$$V = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \psi'} \tag{4.38}$$

then it is easy to see by the coordinate transformation in equation 4.33 that $V_\mu dx^\mu$ is a positive multiple of dy [5]. And hence it follows that $y = -1/\nu$ is a Killing horizon with angular velocity Ω.[6]

The near horizon metric at a constant time slice in the coordinates given by equation 4.33 is

$$ds^2 \sim R^2 \left\{ C^2 \frac{(1-1/\nu)^2}{F(-1/\nu)F(x)} d\psi^2 + \frac{F(x)}{(x-1/\nu)^2} \left[\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right] \right\} \tag{4.39}$$

here it can be seen that the ψ part of the metric has S^1 topology with different values of ring radius depending on the value of x, in general, the S^1 radius is:

$$R\sqrt{\frac{\lambda(1+\lambda)}{\nu(1+\lambda x)}} \tag{4.40}$$

let us consider the $x\phi$ part of the metric which is conformal to

$$ds^2_{x\phi} = \frac{dx^2}{1-x^2} + \frac{(1-x^2)(1+\nu x)^2}{1+\lambda x} d\phi^2 \tag{4.41}$$

doing the transformation $x = -\cos\theta$ with $0 \leq \theta \leq \pi$ we obtain

$$ds^2_{x\phi} = d\theta^2 + \frac{(1-\nu\cos\theta)^2}{1-\lambda\cos\theta} \sin^2\theta d\phi^2 \tag{4.42}$$

For this to be regular at the boundary conditions of $\theta = 0$ and $\theta = \pi$, the the periodicity of ϕ is exactly the same as calculated in equations 4.30 and 4.31. This periodicity leads to a distortion of the 2-sphere metric and hence we say it has distorted S^2 topology. Hence the near horizon topology of the black ring is $S^1 \times S^2$ which is non-spherical, a solution which in 4d wouldn't be allowed! A visualisation [7] of the distortion of the black ring for the same mass and different values of ν is given in figure 4.4.

---

[5] both dt and $d\psi$ contain a transformation of dy

[6] The notion that a Killing horizon requires a vector V be null on a surface at constant y and $V_\mu dx^\mu$ be a multiple dy is a simplification of the definition by Emparan and Reall[11]. For a more rigorous definition, see p244 of Carrol [7].

[7] This is an isometric embedding. It is a visualisation of the 2-sphere in 3d Euclidian space.

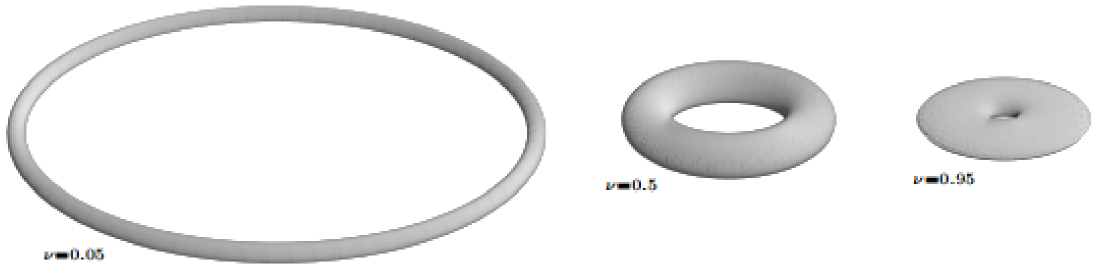

Figure 4: Black rings of same mass with different values of ν. The plot shows the S^2 cross section with the size of the S^1 approximated as the inner radius of the horizon. the rings with $\nu = 0.05$ and $\nu = 0.95$ have the same horizon area [8]

4.5 Physical Magnitudes and Non-Uniqueness

The physical parameters for the black ring, namely: mass (M), angular momentum (J), temperature (T) and horizon area ($\mathcal{A}_H$) are[8]

$$M = \frac{3\pi R^2}{4G}\frac{\lambda}{1-\nu} \tag{4.43}$$

$$J = \frac{\pi R^3}{2G}\frac{\sqrt{\lambda(\lambda-\nu)(1+\lambda)}}{(1-\ nu)^2} \tag{4.44}$$

$$T = \frac{1}{4\pi R}\frac{1+\nu}{\sqrt{\nu}}\sqrt{\frac{1-\lambda}{\lambda(1+\lambda)}} \tag{4.45}$$

$$\mathcal{A}_H = 8\pi^2 R^3 \frac{\nu^{3/2}\sqrt{\lambda(1-\lambda^2)}}{(1-\nu)^2(1+\nu)} \tag{4.46}$$

To fix the a scale we define the dimensionless variables: j, the reduced angular momentum, and a_H, the reduced horizon area, as

$$j = \sqrt{\frac{27\pi}{32G}}\frac{J}{M^{3/2}}, \qquad a_H = \frac{3}{16}\sqrt{\frac{3}{\pi}}\frac{\mathcal{A}_H}{(GM)^{3/2}} \tag{4.47}$$

Requiring that the black ring must be balanced, by equation 4.32, we obtain the equations

$$a_H = 2\sqrt{\nu(1-\nu)}, \qquad j^2 = \frac{(1+\nu)^3}{8\nu} \tag{4.48}$$

From this equation and figure 4.4[8] it is natural to classify black rings using the parameter ν (which in this case can be thought of as the shape parameter) as:

- *Thin black rings*, where $0 < \nu < 1/2$ and $j \to \infty$ as $\nu \to 0$ which corresponds to very thin rings.
- *Fat black rings*, where $1/2 < \nu < 1$ where $\nu = 1 \Rightarrow a_H = 0$ and the solution results in a naked singularity.

For the Myers-Perry black hole in equation 3.10, the relationship between the dimensionless parameters in 4.47 is given in [14] as

$$a_H = 2\sqrt{2(1 - j^2)} \tag{4.49}$$

plotting 4.47 against 4.49 as in figure 4.5 we see some interesting things. For one,

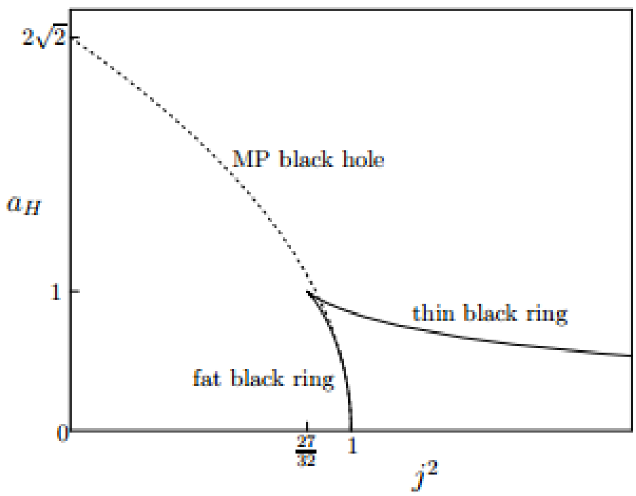

Figure 5: Graph comparing the reduced horizon area as a function of the square of the reduced angular momentum for the black ring and Myers-Perry black hole[11]

the black ring angular momentum is bounded below but not above, unlike the case in Myers-Perry where there are both upper and lower bounds for the angular momentum. Furthermore, $j^2 = 1$ is the location of the naked singularity and $j^2 \to \infty$ are rapidly spinning black rings with decreasing horizon. However, the most interesting location is the part of the graph with $27/32 < j^2 < 1$ where there is the existence of two black rings and a Myers-Perry black hole with the same mass and spin. A clear violation of uniqueness!

[8] It can now also be seen why the black rings with $\nu = 0.05$ and $\nu = 0.95$ have the same horizon area as a_H is invariant under the change $\nu \to 1 - \nu$

5 Further Study

Due to the lack of uniqueness in 5d, we currently do not have a general method to find solutions to Einstein's equations in higher dimensions. In 4d we can assume spherical symmetry and the solution is unique, however, as shown in the previous sections we can have a solution with spherical symmetric (Kerr) and a ring solution for the same system. Thus it would be beneficial to further study higher dimensional spacetimes and find a general solution for horizon topology. Black rings are also relatively new and unstudied. The existence of such a solution leads to even more interesting solutions, such as the Black Saturn[9], which consists of a spherical black hole surrounded by a rotating black ring, which turns out to be a stable solution due to the angular momentum of the Black Saturn.

Furthermore, studying the near horizon limit and microstates of the black ring system has been discussed [4, 2] and further research into black rings and AdS/CFT could lead to interesting results in the dual field theory.

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A Proving Hodge Dual

$B_{t\psi}$

> $B_{t\psi} := -\frac{1}{2}\left(1 - \frac{\left(R^2 + r_1^2 + r_2^2\right)}{\text{Sigma}}\right); B_{\psi t} := \frac{1}{2}\left(1 - \frac{\left(R^2 + r_1^2 + r_2^2\right)}{\text{Sigma}}\right)$

$$B_{t\psi} := -\frac{1}{2} + \frac{1}{2}\,\frac{R^2 + r_1^2 + r_2^2}{\Sigma}$$

$$B_{\psi t} := \frac{1}{2} - \frac{1}{2}\,\frac{R^2 + r_1^2 + r_2^2}{\Sigma} \qquad \textbf{(1.1)}$$

> $\text{Sigma} := \text{sqrt}\left(\left(r_1^2 + r_2^2 + R^2\right)^2 - 4 \cdot R^2 \cdot r_2^2\right)$

$$\Sigma := \sqrt{\left(R^2 + r_1^2 + r_2^2\right)^2 - 4\,R^2\,r_2^2} \qquad \textbf{(1.2)}$$

>

> $H[r1, t, \psi] := \frac{1}{6} \cdot \left(\mathit{diff}\left(B_{t\psi}, r_1\right) - \mathit{diff}\left(B_{\psi t}, r_1\right)\right)$

$$H_{r1,\,t,\,\psi} := \frac{1}{3}\,\frac{r_1}{\sqrt{\left(R^2 + r_1^2 + r_2^2\right)^2 - 4\,R^2\,r_2^2}} - \frac{1}{3}\,\frac{\left(R^2 + r_1^2 + r_2^2\right)^2 r_1}{\left(\left(R^2 + r_1^2 + r_2^2\right)^2 - 4\,R^2\,r_2^2\right)^{3/2}} \qquad \textbf{(1.3)}$$

> $H[r1, \psi, t] := \frac{1}{6} \cdot \left(\mathit{diff}\left(B_{\psi t}, r_1\right) - \mathit{diff}\left(B_{t\psi}, r_1\right)\right)$

$$H_{r1,\,\psi,\,t} := -\frac{1}{3}\,\frac{r_1}{\sqrt{\left(R^2 + r_1^2 + r_2^2\right)^2 - 4\,R^2\,r_2^2}} + \frac{1}{3}\,\frac{\left(R^2 + r_1^2 + r_2^2\right)^2 r_1}{\left(\left(R^2 + r_1^2 + r_2^2\right)^2 - 4\,R^2\,r_2^2\right)^{3/2}} \qquad \textbf{(1.4)}$$

> $H[r2, t, \psi] := \frac{1}{6} \cdot \left(\left(\mathit{diff}\left(B_{t\psi}, r_2\right) - \mathit{diff}\left(B_{\psi t}, r_2\right)\right)\right)$

$$H_{r2,\,t,\,\psi} := \frac{1}{3}\,\frac{r_2}{\sqrt{\left(R^2 + r_1^2 + r_2^2\right)^2 - 4\,R^2\,r_2^2}} - \frac{1}{12}\,\frac{\left(R^2 + r_1^2 + r_2^2\right)\left(4\left(R^2 + r_1^2 + r_2^2\right) r_2 - 8\,R^2\,r_2\right)}{\left(\left(R^2 + r_1^2 + r_2^2\right)^2 - 4\,R^2\,r_2^2\right)^{3/2}} \qquad \textbf{(1.5)}$$

> $Hup[r1, t, \psi] := -\frac{1}{r_2^2} \cdot H[r1, t, \psi]$

$$Hup_{r1,\,t,\,\psi} := \qquad \textbf{(1.6)}$$

$$-\frac{\frac{1}{3}\frac{r_1}{\sqrt{\left(R^2+r_1^2+r_2^2\right)^2-4R^2r_2^2}}-\frac{1}{3}\frac{\left(R^2+r_1^2+r_2^2\right)^2 r_1}{\left(\left(R^2+r_1^2+r_2^2\right)^2-4R^2r_2^2\right)^{3/2}}}{r_2^2}$$

> $Hup[r2, t, \psi] := -\frac{1}{r_2^2}\cdot H[r2, t, \psi]$

$$Hup_{r2, t, \psi} := -\frac{1}{r_2^2}\left(\frac{1}{3}\frac{r_2}{\sqrt{\left(R^2+r_1^2+r_2^2\right)^2-4R^2r_2^2}} - \frac{1}{12}\frac{\left(R^2+r_1^2+r_2^2\right)\left(4\left(R^2+r_1^2+r_2^2\right)r_2-8R^2r_2\right)}{\left(\left(R^2+r_1^2+r_2^2\right)^2-4R^2r_2^2\right)^{3/2}}\right) \quad \textbf{(1.7)}$$

> $simplify\left(expand\left(\left(diff\left(r_1\cdot r_2\cdot Hup[r1, t, \psi], r_1\right)+diff\left(r_1\cdot r_2\cdot Hup[r2, t, \psi], r_2\right)\right)\right)\right)$

$$0 \quad \textbf{(1.8)}$$

Exterior Derivative of A_ϕ

> $A_\phi := -\frac{1}{2}\left(1+\frac{\left(R^2-r_1^2-r_2^2\right)}{\text{Sigma}}\right)$

$$A_\phi := -\frac{1}{2}-\frac{1}{2}\frac{R^2-r_1^2-r_2^2}{\sqrt{\left(R^2+r_1^2+r_2^2\right)^2-4R^2r_2^2}} \quad \textbf{(2.1)}$$

> $simplify\left(-\frac{1}{2}\cdot diff\left(A_\phi, r_1\right)\right)$

$$-\frac{r_1R^2\left(R^2+r_1^2-r_2^2\right)}{\left(R^4+2R^2r_1^2-2R^2r_2^2+r_1^4+2r_1^2r_2^2+r_2^4\right)^{3/2}} \quad \textbf{(2.2)}$$

> $simplify\left(-\frac{1}{2}\cdot diff\left(A_\phi, r_2\right)\right)$

$$-\frac{2r_2R^2r_1^2}{\left(R^4+2R^2r_1^2-2R^2r_2^2+r_1^4+2r_1^2r_2^2+r_2^4\right)^{3/2}} \quad \textbf{(2.3)}$$

> $simplify\left(\frac{r_1\cdot r_2}{2}\cdot\left(3\cdot Hup[r2, t, \psi]\right)\right)$

$$-\frac{r_1R^2\left(R^2+r_1^2-r_2^2\right)}{\left(R^4+2R^2r_1^2-2R^2r_2^2+r_1^4+2r_1^2r_2^2+r_2^4\right)^{3/2}} \quad \textbf{(2.4)}$$

> $simplify\left(-\frac{r_1\cdot r_2}{2}\cdot\left(3\cdot Hup[r1, t, \psi]\right)\right)$

$$-\frac{2\, r_2\, R^2\, r_1^{\,2}}{\left(R^4 + 2\, R^2\, r_1^{\,2} - 2\, R^2\, r_2^{\,2} + r_1^{\,4} + 2\, r_1^{\,2}\, r_2^{\,2} + r_2^{\,4}\right)^{3/2}} \qquad \textbf{(2.5)}$$

> **(2.2)−(2.4)**

$$0 \qquad \textbf{(2.6)}$$

> **(2.3)−(2.5)**

$$0 \qquad \textbf{(2.7)}$$

>

B Black Ring Metric Solution to Einstein's Equations

```
> with(Physics) :
> with(LinearAlgebra) :
> Setup(mathematicalnotation = true) :
> Setup(dimension = 5) :
```

The dimension and signature of the tensor space are set to: [5, - - - - +] **(1)**

```
> Coordinates(X = [ψ, ϕ, x, y, t]) :
```

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (\psi, \phi, x, y, t)\}$

Systems of spacetime Coordinates are: $\{X = (\psi, \phi, x, y, t)\}$ **(2)**

```
> G := Array(symmetric, 1 ..5, 1 ..5) :
> F(z) := 1 + λ·z:
> H(z) := (1 − z^2) (1 + ρ·z) :
> C := sqrt(λ·(λ − ρ)·(1 + λ)/(1 − λ)) :
> #coords order: ψ, ϕ, x, y, t
```

$$> ds2 := -\frac{F(y)}{F(x)}\left(d\_(t) - \frac{C\cdot R\cdot(1+y)}{F(y)}d\_(\psi)\right)^2 + \frac{R^2}{(x-y)^2}\cdot F(x)\cdot\left(-\frac{H(y)}{F(y)}\left(d\_(\psi)\right)^2 - \frac{\left(d\_(y)\right)^2}{H(y)} + \frac{\left(d\_(x)\right)^2}{H(x)} + \frac{H(x)}{F(x)}\left(d\_(\phi)\right)^2\right)$$

$$ds2 := -\frac{(\lambda y+1)\left(\partial(t) - \frac{\sqrt{\frac{\lambda(\lambda-\rho)(1+\lambda)}{1-\lambda}}\,R\,(1+y)\,(\partial(\psi))}{\lambda y+1}\right)^2}{\lambda x+1} + \frac{1}{(x-y)^2}\left(R^2(\lambda x+1)\left(-\frac{(-y^2+1)(\rho y+1)(\partial(\psi))^2}{\lambda y+1} - \frac{(\partial(y))^2}{(-y^2+1)(\rho y+1)} + \frac{(\partial(x))^2}{(-x^2+1)(\rho x+1)} + \frac{(-x^2+1)(\rho x+1)(\partial(\phi))^2}{\lambda x+1}\right)\right) \quad \textbf{(3)}$$

```
> Setup(metric = ds2) :
> g_[ ]
```

$$g_{\mu,\nu} = \left[\left[\frac{1}{(x-y)^2(\lambda-1)(\lambda x+1)}\left((1+y)\left((y^2+(\rho x^2-2x+1)y+(-\rho+2)x^2 - 2x)\lambda^2 + ((-\rho+1)y^2+(-\rho x^2+(4\rho-2)x-\rho)y-\rho x^2+2x-1)\lambda - (y-1)(\rho y+1)\right)R^2\right), 0, 0, 0, \frac{\sqrt{\frac{\lambda(-\lambda+\rho)(1+\lambda)}{\lambda-1}}\,R\,(1+y)}{\lambda x+1}\right], \quad \textbf{(4)}$$

$$\left[0,\ -\frac{R^2\,(x^2-1)\,(\rho\,x+1)}{(x-y)^2},\ 0,\ 0,\ 0\right],$$

$$\left[0,\ 0,\ -\frac{R^2\,(\lambda\,x+1)}{(x-y)^2\,(x^2-1)\,(\rho\,x+1)},\ 0,\ 0\right],$$

$$\left[0,\ 0,\ 0,\ \frac{R^2\,(\lambda\,x+1)}{(x-y)^2\,(y^2-1)\,(\rho\,y+1)},\ 0\right],$$

$$\left[\frac{\sqrt{\frac{\lambda\,(-\lambda+\rho)\,(1+\lambda)}{\lambda-1}}\,R\,(1+y)}{\lambda\,x+1},\ 0,\ 0,\ 0,\ \frac{-\lambda\,y-1}{\lambda\,x+1}\right]\Bigg]$$

> *Einstein*[ ]

$$G_{\mu,\,\nu}=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\tag{5}$$

>

C V is Null

> Omega := $\frac{1}{R}\cdot\text{sqrt}\left(\frac{(\text{lambda}-\text{mu})}{\text{lambda}(1+\text{lambda})}\right)$:

> $deltapsi$:= $\left(\frac{2\cdot\text{Pi}\cdot\text{sqrt}(1-\text{lambda})}{1-\text{mu}}\right)$:

> $F(k)$:= $1+\text{lambda}\cdot k$:

> $G(k)$:= $(1-k^2)\,(1+\text{mu}\cdot k)$:

> C := sqrt$\left(\frac{\text{lambda}\cdot(\text{lambda}-\text{mu})\,(1+\text{lambda})}{1-\text{lambda}}\right)$:

>

>

> g_mumu := $-\frac{F(y)}{F(x)}$:

> g_psimu := $\left(\frac{(2\,C\cdot R\cdot(1+y))}{F(x)}\right)$:

> g_mupsi := g_psimu :

> g_psipsi := $\left(-\frac{C^2\cdot R^2\cdot(1+y)^2}{F(y)\cdot F(x)}-\frac{R^2}{(x-y)^2}\left(\frac{F(x)\cdot G(y)}{F(y)}\right)\right)$:

> $Vmumu$:= $g\_mumu+\frac{g\_psimu\cdot\text{Omega}\cdot deltapsi}{2\cdot\text{Pi}}+g\_psipsi\cdot\left(\frac{\Omega\cdot deltapsi}{2\cdot\text{Pi}}\right)^2$:

> $factor(\%)$

$$-\frac{2\,\pi\sqrt{1-\lambda}}{-1+\mu} \tag{1}$$

> $subs\left(y=-\frac{1}{\text{mu}},\%\right)$

$$-\frac{1}{(-1+\mu)^2\,\lambda(1+\lambda)\left(x+\frac{1}{\mu}\right)^2(\lambda x+1)}\Bigg(\lambda+2\,\lambda(1 \tag{2}$$

$$+\lambda)\sqrt{-\frac{\lambda\,(\lambda-\mu)\,(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}\,\mu x^2$$

$$+\frac{2\,\lambda(1+\lambda)\sqrt{-\frac{\lambda\,(\lambda-\mu)\,(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}\,x^2}{\mu}$$

$$-\frac{8\,\lambda(1+\lambda)\sqrt{-\frac{\lambda\,(\lambda-\mu)\,(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}\,x}{\mu}-2\,\lambda(1+\lambda)\,\lambda x$$

$$+\frac{4\lambda(1+\lambda)\lambda x}{\mu}+\lambda(1+\lambda)-\frac{\lambda(1+\lambda)\lambda x^2}{\mu}-\frac{2\lambda}{\mu}-\frac{2\lambda(1+\lambda)}{\mu}-\frac{2\lambda^3 x}{\mu^2}$$

$$-\frac{2\lambda^2 x}{\mu^2}-\lambda(1+\lambda)\lambda\mu x^2+2\lambda(1+\lambda)\mu x+\frac{\lambda(1+\lambda)}{\mu^2}+\frac{2\lambda^3}{\mu^2}+2\lambda(1+\lambda)\lambda x^2$$

$$+\frac{\lambda}{\mu^2}-\frac{\lambda(1+\lambda)\lambda}{\mu^3}+\frac{4\lambda^3 x}{\mu}+\frac{3\lambda^2}{\mu^2}+\frac{2\lambda(1+\lambda)x}{\mu}-2\lambda\mu x^2-\frac{\lambda(1+\lambda)\lambda}{\mu}$$

$$+\frac{2\lambda(1+\lambda)\lambda}{\mu^2}-4\lambda(1+\lambda)x-\frac{\lambda^3}{\mu^3}-\frac{\lambda^2}{\mu^3}-\frac{\lambda^3}{\mu}-\frac{3\lambda^2}{\mu}+4\lambda(1$$

$$+\lambda)\sqrt{-\frac{\lambda(\lambda-\mu)(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}\,x-\frac{\lambda^3x^2}{\mu}+3\lambda^2x^2-\frac{\lambda^2x^2}{\mu}$$

$$+\lambda x^2+\frac{6\lambda^2 x}{\mu}+\frac{2\lambda x}{\mu}-\frac{2\lambda(1+\lambda)\lambda x}{\mu^2}$$

$$-\frac{4\lambda(1+\lambda)\sqrt{-\frac{\lambda(\lambda-\mu)(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}}{\mu^2}$$

$$+\frac{2\lambda(1+\lambda)\sqrt{-\frac{\lambda(\lambda-\mu)(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}}{\mu^3}+2\lambda^3x^2-2\lambda^3x$$

$$-6\lambda^2x+\lambda(1+\lambda)x^2+\lambda^2+2\lambda^2\mu x+2\lambda\mu x+\lambda(1+\lambda)\mu^2x^2-2\lambda(1+\lambda)\mu x^2$$

$$-\lambda^3\mu x^2+\lambda^2\mu^2x^2-3\lambda^2\mu x^2+\lambda\mu^2x^2-4\lambda(1$$

$$+\lambda)\sqrt{-\frac{\lambda(\lambda-\mu)(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}\,x^2-4\lambda x$$

$$+\frac{2\lambda(1+\lambda)\sqrt{-\frac{\lambda(\lambda-\mu)(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}}{\mu}$$

$$+\frac{4\lambda(1+\lambda)\sqrt{-\frac{\lambda(\lambda-\mu)(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}\,x}{\mu^2}\Bigg)$$

> *factor*(%)

$$-\frac{1}{\mu(\lambda x+1)\lambda(1+\lambda)}\left(2\sqrt{-\frac{\lambda(\lambda-\mu)(1+\lambda)}{-1+\lambda}}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}}\sqrt{1-\lambda}\,\lambda(1+\lambda)-\lambda^3\right.$$
$$\left.+\lambda^2\mu-\lambda(1+\lambda)\lambda+\lambda(1+\lambda)\mu-\lambda^2+\mu\lambda\right) \quad \textbf{(3)}$$